Facts about limits.

Definition 1 (Formal). We say $\lim_{n\to\infty} a_n = L$ if

for all $\epsilon > 0$,
there exists $K \in \mathbb{N}$,
so that if $n \geq K$,
then $|a_n - L| < \epsilon$.

Definition 2 (Intuitive). We say $\lim_{n\to\infty} a_n = L$ if

however close we want to be,
there's a place we can go,
so that beyond that place,
we are that close.

Proposition 3 (Limits are unique). *If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$, then $L = M$.*

Proposition 4. *If the sequence a_n converges, then a_n is bounded.*

Corollary 5. *If the sequence a_n is not bounded, then a_n diverges.*

Proposition 6. *If $\lim_{n\to\infty} a_n = L$, and $c \in \mathbb{R}$, then $\lim_{n\to\infty} c \cdot a_n = c \cdot L$.*

Proposition 7. *If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then $\lim_{n\to\infty} a_n + b_n = L + M$.*

Proposition 8. *If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then $\lim_{n\to\infty} a_n \cdot b_n = L \cdot M$.*

Proposition 9. *If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, and $b_n \neq 0$ and $M \neq 0$, then $\lim_{n\to\infty} a_n / b_n = L/M$.*

Proposition 10. *If $\lim_{n\to\infty} a_n = L$ and $m \in \mathbb{N}$, then $\lim_{n\to\infty} a_{n+m} = L$.*

Proposition 11. *If $\lim_{n\to\infty} a_n = L$, and b_n is a sequence differing from a_n in finitely many terms, then $\lim_{n\to\infty} b_n = L$.*

Theorem 12. *If a_n is a nondecreasing bounded above sequence, then a_n converges.*

Theorem 13. *If a_n is a nonincreasing bounded below sequence, then a_n converges.*

Theorem 14 (Squeezing theorem). *If a_n, b_n, and c_n are sequences of real numbers, and for all $n \in \mathbb{N}$, we have $a_n \leq b_n \leq c_n$, and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Theorem 15 (Sequences and continuity). *If a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at L, and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$.*

Example 16. For a real number $x > 0$,

$$\lim_{n\to\infty} x^{1/n} = 1.$$

Example 17. For a real number x with $-1 < x < 1$,

$$\lim_{n\to\infty} x^n = 0.$$

Example 18. For a real number $x > 0$,

$$\lim_{n\to\infty} \frac{1}{n^x} = 0.$$

Example 19. For $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Example 20. $\lim_{n\to\infty} \dfrac{\log n}{n} = 0.$

Example 21. $\lim_{n\to\infty} n^{1/n} = 0.$

Example 22. For $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$