Point nine repeating.

No maximum element.

Every nonempty bounded set has a least upper bound, but not every set contains a “maximum.”

Does the set $(0,1)$ contains a largest element? **No!** For every number in $(0,1)$, I can find a larger one: namely, if you say that $x \in (0,1)$ is the largest number in $(0,1)$, then I will tell you that $(1+x)/2$ is also in $(0,1)$, but it is bigger.

It helps to think of a concrete example: you might say that 0.963 is the largest number in $(0,1)$, but I will retort that

$$0.963 < \frac{1+0.963}{2} = .9815 \in (0,1)$$

and 0.9815 is bigger than your number.

Repeating decimals.

You might claim that $0.\overline{9}$ is the “biggest” number in $(0,1)$. But I will say that $0.\overline{9} = 1$, so $0.\overline{9}$ is not in the set $(0,1)$.

Why? What might we mean by $0.\overline{9}$? Take a look at the following:

$$\begin{aligned}
0.9 &= 9 \cdot 10^{-1} \\
0.99 &= 9 \cdot 10^{-1} + 9 \cdot 10^{-2} \\
0.999 &= 9 \cdot 10^{-1} + 9 \cdot 10^{-2} + 9 \cdot 10^{-3} \\
0.9999 &= 9 \cdot 10^{-1} + 9 \cdot 10^{-2} + 9 \cdot 10^{-3} + 9 \cdot 10^{-4} \\
\vdots & \quad \vdots \\
0.\overline{9} &= 9 \cdot 10^{-1} + 9 \cdot 10^{-2} + 9 \cdot 10^{-3} + 9 \cdot 10^{-4} + \cdots,
\end{aligned}$$

or in the fancier notation we’ll be introducing soon, $\sum_{n=1}^{\infty} 9 \cdot 10^{-n}$.

In any case, $10 \cdot 0.\overline{9} = 9.\overline{9}$, so

$$\begin{aligned}
9 \cdot 0.\overline{9} &= (10-1) \cdot 0.\overline{9} \\
&= 10 \cdot 0.\overline{9} - 0.\overline{9} \\
&= 9.\overline{9} - 0.\overline{9} = 9.
\end{aligned}$$

Divide both sides by 9 to see that $0.\overline{9}$ must be another name for 1.

A much shorter argument.

You might already believe that $0.\overline{3} = 1/3$. Multiply both sides by three, to get $0.\overline{9} = 1$.

There are many ways to write a number.

Just because 1 looks different than $0.\overline{9}$ doesn’t mean it is different: **IV** is not 4, which is not “four,” which is not ∷, but all of these (might) mean the same thing. *The signifier is not the signified.*