Fake Midterm 2 — Math 345

November 2010

> Practical application is found by not looking for it, and one can say that the whole progress of civilization rests on that principle. —Hadamard

Name: ______________________________

Lecture time (circle one): 12:30–1:18 P.M.

2:30–3:18 P.M.

1. Write your name above.
2. Calculators are forbidden.
3. Look inside the fake exam before taking the real exam.
4. Justify your answers.
5. Show your work.
6. Write your answers down to practice.
7. Answer all questions.
8. To prevent fire, do not divide by zero.

Problem 1	/360
Problem 2	/360
Problem 3	/360
Problem 4	/360
Problem 5	/360
Problem 6	/360
Problem 7	/360
Problem 8	/360
Problem 9	/360
Problem 10	/360
Problem 11	/360
Problem 12	/360
Problem 13	/360
Problem 14	/360
Problem 15	/360
Problem 16	/360
Problem 17	/360
Problem 18	/360
Problem 19	/360
Problem 20	/360
Total	/7200

Problem 1 /360

Is it the case for all integers x that

$$12 \text{ divides } x(x+1)(x+2)(x+3)?$$

If so, prove it. If not, provide a counterexample.

Solution

Problem 2 /360

Is it the case for all integers x that

$$16 \text{ divides } 17^n - 1?$$

If so, prove it. If not, provide a counterexample.

Solution

Problem 3 /360

Suppose $S \subset \mathbb{Z}$ and that for all $x \in S$, $x > -10$. Does S have a least element? If so, prove it. If not, give a counterexample.

Solution

Problem 4 /360

Suppose $a \equiv b \mod m$ and $c \equiv d \mod m$. Is it the case that $ac \equiv bd \mod m$? If so, prove it. If not, give a counterexample.

Solution

Problem 5 /360

Suppose $a \equiv b \mod m$ and $c \equiv d \mod m$. Is it the case that $a + c \equiv b + d \mod m$? If so, prove it. If not, give a counterexample.

Solution

Problem 6 /360

Suppose $a \equiv b \mod m$ and $c \equiv d \mod m$. Is it the case that $a^c \equiv b^d \mod m$? If so, prove it. If not, give a counterexample.

Solution

Problem 7 /360

Find a polynomial $f(n)$ so that

$$f(n) = \sum_{k=1}^{n} \left(k^2 + k\right).$$

Solution

Problem 8 /360

Find a polynomial $f(n)$ so that

$$f(n) = \sum_{k=1}^{n} k^3$$

Solution

Problem 9 /360

Prove by induction that for all $n \in \mathbb{N}$,

$$\binom{n}{2} = \frac{(n)(n-1)}{2}.$$

(I thank Marilyn Rayner for correcting an error in a previous version of this problem.)

Solution

Problem 10 /360

Prove by induction that for all $n \in \mathbb{N}$ and for all integers x and y,

$$\frac{x^n - y^n}{x - y}$$

is an integer.

Solution

Problem 11 /360

Prove by induction that, for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} (2k+1)$$

is a perfect square. (I thank Marilyn Rayner for correcting an error in a previous version of this problem.)

Solution

Problem 12 /360

Prove by induction that, for all $n \in \mathbb{N}$,

$$\sqrt{2\sqrt{3\sqrt{4\cdots\sqrt{(n-1)\sqrt{n}}}}} < 3.$$

Solution

Problem 13 /360

Prove by induction that, for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n.$$

(This problem might be very hard)

Solution

Problem 14 /360

Let F_n be the Fibonacci numbers. For wich values of n does F_n end in a zero?

Solution

Problem 15 /360

Let F_n be the Fibonacci numbers (here, $F_1 = 1$ and $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$). Suppose x is a real number for which $x^2 = 1 - x$. Is it the case that $x^{100} = F_{99} - F_{100}x$? If so, prove it. (This is a situation where you might want to prove a much stronger statement by induction).

Solution

Problem 16 /360

Define a sequence G_n so that $G_1 = 1$, $G_2 = 1$, $G_3 = 1$, and $G_{n+3} = G_{n+2} + G_{n+1} + G_n$. For which n is G_n even? Prove your claim by using induction.

Solution

Problem 17 /360

Prove that there are infinitely many prime numbers.

Solution

Problem 18 /360

State and prove the binomial theorem.

Solution

Problem 19 /360

Use the Binomial theorem to expand $(1+x)^8$.

Solution

Problem 20 /360

Show that for every integer $x \geq 2$, there is a prime number p so that p divides x.