

State the binomial theorem, and then apply it to prove that, for nonnegative integers n ,

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

Be sure to explain your argument carefully.

Solution

First, I state the binomial theorem:

For $x, y \in \mathbb{R}$ and a nonnegative integer n ,

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

Claim.

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

Proof. Let $x = y = 1$; then the binomial theorem gives

$$(1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 1^k.$$

But $1^k = 1$ and $1^{n-k} = 1$, so

$$2^n = (1 + 1)^n = \sum_{k=0}^n \binom{n}{k},$$

which is what we wanted to prove. □

Commentary

Some people gave an inductive proof of this fact; this is not needed. All you need to do is apply the binomial theorem.

Prove, by complete induction, that every integer $x \geq 2$ is either prime, or is the product of primes.

Solution

Let $P(x)$ be the statement that x is either prime, or a product of primes. In other words, $P(x)$ is the statement that $x = p_1 \cdots p_n$ for primes p_i .

Claim. For all integers $x \geq 2$, the statement $P(x)$ is true.

Proof. We proceed by induction.

Base case. Let $x = 2$. Then two is prime, so $P(2)$ is true.

Inductive step. We assume $P(k)$ holds for $2 \leq k \leq x$. We want to show $P(x + 1)$.

If $x + 1$ is prime, then $P(x + 1)$ is true.

If $x + 1$ is not prime, then $x + 1 = ab$ for $a, b \in \mathbb{N}$ with $a \neq 1$ and $b \neq 1$. Then $2 \leq a < x + 1$ and $2 \leq b < x + 1$, so $P(a)$ and $P(b)$ are true. This means

$$\begin{aligned} a &= p_1 \cdots p_n \text{ and} \\ b &= q_1 \cdots q_m \end{aligned}$$

for primes p_i and q_j , so

$$x + 1 = ab = (p_1 \cdots p_n)(q_1 \cdots q_m)$$

so $P(x + 1)$ is true.

Thus, by strong induction, the statement $P(x)$ holds for all integers $x \geq 2$. □

Commentary

Some people did not use complete induction—you really need to use complete induction here.

Prove that there are infinitely many prime numbers.

Solution

Claim. *There are infinitely primes numbers.*

Proof. Suppose not—then there are finitely many prime numbers, say p_1, \dots, p_n .

Consider $x = p_1 \cdots p_n + 1$. By the previous problem, x can be written as a product of primes, so in particular, there is a prime number p_i which divides x . But p_i also divides the product of all the primes, $p_1 \cdots p_n$. Therefore, by the theorem that an integer dividing a and b divides $a - b$, we conclude that p_i divides $x - p_1 \cdots p_n$. But then p_i divides 1, but $p_i > 0$, so $p_i \leq 1$. This implies $p_i = 1$, which is a contradiction. \square

Commentary

Many people gave very nicely written solutions for this problem.

Let $A = \{k \in \mathbb{Z} \mid k \geq 3\}$. Prove by induction for each $n \in A$ that

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

You may use the recurrence relation for binomial coefficients, and the fact that

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

You will get more points if you do the algebra in the induction efficiently than if you do it correctly but inefficiently—don't just expand everything in sight. Look for a way to apply the distributive law to make your work easier to understand.

Solution

Let $P(n)$ be the statement that

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

I claim, for an integer n with $n \geq 3$, the statement $P(n)$ holds.

Proof. We proceed by induction.

Base case. Consider $n = 3$. The statement $P(3)$ is

$$\binom{3}{3} = \frac{3(3-1)(3-2)}{6} = 1,$$

which is true by inspecting Pascal's triangle.

Inductive step. Assume $P(n)$; we will show $P(n+1)$. By the recurrence relation for binomial coefficients,

$$\begin{aligned} \binom{n+1}{3} &= \binom{n}{3} + \binom{n}{2} \\ &= \binom{n}{3} + \frac{n(n-1)}{2} && \text{(as we are told we may assume)} \\ &= \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)}{2} && \text{(by the inductive hypothesis)} \\ &= (n)(n-1) \left(\frac{n-2}{6} + \frac{1}{2} \right) = (n)(n-1) \frac{n+1}{6} && \text{(by the distributive law)} \end{aligned}$$

which is the statement $P(n+1)$. Therefore, by induction, $P(n)$ holds for all $n \geq 3$. □

Commentary

Many people used the fact that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

to prove this; however, the problem requires that you do this by induction.

Give a proof by induction that, for all $n \in \mathbb{N}$,

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}.$$

Then, explain why $1^3 + 2^3 + 3^3 + \dots + (2^{10})^3 \equiv 1 \pmod{3}$.

Solution

Let $P(n)$ be the statement that

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}.$$

I claim $P(n)$ holds for all $n \in \mathbb{N}$.

Proof. We proceed by induction. **Base case.** Let $n = 1$. Then $P(1)$ asserts

$$\sum_{k=1}^1 k^3 = \frac{1^2(1+1)^2}{4} = 1,$$

which is true. **Inductive step.** Suppose $P(n)$; we prove $P(n+1)$. Since $P(n)$, we know

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}.$$

Adding $(n+1)^3$ to both sides yields

$$\begin{aligned} \sum_{k=1}^{n+1} k^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2 \left(\frac{n^2}{4} + (n+1) \right) \\ &= (n+1)^2 \left(\frac{n^2}{4} + \frac{4n+4}{4} \right) \\ &= (n+1)^2 \left(\frac{n^2 + 4n + 4}{4} \right) \\ &= (n+1)^2 \left(\frac{(n+2)^2}{4} \right), \end{aligned}$$

which is the statement $P(n+1)$, so, by induction, we may conclude $\forall n \in \mathbb{N} P(n)$. □

Claim. $1^3 + 2^3 + 3^3 + \cdots + (2^{10})^3 \equiv 1 \pmod{3}$

Proof. By the preceding claim, $1^3 + 2^3 + 3^3 + \cdots + (2^{10})^3 = \frac{(2^{10})^2(2^{10} + 1)^2}{4}$, and

$$\begin{aligned} \frac{(2^{10})^2(2^{10} + 1)^2}{4} &= \frac{2^{20}(2^{20} + 2 \cdot 2^{10} + 1)}{4} \\ &= 2^{18}(2^{20} + 2 \cdot 2^{10} + 1) \\ &= 4^9(4^{10} + 2 \cdot 4^5 + 1) \\ &\equiv 1^9(1^{10} + 2 \cdot 1^5 + 1) \pmod{3} \\ &\equiv 1(1 + 2 + 1) \pmod{3} \\ &\equiv 4 \equiv 1 \pmod{3} \end{aligned}$$

□