

2.5. EBRT in A, B, fA, fB, \subseteq on (ELG, INF) .

In this section, we use the tree methodology described in section 2.1 to analyze EBRT in A, B, fA, fB, \subseteq on (ELG, INF) and $(EVSD, INF)$. We handle both BRT settings at once, as they behave the same way for EBRT in A, B, fA, fB, \subseteq . In particular, we show that they are RCA_0 secure (see Definition 1.1.43).

Some of this treatment is the same as for EBRT in A, B, fA, fB, \subseteq on (SD, INF) given in section 2.4. However, many new features appear that makes this section considerably more involved than section 2.4.

A key difference between EBRT in A, B, fA, fB, \subseteq on (SD, INF) and on (ELG, INF) is that the Compelmentation Theorem holds on (SD, INF) , yet fails on (ELG, INF) . E.g., it fails for $f(x) = 2x$.

Let $f: N^k \rightarrow N$ be partial. Define the following series of sets by induction $i \geq 1$.

$$\begin{aligned} S_1 &= N. \\ S_{i+1} &= N \setminus fS_i. \end{aligned}$$

LEMMA 2.5.1. $S_2 \subseteq S_4 \subseteq S_6 \subseteq \dots \subseteq \dots \subseteq S_5 \subseteq S_3 \subseteq S_1$. I.e., for all $i \geq 1$, $S_{2i} \subseteq S_{2i+2} \subseteq S_{2i+1} \subseteq S_{2i-1}$.

Proof: We argue by induction on $i \geq 1$. The basis case is

$$S_2 \subseteq S_4 \subseteq S_3 \subseteq S_1.$$

To see this, clearly

$$\begin{aligned} S_3 &\subseteq S_1. \\ N \setminus S_1 &\subseteq N \setminus S_3. \\ S_2 &\subseteq S_4. \\ S_2 &\subseteq S_1. \\ N \setminus S_1 &\subseteq N \setminus S_2. \\ S_2 &\subseteq S_3. \\ fS_2 &\subseteq fS_3. \\ N \setminus fS_3 &\subseteq N \setminus fS_2. \\ S_4 &\subseteq S_3. \end{aligned}$$

Now assume the induction hypothesis

$$S_{2i} \subseteq S_{2i+2} \subseteq S_{2i+1} \subseteq S_{2i-1}.$$

Then

$$\begin{aligned}
fS_{2i} &\subseteq fS_{2i+2} \subseteq fS_{2i+1} \subseteq fS_{2i-1}. \\
N \setminus fS_{2i-1} &\subseteq N \setminus fS_{2i+1} \subseteq N \setminus fS_{2i+2} \subseteq N \setminus fS_{2i}. \\
S_{2i} &\subseteq S_{2i+2} \subseteq S_{2i+3} \subseteq S_{2i+1}. \\
fS_{2i} &\subseteq fS_{2i+2} \subseteq fS_{2i+3} \subseteq fS_{2i+1}. \\
N \setminus fS_{2i+1} &\subseteq N \setminus fS_{2i+3} \subseteq N \setminus fS_{2i+2} \subseteq N \setminus fS_{2i}. \\
S_{2i+2} &\subseteq S_{2i+4} \subseteq S_{2i+3} \subseteq S_{2i+1}.
\end{aligned}$$

QED

LEMMA 2.5.2. Let $f:N^k \rightarrow N$ be partial, where each $f^{-1}(n)$ is finite. Let $A = S_2 \cup S_4 \cup \dots$, and $B = S_1 \cap S_3 \cap \dots$. Then $A \subseteq B$, $A = N \setminus fB$, $B = N \setminus fA$.

Proof: Let A, B be as given. By Lemma 2.5.1, $A \subseteq B$.

Fix $i \geq 1$. $S_{2i} = N \setminus fS_{2i-1}$, $S_{2i} \cap fS_{2i-1} = \emptyset$, $S_{2i} \cap fB = \emptyset$. Since $i \geq 1$ is arbitrary, $A \cap fB = \emptyset$. I.e., $A \subseteq N \setminus fB$.

