

Das Unendliche hat wie keine andere Frage von jeher so tief das Gemüt der Menschen bewegt. *The infinite, has, like no other question, moved the human mind so deeply.*

—David Hilbert

Problem 1

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Let A be the set of English words, and B be the set of words (nonsense or not) made up of (possibly zero!) letters in the Latin alphabet. Consider the function $f : A \rightarrow B$ which sends an English word to the word without its last letter.

Is f injective? Is f surjective?

Solution

Problem 2

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Let A be the set of all strings of at least 2 letters in the Latin alphabet. Let $r : A \rightarrow A$ be the function which reverses its input (so $r(\text{hello}) = \text{olleh}$) and $s : A \rightarrow A$ be the function which takes the first letter and makes it the last letter (so $s(\text{hello}) = \text{elloh}$).

By applying r and s repeatedly, is it possible to transform game into mage? Is it possible to transform pets to pest? Is it possible to transform maple into ample?

Solution

Problem 3

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Let $A = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $B = \{x \in \mathbb{Z} : x \text{ is odd}\}$. Describe a bijection $f : A \rightarrow B$.
Be sure to justify your answer completely.

Solution

Problem 4

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Use complete induction to prove that every natural number can be written as a product of prime numbers.

Solution

Problem 5

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Consider the statement: if $f : A \rightarrow A$ is injective, then f is surjective. What is the converse of this statement? The contrapositive of this statement? If A is a finite set, which of these statements are true?

Solution

Problem 6

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Let $f : \mathbb{R} \rightarrow \mathbb{R}$, and suppose $\exists x \in \mathbb{R}, \forall y \in \mathbb{R} \exists z \in \mathbb{R} f(x+z) = y$. Does it follow that f is surjective?

Solution

Problem 7

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Suppose $f : \mathbb{N} \rightarrow \mathbb{N}$ is given by

$$f(n) = \begin{cases} n + 1 & \text{if } n \text{ is odd} \\ n - 1 & \text{if } n \text{ is even} \end{cases}$$

Is f bijective? If so, find an inverse function.

Solution

Problem 8

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Suppose A, B, C are sets, and $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions. If f is surjective, and g is injective, what can you say about the composition $g \circ f$? Need it be injective? Need it be surjective?

Solution

Suppose $f : \mathbb{N} \rightarrow \mathbb{N}$ is a function with the properties that

- $f(xy) = f(x) \cdot f(y)$ and
- $f(p) = 2$ if p is prime.

Among two-digit numbers x , how large can $f(x)$ be?

Solution

Problem 10

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Find three sets of real numbers, A , B , and C , so that $A \cap B \neq \emptyset$, and $A \cap C \neq \emptyset$, and $B \cap C \neq \emptyset$, but $A \cap B \cap C = \emptyset$.

Solution

Let $P_2(A)$ be the set of two element subsets of A . If A is an n -element set, show that $P_2(A)$ has $\binom{n}{2}$ elements.

Solution

Problem 12

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Let A be the set of natural numbers which only use the digits 1 and 2. Define B by

$$\{n \in \mathbb{N} : 9n \in A\}.$$

Does B have a least element?

Solution

Problem 13

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Let A be a set. Suppose $P(n)$ is the proposition $(n \in A) \rightarrow ((n + 1) \in A)$. If $P(n)$ is true for all n , does it follow that $A = \mathbb{N}$?

Solution

Let P, Q, R, S be propositions.

Rewrite $(\neg(P \vee (Q \wedge R))) \wedge S$ in a simpler way.

Also, rewrite $(\neg(P \vee (Q \wedge R))) \wedge P$ in a simpler way.

Solution

Problem 15

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Show that every k^{th} Fibonacci number is a multiple of F_k .

Solution

For which $n \in \mathbb{Z}$ is there a number $x \in \mathbb{Z}$ so that $x^2 \equiv n \pmod{1}$?

