Problem 1.

Let p_1 be the point having polar coordinates $r = 1$ and $\theta = \pi$.
Let p_2 be the point having polar coordinates $r = -1$ and $\theta = \pi/2$.
Find the Euclidean distance between p_1 and p_2.

Solution.

The relationship between cartesian coordinates (x, y) and polar coordinates (r, θ) is given by

$$(x, y) = (r\cos\theta, r\sin\theta).$$

So the cartesian coordinates for p_1 are

$$(1\cos\pi, 1\sin\pi) = (-1, 0).$$

The cartesian coordinates for p_2 are

$$(-1\cos(\pi/2), -1\sin(\pi/2)) = (0, -1).$$

A vector starting at p_1 and ending at p_2 is

$$p_2 - p_1 = (0, -1) - (-1, 0) = (1, -1)$$

which has length

$$||(1, -1)|| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}.$$

The distance between p_1 and p_2 is $\sqrt{2}$.

Problem 2.

Define functions $f_1, \ldots, f_4 : \mathbb{R} \to \mathbb{R}^2$ by

$$\begin{aligned} f_1(t) &= (t^2 + t, t^2 + t), \\ f_2(t) &= (t^2 + t, t^2 - t), \\ f_3(t) &= (t^2 - t, t^2 + t), \text{ and} \\ f_4(t) &= (t^2 - t, t^2 - t). \end{aligned}$$

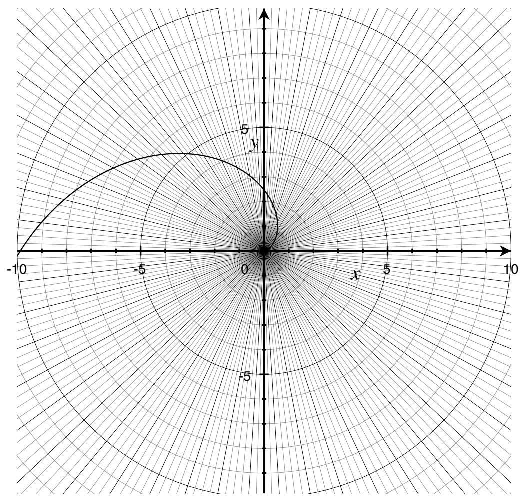
Below are shown graphs of vector-valued functions; label the graphs with the corresponding function, and explain why you made the choices you did. *Hint:* To get started, calculate the deriative at $t = 0$.

Solution.

This problem was ill-posed; everyone received full credit.

Problem 3.

Consider the curve given, in polar coordinates, by $r = \theta^2$. Find the **slope** of the tangent line to the curve when $\theta = \pi/2$.



Solution.

Recall that the relationship between polar and cartesian coordinates is

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}$$

Using this, we convert $r = \theta^2$ into a parametric equation

$$\begin{aligned} x &= \theta^2\cos\theta, \\ y &= \theta^2\sin\theta. \end{aligned}$$

This gives the position of a point (x, y) depending on θ. We differentiate to find

$$\begin{aligned} \frac{dx}{d\theta} &= 2\theta\cos\theta - \theta^2\sin\theta, \\ \frac{dy}{d\theta} &= 2\theta\sin\theta + \theta^2\cos\theta. \end{aligned}$$

The ratio of these derivatives gives

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\theta\cos\theta - \theta^2\sin\theta}{2\theta\sin\theta + \theta^2\cos\theta}.$$

The problem asks us to find the slope of the tangent line, dy/dx, when $\theta = \pi/2$. In the graph, this looks to be roughly -1. In reality, this is

$$\begin{aligned} \frac{dy}{dx} &= \frac{2(\pi/2)\cos(\pi/2) - (\pi/2)^2\sin(\pi/2)}{2(\pi/2)\sin(\pi/2) + (\pi/2)^2\cos(\pi/2)} \\ &= \frac{\pi\cdot 0 - (\pi/2)^2\cdot 1}{\pi\cdot 1 + (\pi/2)^2\cdot 0} \\ &= \frac{-(\pi/2)^2}{\pi} = -\frac{\pi}{4}, \end{aligned}$$

which is a bit less than -1, consistent with the picture.

Problem 4.

Consider the functions $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R} \to \mathbb{R}^2$ given by

$$\begin{aligned} f(t) &= (6t, 8t) \\ g(s) &= (3s, 2s^2). \end{aligned}$$

The graphs intersect when $t = 1$ and $s = 2$; find the cosine of the angle of intersection of these two curves.

Solution.