Since $S_{2i+1} = N \setminus fS_{2i}$, we see that for all $j \geq i$, $S_{2i+1} \cap fS_{2j} = \emptyset$. Hence $S_{2i+1} \cap fA = \emptyset$. Since $i \geq 1$ is arbitrary, $B \cap fA = \emptyset$. I.e., $B \subseteq N \setminus fA$.

Now let $n \in N \setminus fB$. We claim that for some $j \geq 0$, $n \notin fS_{2j+1}$. Suppose that for all $j \geq 0$, $n \in fS_{2j+1}$. Since $f^{-1}(n)$ is finite, there exists $x \in f^{-1}(n)$ which lies in infinitely many S_{2j+1} . Hence there exists $x \in f^{-1}(n)$ such that $x \in B$. Therefore $n \in fB$. This establishes the claim. Fix $j \geq 0$ such that $n \notin fS_{2j+1}$. Then $n \in S_{2j+2}$, and so $n \in A$. This establishes that $A = N \setminus fB$.

Finally, let $n \in N \setminus fA$. Then for all i , $n \notin fS_{2i}$. Hence for all j , $n \in S_{2j+1}$. Therefore $n \in B$. This establishes that $B = N \setminus fA$. QED

LEMMA 2.5.3. Let $f:[0,n]^k \rightarrow [0,n]$ be partial, $n \geq 0$. There exist $A \subseteq B \subseteq [0,n]$ such that $A = [0,n] \setminus fB$ and $B = [0,n] \setminus fA$.

Proof: Let n, f be as given. Obviously $f:N^k \rightarrow N$ is partial, and each $f^{-1}(n)$ is finite. By Lemma 2.5.2, let $A = S_2 \cup S_4 \cup \dots$, and $B = S_1 \cap S_3 \cap \dots$. Then $A \subseteq B$, $A = N \setminus fB$, $B = N \setminus fA$. Note that $A \cap [0,n] \subseteq B \cap [0,n]$, $A \cap [0,n] = [0,n] \setminus fB$, $B \cap [0,n] \setminus fA$. QED

LEMMA 2.5.4. For all $f \in \text{EVSD}$ there exist infinite $A \subseteq B \subseteq \mathbb{N}$ such that $B \cup fA = A \cup fB = \mathbb{N}$.

Proof: Let $f \in \text{EVSD}$. Let $n \geq 1$ be such that $|x| \geq n \rightarrow f(x) > |x|$. Let f' be the restriction of f to those elements of $[0, n-1]^k$ whose value lies in $[0, n-1]$. Then $f': [0, n-1]^k \rightarrow [0, n-1]$ is partial.

By Lemma 2.5.3, let $A' \subseteq B' \subseteq [0, n-1]$, where $A' = [0, n-1] \setminus f'B'$ and $B' = [0, n-1] \setminus f'A'$.

We now define the required A, B by induction. Membership in A, B for $m < n$ is just membership in A', B' . Thus for all $m < n$,

$$\begin{aligned} m \in B &\leftrightarrow m \in B' \leftrightarrow m \notin f'A' \leftrightarrow m \notin fA. \\ m \in A &\leftrightarrow m \in A' \leftrightarrow m \notin f'B' \leftrightarrow m \notin fB. \end{aligned}$$

Now suppose membership in A, B has been defined for all $0 \leq i < m$, where $m \geq n$, and we have $A \subseteq B$ thus far.

case 1. $m \notin fA$ thus far. Put $m \in A, B$.
case 2. $m \in fA$ thus far. Put $m \notin A, B$.

This defines membership of m in A, B . Note that we still have $A \subseteq B$.

Now let A, B be the result of this inductive construction. Note that by the choice of n , all of the "thus far" remain true of the actual A, B , where $m \geq n$. Thus we have for all $m \geq n$,

$$\begin{aligned} A &\subseteq B. \\ m \notin fA &\leftrightarrow m \in A \leftrightarrow m \in B. \\ m \notin A &\rightarrow m \in fA \rightarrow m \in fB. \end{aligned}$$

Hence for all $m \geq n$, $m \in B \cup fA$ and $m \in A \cup fB$. Since this also holds for $m < n$, this holds for all $m \in \mathbb{N}$.

Finally, suppose A is finite. Then fA is finite, and so eventually all m are placed in A . Thus A is infinite. Hence A is infinite. QED

LEMMA 2.5.5. For all $f \in \text{EVSD}$ there exist infinite $A \subseteq B \subseteq \mathbb{N}$ such that $A \cup fB = \mathbb{N}$ and $B \cap fA = \emptyset$.