Solution

Problem 17

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For which prime numbers p is it possible to find an integer x so that $x^2 \equiv -1 \pmod{p}$? You might not be able to prove your statement, but you will probably be able to come up with enough evidence to support your conjecture.

Solution

Solution

Problem 19

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Let A be the set

$$\{n \in \mathbb{N} : n \text{ is prime and } n \equiv 3 \pmod{4}\}.$$

Prove that A is infinite.

Solution

Let A be the set of prime numbers. Can you find a million consecutive numbers in the set $\mathbb{N} \setminus A$?

Solution

Problem 21

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Explain why there is a bijection between \mathbb{N} and \mathbb{Q} .

Solution

Prove that, if A is an infinite set, then there exists a proper subset $B \subset A$ with B equinumerous to A .

Solution

Problem 23

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Let A, B, C be sets. Prove that $A \cap (B \cup C)$ is equal to $(A \cap B) \cup (A \cap C)$.

Solution

Problem 24

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For $q \in \mathbb{Q}$, let A_q be the interval $(q - \epsilon, q + \epsilon)$. Prove that $\bigcup_{q \in \mathbb{Q}} A_q = \mathbb{R}$.

Solution

Problem 25

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Show that 101 divides $(15 + 17)^{101} - 15 - 17$.

Solution

Describe a collection of eight sets, any seven of which intersect, but for which all eight do not intersect.
Describe a collection of k sets, any $(k - 1)$ of which intersect, but for which all k do not intersect.

Solution

Define functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ and $g : \mathbb{Z} \rightarrow \mathbb{Z}$ by $f(n) = n + 1$ and

$$g(n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n = 1, \\ n & \text{otherwise.} \end{cases}$$

Describe a bijective function $h : \mathbb{Z} \rightarrow \mathbb{Z}$ which cannot be written as a composition of f and g and f^{-1} .

Solution

In the statement

$$\forall x \in \mathbb{R} ((\exists y \in \mathbb{R} (x^2 > y)) \vee (\exists z \in \mathbb{R} (z^2 < w))),$$

which variables are bound? Which variables are free?

Solution

Problem 29

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Prove the binomial theorem.

Solution

Show that

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}.$$

Solution

Problem 31

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Describe a proposition $P(x, y)$ so that

$$\forall x \exists y P(x, y)$$

is true, but

$$\exists x \forall y P(x, y)$$

is false.

Solution

Let ω be the set of nonnegative integers, and define a function $A : \omega \times \omega \rightarrow \omega$,

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0 \\ A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m - 1, A(m, n - 1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

Compute $A(4, 4)$.

Solution

Problem 33

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Define a sequence a_n by the rule that $a_0 = 1$ and $a_1 = 2$ and $a_{n+2} = 2a_{n+1} + a_n$. For which values of n is a_n a multiple of 7?

Solution

Problem 34

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Show that the set of functions $f : \mathbb{N} \rightarrow \mathbb{N}$ is not countable.

Solution

Problem 35

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Let A be the set of functions $\mathbb{N} \rightarrow \mathbb{N}$, and let $g : \mathbb{N} \rightarrow A$ be a function. Define a function $f : \mathbb{N} \rightarrow \mathbb{N}$ by $f(n) = g(n)(n) + 1$. Does there exist a number $F \in \mathbb{N}$ so that $g(F) = f$?

Solution

Define $A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, and suppose B_x is the interval $(x, 10)$. Determine

$$\bigcup_{x \in A} B_x \text{ and } \bigcap_{x \in A} B_x \text{ and}$$

Solution

Let A, B, C be three subsets of \mathbb{R}^2 . Rewrite

$$\mathbb{R}^2 \setminus ((\mathbb{R}^2 \setminus (A \cup B)) \cap C)$$

in a shorter form.

Solution

Problem 38

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Let A be the power set of \mathbb{R} ; show that there is no bijection between A and \mathbb{R} .

Solution