Note that $f(1) = (6, 8)$ and $g(2) = (6, 8)$, so they do intersect. A common mistake was to find the angle between the vectors $f(1)$ and $g(2)$, when in fact you must find the angle between $f'(1)$ and $g'(2)$. Since

$$f'(t) = (6, 8) \text{ and } g'(s) = (3, 4s)$$

we get

$$f'(1) = (6, 8) \text{ and } g'(2) = (3, 8)$$

The angle between the curves at the point of intersection is the angle between the vectors $(6, 8)$ and $(3, 8)$. The dot product is

$$(6, 8) \cdot (3, 8) = 18 + 64 = 82$$

The length of the first vector is

$$||f'(1)|| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10.$$

The length of the second vector is

$$||g'(2)|| = \sqrt{3^2 + 8^2} = \sqrt{9 + 64} = \sqrt{73}.$$

Consequently, using the geometric interpretation of the dot product,

$$\cos\theta = \frac{f'(1) \cdot g'(2)}{||f'(1)||\,||g'(2)||} = \frac{82}{10 \cdot \sqrt{73}}.$$

Some people wrote this in lowest terms, but that is, of course, not necessary.

Problem 5.

If possible, write the vector $(8, 9) \in \mathbb{R}^2$ as a scalar multiple of $(1, 1)$ added to a scalar multiple of $(1, 2)$. If it is not possible, explain why.

Solution.

We want to find $\alpha, \beta \in \mathbb{R}$ so that

$$(8, 9) = \alpha \cdot (1, 1) + \beta \cdot (1, 2).$$

In other words, we want to solve the system of equations

$$\begin{aligned} 8 &= \alpha \cdot 1 + \beta \cdot 1, \\ 9 &= \alpha \cdot 1 + \beta \cdot 2. \end{aligned}$$

Subtracting the second equation from the first implies $\beta = 1$. Solving then for α, we find $\alpha = 7$. A quick check

$$(8, 9) = 7 \cdot (1, 1) + 1 \cdot (1, 2) = (7 \cdot 1 + 1 \cdot 1, 7 \cdot 1 + 1 \cdot 2)$$

shows that this works.

Problem 6.

Consider two vectors in $\mathbb{R}^2$,

$$v = (1, 2) \text{ and } w = (1, 1).$$

(a) Find a unit length vector pointing in the same direction as v.

(b) Calculate $w \cdot (100v + 1000w)$.

Solution.

The lengh of v is

$$||v|| = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

To find a unit length vector, we divide v by its length, to get

$$\frac{v}{||v||} = \frac{(1, 2)}{\sqrt{5}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right).$$

Some people chose to rationalize the denominator (though this is not necessary).

Problem 7.

Compute the angle between the vectors $(1, 0, 0)$ and $(\sqrt{3}, 0, 3)$. Are they orthogonal?

Solution.

If θ is the angle between vectors v and w, then

$$\cos\theta = \frac{v \times w}{||v|| \, ||w||}.$$

If $v = (1, 0, 0)$ and $w = (\sqrt{3}, 0, 3)$, then

$$v \times w = 1 \cdot \sqrt{3} + 0 \cdot 0 + 0 \cdot 3 = \sqrt{3}.$$

The length of v is $||v|| = 1$. The length of w is

$$||w|| = \sqrt{(\sqrt{3})^2 + 0^2 + 3^2} = \sqrt{3 + 9} = \sqrt{12}.$$

Therefore,

$$\cos\theta = \frac{v \times w}{||v||\,||w||} = \frac{\sqrt{3}}{1 \cdot \sqrt{12}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

But the problem does not ask us to find the cosine of the angle—the problem asks for the angle between the given vectors. Many people made the mistake of only determining whether or not the angles were orthogonal without determining θ. If $\cos\theta = 1/2$, then $\theta = \pi/3$.

Since the dot product is not zero (equivalently, since $\cos\theta$ is not zero), the given vectors are not orthogonal.

Problem 8.

Let $v = (1, 1, 1) \in \mathbb{R}^3$, and $w = (2, 0, 0) \in \mathbb{R}^3$.

(a) Compute $v \times w$.

(b) Calculate the area of the parallelogram having vertices

$$(0, 0, 0), \quad (1, 1, 1), \quad (3, 1, 1), \quad (2, 0, 0).$$

Solution.

To calculate the cross product in **(a)**, we rewrite v as $i+j+k$ and w as $2i$. Then, the cross product can be computed as

$$\begin{aligned} v \times w &= (i + j + k) \times (2i) \\ &= (i \times (2i)) + (j \times (2i)) + (k \times (2i)) \\ &= 0 + 2(j \times i) + 2(k \times i) \\ &= 0 - 2k + 2j, \end{aligned}$$

which, written as a tuple, is $(0, 2, -2)$.

Next, we tackle part **(b)**. The area of the parallelogram with vertices 0,v,w, and $v + w$ is the length of the cross product, i.e., $||v \times w||$. Since $v \times w = (0, 2, -2)$,the area of the parallelogram is the length of $(0, 2, -2)$, namely,

$$||v \times w|| = \sqrt{0^2 + 2^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}.$$

Of course, $\sqrt{8}$ is a perfectly good answer.

Problem 9.