Proof: Let $f \in \text{EVSD}$. Let n, A', B' be as in the first paragraph of the proof of Lemma 2.5.4.

We now define the required A, B by induction. Membership in A, B for $m < n$ is just membership in A', B' . Thus for all $m < n$,

$$\begin{aligned} m \in B &\leftrightarrow m \in B' \leftrightarrow m \notin f'A' \leftrightarrow m \notin fA. \\ m \in A &\leftrightarrow m \in A' \leftrightarrow m \notin f'B' \leftrightarrow m \notin fB. \end{aligned}$$

Now suppose membership in A, B has been defined for all $i < m$, where $m \geq n$, and we have $A \subseteq B$ thus far.

case 1. $m \notin fB$ thus far. Put $m \in A, B$.
case 2. $m \in fB$ thus far. Put $m \notin A, B$.

This defines membership of m in A, B . Note that we still have $A \subseteq B$.

Now let A, B be the result of this inductive construction. Note that by the choice of n , all of the "thus far" remain true of the actual A, B , where $m \geq n$. Thus we have for all $m \geq n$,

$$\begin{aligned} A &\subseteq B. \\ m \notin fB &\leftrightarrow m \in A \leftrightarrow m \in B. \\ m \in B &\rightarrow m \notin fB \rightarrow m \notin fA. \end{aligned}$$

Hence for all $m \geq n$, $m \in A \cup fB$ and $m \notin B \cap fA$. Since this also holds for $m < n$, this holds for all $m \in N$.

Finally, suppose A is finite. Then eventually all m are placed in fB . Hence eventually all m are placed outside B . Hence B is finite. So fB is finite. Then eventually all m are put in A, B . This is a contradiction. QED

LEMMA 2.5.6. There exists $f \in \text{ELG}$ such that $f^{-1}(0) = \{(0, \dots, 0)\}$, $f(N \setminus \{0\}) \subseteq 2N+1$, and for all $A \subseteq N$ containing 0, $fA \cap 2N \subseteq A \rightarrow fA$ is cofinite.

Proof: Let $g \in \text{ELG} \cap \text{SD}$ be given by Lemma 3.2.1. We define 4-ary $f \in \text{ELG}$ as follows. $f(0, 0, 0, 0) = 0$. $f(0, n, m, r) = g(n, m, r)$ if $(n, m, r) \neq (0, 0, 0)$. $f(t, n, m, r) = 2|t, n, m, r|+1$ if $t \neq 0$. Obviously $f \in \text{ELG} \cap \text{SD}$, $f(N \setminus \{0\}) \subseteq 2N+1$, and $f^{-1}(0) = \{(0, 0, 0, 0)\}$.

Now let $A \subseteq \mathbb{N}$, $0 \in A$, where $fA \cap 2\mathbb{N} \subseteq A$. Since $gA \subseteq fA$, we have $gA \cap 2\mathbb{N} \subseteq A$, and so by Lemma 3.2.1, gA is cofinite. Hence fA is cofinite. QED

LEMMA 2.5.7. The following is false. For all $f \in \text{ELG}$ there exist infinite $A \subseteq B \subseteq \mathbb{N}$ such that $A \cap fB = \emptyset$, $B \cup fB = \mathbb{N}$, and $fB \subseteq B \cup fA$.

Proof: Let $f \in \text{ELG}$ be given by Lemma 2.5.6. Let $A \cap fB = \emptyset$, $B \cup fB = \mathbb{N}$, and $fB \subseteq B \cup fA$, where A is infinite. Now $0 \in B \vee 0 \in fB$. Since $f^{-1}(0) = \{(0,0,0,0)\}$, we have $0 \in B$, $0 \in fB$, $0 \notin A$. Therefore $fA \subseteq 2\mathbb{N}+1$. Since $fB \subseteq B \cup fA$, we have $fB \cap 2\mathbb{N} \subseteq B$. Therefore fB is cofinite. This contradicts $A \cap fB = \emptyset$. QED

LEMMA 2.5.8. The following is false. For all $f \in \text{ELG}$ there exist infinite $A \subseteq B \subseteq \mathbb{N}$ such that $B \cup fA = \mathbb{N}$ and $A \cap fB = \emptyset$.

