





Consider the proposition:

$$(Q \Rightarrow R) \Rightarrow ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))$$

Is the proposition a tautology? If not, explain why not. If it is a tautology, use the method of conditional proof to prove that it is a tautology. Do not use cases, and be careful not to skip any steps.

## Solution

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**Claim.** *The given proposition is a tautology.*

*Proof.* Assume  $Q \Rightarrow R$ . (A1)

We want to prove that  $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$ .

To prove this latter statement, we will assume  $P \Rightarrow Q$ . (A2)

we want to prove  $P \Rightarrow R$ .

To prove this final statement, we will assume  $P$ . (A3)

We want to prove  $R$ .

Assumptions (A3) and (A2) are that  $P$  is true and that  $P \Rightarrow Q$   
so by modus ponens,  $Q$  is true.

But assumption (A1) is that  $Q \Rightarrow R$ ,  
so by modus ponens,  $R$  is true.

We have proved  $R$  is true, which is what we wanted to prove. □

## Commentary

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Many people submitted solutions for this problem that were much more complicated than the proof I offer above; the problem text asks you not to split into cases. The structure of the proof is modeled on the structure of the proposition: the proof begins by considering the outermost implication arrow, which involves proving another implication, which itself involves proving an implication.

Let  $x$  be an integer.

Write down the **contrapositive** of the conditional sentence

If  $x$  is even, then  $x^2$  is even.

Is this a true statement? Is the contrapositive a true statement?

If yes, prove it. If not, find a counterexample.

### Solution

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Let  $P$  be the proposition “ $x$  is even” and  $Q$  be the proposition “ $x^2$  is even.” The contrapositive of  $P \Rightarrow Q$  is  $(\neg Q) \Rightarrow (\neg P)$ , so the contrapositive of the given statement is “If  $x^2$  is not even, then  $x$  is not even.”

The original statement is true.

**Claim.** *If  $x$  is even, then  $x^2$  is even.*

*Proof.* Assume  $x$  is even.

Then there exists an integer  $k$  so that  $x = 2k$ ,  
and so  $x^2 = (2k)^2 = 4k^2$ .

But  $4k^2 = 2(2k^2)$ ,

so  $x^2$  is also twice an integer, and therefore even. □

The contrapositive is also true

(because an implication holds if and only if the contrapositive of the implication holds).

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### Commentary

Many people submitted separate proofs for the original statement and the contrapositive: this is unnecessary. It is easier to prove the original statement than the contrapositive in my opinion.

Consider the four propositions

$$P \vee (Q \wedge R), \tag{1}$$

$$(P \wedge Q) \Rightarrow R, \tag{2}$$

$$(\neg P) \vee (\neg Q) \vee R, \text{ and} \tag{3}$$

$$(P \vee Q) \wedge (P \vee R). \tag{4}$$

Exactly which of these propositions are logically equivalent to which other propositions?

Provide justification for all the claims you make; in particular, if you claim that two propositions are logically equivalent, you must prove this, and if you claim that they are *not* equivalent, you must explain why not.

I prefer arguments that don't involve cases.

## Solution

I claim that (1) and (4) are equivalent, and (2) and (3) are equivalent, but neither (1) nor (4) are equivalent to (2) or (3).

**Claim.**  $(1) \equiv (4)$ .

*Proof.* This is precisely the distributive law. □

**Claim.**  $(2) \equiv (3)$ .

*Proof.*

$$\begin{aligned} (2) &\equiv (P \wedge Q) \Rightarrow R \\ &\equiv (\neg(P \wedge Q)) \vee R && \text{(definition of } \Rightarrow \text{)} \\ &\equiv ((\neg P) \vee (\neg Q)) \vee R && \text{(de Morgan's law)} \\ &\equiv (3). \end{aligned}$$

□

**Claim.**  $(1) \not\equiv (3)$ .

*Proof.* If  $P$  is true,  $Q$  is true, and  $R$  is false, then (3) is false, but (1) is true. So  $(1) \not\equiv (3)$ . Providing a specific example is the quickest way to verify this. □

## Commentary

A complete solution to this problem requires proving three statements (namely that  $(1) \equiv (4)$ , that  $(2) \equiv (3)$ , and that  $(1) \not\equiv (3)$ ). You received 120 points for each claim you proved. Many people failed to discuss why  $(1) \not\equiv (3)$  and lost points because of this; the problem text asks “exactly which” propositions are equivalent, so you must discuss both which are equivalent and which are inequivalent.

Let  $x$  and  $y$  be real numbers, and consider the following proposition:

If  $x$  is rational and  $y$  is irrational, then  $x + 2y$  is irrational.

If the proposition is true, prove it; if not, give a counterexample.

## Solution

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This is a true proposition.

**Claim.** *If  $x$  is rational and  $y$  is irrational, then  $x + 2y$  is irrational.*

*Proof.* Assume that  $x$  is rational,  $y$  is irrational. For a contradiction, we assume that  $x + 2y$  is rational.

Since the difference of rational numbers is rational,  $(x + 2y) - x$  is rational, so  $2y$  is rational. But  $1/2$  is rational, and since the product of rational numbers is rational,  $(1/2) \cdot 2y = y$  is rational. But this is a contradiction— $y$  is irrational.  $\square$

For completeness, I include proofs of two results that I used.

**Claim.** *The difference of rational numbers is rational.*

*Proof.* Assume  $x, y \in \mathbb{Q}$ . Then there exist integers  $a, b, c, d \in \mathbb{Z}$  with  $b, d \neq 0$  so that

$$x = \frac{a}{b} \text{ and } y = \frac{c}{d}$$

and, combining denominators,

$$x - y = \frac{ad - bc}{bd}$$

is a rational number, since  $ad - bc \in \mathbb{Z}$ ,  $bd \in \mathbb{Z}$ , and  $bd \neq 0$ .  $\square$

**Claim.** *The product of rational numbers is rational.*

*Proof.* Assume  $x, y \in \mathbb{Q}$ . Then there exist integers  $a, b, c, d \in \mathbb{Z}$  with  $b, d \neq 0$  so that

$$x = \frac{a}{b} \text{ and } y = \frac{c}{d}$$

and, multiplying,

$$xy = \frac{ac}{bd}$$

is a rational number, since  $ac, bd \in \mathbb{Z}$  and  $bd \neq 0$ .  $\square$

## Commentary

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A number of people stated that  $2y$  is irrational by claiming that a rational number times an irrational number is irrational; this is not a true statement (consider when the rational number is zero).

Is the statement

$$\forall x \in \mathbb{R} (\exists y \in \mathbb{R} (y^2 < x))$$

true or false? For full credit, justify your answer.

### Solution

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The proposition is false.

I claim that the negation of  $\forall x \in \mathbb{R} (\exists y \in \mathbb{R} (y^2 < x))$  is true, namely

**Claim.**  $\exists x \in \mathbb{R} (\forall y \in \mathbb{R} (y^2 \geq x))$

*Proof.* Set  $x = -1$ .

Let  $y$  be a real number.

Then  $y^2 \geq 0$ , so  $y^2 \geq 0 > -1 = x$ , which is what I wanted to prove. □

### Commentary

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Many students gave a specific choice of  $x$  and  $y$  for which  $y^2 \not< x$ ; this is not enough; you need to explain why there is a value of  $x$  for which no value of  $y$  will satisfy  $y^2 < x$ .