Consider the lines

$$\begin{aligned} L_1 &= \{v \in \mathbb{R}^3 : v = (1, 1, 1)t + (0, 0, 0) \text{ for } t \in \mathbb{R}\ \}, \\ L_2 &= \{v \in \mathbb{R}^3 : v = (1, 1, 2)t + (2, 2, 0) \text{ for } t \in \mathbb{R}\ \}. \end{aligned}$$

Do L_1 and L_2 intersect? If they intersect, find the point of intersection; if not, determine whether L_1 and L_2 are parallel or skew.

Solution.

They do intersect. We search for $t, s \in \mathbb{R}$ so that

$$(1,1,1)t + (0,0,0) = (1,1,2)s + (2,2,0).$$

Next, replace this single vector equation with three equations,

$$\begin{aligned} 1 \cdot t + 0 &= 1 \cdot s + 2, \\ 1 \cdot t + 0 &= 1 \cdot s + 2, \\ 1 \cdot t + 0 &= 2 \cdot s + 0. \end{aligned}$$

Subtracting the third equation from the second gives $0 = s - 2$, so $s = 2$. Solving for t then, $t = 4$.

The problem asks us to find the point where these lines intersect (and a common mistake was forgetting to do this). Substituting $t = 4$ into the first equation, we find that the point of intersection is

$$(4,4,4) = (1,1,1) \cdot 4 + (0,0,0).$$

Problem 10.

Suppose P is a plane containing the point $(1,1,1)$ and having normal vector $(4,5,-6)$.

(a) Write down an equation for the plane P.

(b) Find the intersection of P with the line

$$L = \{v \in \mathbb{R}^3 : v = t \cdot (1,1,1) + (0,0,0) \text{ for } t \in \mathbb{R}\},$$

or, if they do not intersect, prove it.

Solution.

For part **(a)**, you should write down the formula for a plane through a point b and with normal vector n, namely,

$$P = \{v \in \mathbb{R}^3 : (v - b) \cdot n = 0\}.$$

So in this case,

$$P = \{v \in \mathbb{R}^3 : (v - (1,1,1)) \cdot (4,5,-6) = 0\}.$$

Of course, there are other ways of writing down an equation for a plane, and I gave credit for any of them. For instance, you might rewrite this as

$$((x,y,z) - (1,1,1)) \cdot (4,5,-6) = 0$$

and then expand out to get

$$4x + 5y - 6z = 3.$$

I meant part **(b)** to be really easy: the line L includes the point $(1,1,1)$, and the plane P also contains the point $(1,1,1)$, so they intersect at that point. You could have written down a system of linear equations and solved it, too, but you don't need to if you can see the solution right away.

Problem 11.

The following ten questions are for extra credit. **Circle** the best answer.

True $\boxed{\textbf{False}}$ There are four vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ so that v_i and v_j are orthogonal for $i \neq j$. *I meant this problem to ask for nonzero vectors; as stated, it is true, if you use the zero vector. I gave credit for either answer.*

$\boxed{\textbf{True}}$ **False** If $v_1, v_2 \in \mathbb{R}^3$, then $v_1 \times v_2$ is orthogonal to both v_1 and v_2. *This is a property of the cross product.*

$\boxed{\textbf{True}}$ **False** There are two lines in $\mathbb{R}^3$ which do not intersect and are not parallel. *This is the definition of skew lines.*

True $\boxed{\textbf{False}}$ If $f(t) = (t, t^2, t^3)$, then $\frac{df}{dt} = (1, 2, 3)$ when $t = 0$. *This is actually the derivative evaluated at $t = 1$ not $t = 0$.*

$\boxed{\textbf{True}}$ **False** Given a vector $v \in \mathbb{R}^3$, it is possible to find six unit vectors in $\mathbb{R}^3$ which are each orthogonal to v. *You can find infinitely many vectors perpendicular to a given vector*

True $\boxed{\textbf{False}}$ If v, w, u are vectors in $\mathbb{R}^3$, then $((v \cdot u) \cdot w) \times v$ is zero.

True $\boxed{\textbf{False}}$ Suppose $v \in \mathbb{R}^2$ is a unit vector; there does not exist a vector $w \in \mathbb{R}^2$ so that $v \cdot w = 10^{10}$. *There does exist such a vector—you can choose $w = 10^{10} \cdot v$.*

$\boxed{\textbf{True}}$ **False** Given a line L in $\mathbb{R}^3$, it is possible to find two planes P_1 and P_2 so that P_1 and P_2 do not intersect L, but P_1 intersects P_2 in a line. *Choose two planes that are not parallel, but which do not contain the line L.*

$\boxed{\textbf{True}}$ **False** For all vectors $v, w \in \mathbb{R}^3$, it is the case that

$$((4w) \times (3v)) \cdot (5w) = ((3v) \times (4w)) \cdot (4w).$$

In fact, both sides are zero.

$\boxed{\textbf{True}}$ **False** For vectors $v, w \in \mathbb{R}^4$, it is always the case that

$$(v + w) \cdot (v + w) = ||v||^2 + ||w||^2 + 2 \cdot (v \cdot w).$$

You can prove this by using the distributive property.