Proof: Let f be as given by Lemma 2.5.6. Let $A \subseteq B \subseteq \mathbb{N}$, $B \cup fA = \mathbb{N}$, $A \cap fB = \emptyset$, where A is infinite. Since $0 \in B \cup fA$, we have $0 \in B \vee 0 \in fA$. If $0 \in fA$ then $0 \in A, B$, because $f^{-1}(0) = \{(0,0,0,0)\}$. Hence $0 \notin fA$, $0 \notin A$. Therefore $fA \subseteq 2\mathbb{N}+1$. Since $B \cup fA = \mathbb{N}$, we have $2\mathbb{N} \subseteq B$. By Lemma 3.2.1, fB is cofinite. By $A \cap fB = \emptyset$, A is finite. But A is infinite. QED

LEMMA 2.5.9. For all $f \in \text{EVSD}$ there exist infinite $A \subseteq B \subseteq \mathbb{N}$ such that $B \cup fA = \mathbb{N}$ and $A \subseteq fB$.

Proof: Let n be such that $|x| \geq n \rightarrow f(x) > |x|$. We can use Lemma 2.4.1 with \mathbb{N} replaced by $[n, \infty)$. Let $A, B \subseteq [n, \infty)$, $A \subseteq B$, $B \cup fA = [n, \infty)$ and $A = B \cap fB$, where A is infinite. Then $B \cup fA = [n, \infty)$, $A \subseteq fB$. Replace B with $B \cup [0, n-1]$. QED

LEMMA 2.5.10. The following is false. For all $f \in \text{ELG}$ there exist infinite $A \subseteq B \subseteq \mathbb{N}$ such that $A \cap fA = \emptyset$, $B \cup fB = \mathbb{N}$, $B \cap fB \subseteq A \cup fA$.

Proof: Let f be as given by Lemma 2.5.6. Let $A \subseteq B \subseteq \mathbb{N}$ such that $A \cap fA = \emptyset$, $B \cup fB = \mathbb{N}$, $B \cap fB \subseteq A \cup fA$, where A, B are infinite. Then $0 \in B \cup fB$, and so $0 \in B \cap fB$. Hence $0 \in A \cup fA$, in which case $0 \in A \cap fA$. QED

LEMMA 2.5.11. For all $f \in \text{EVSD}$ there exist infinite $A \subseteq B \subseteq \mathbb{N}$ such that $A \cup fB = \mathbb{N}$ and $fA \subseteq B$.

Proof: Let f' be the restriction of f to $\{x: f(x) > |x|\}$. Then f' is defined at all but finitely many elements of $\text{dom}(f)$. As remarked right after Lemma 2.4.5, Lemma 2.4.2 holds even for partial functions, and so in particular for f' . Let $A \subseteq B \subseteq \mathbb{N}$, where $A \cup f'B = \mathbb{N}$ and $f'A \subseteq B$ and A is infinite. Let $A' = \mathbb{N} \setminus fB \subseteq A$. Since $f'B$ contains all but finitely many elements of fB , we see that A' remains infinite. Then A', B are as required. QED

LEMMA 2.5.12. Let $f \in \text{EVSD}$. There exist infinite $A \subseteq B \subseteq \mathbb{N}$ such that $fB \subseteq B \cup fA$ and $A = B \cap fB$.

Proof: Let n be such that $|x| \geq n \rightarrow f(x) > |x|$. We can use Lemma 2.4.1 with \mathbb{N} replaced by $[n, \infty)$. Let $A, B \subseteq [n, \infty)$, $A \subseteq B$, $B \cup fA = [n, \infty)$, and $A = B \cap fB$, where A is infinite. Since $fB \subseteq [n, \infty)$, the proof is complete. QED

LEMMA 2.5.13. Let $f \in \text{EVSD}$. There exist infinite $A \subseteq \mathbb{N}$ such that $A \cap f(A \cup fA) = \emptyset$.

Proof: Let n be such that $|x| \geq n \rightarrow f(x) > |x|$. Define $n_0 < n_1 < \dots$ by induction as follows. Let $n_0 = n$. Suppose n_i has been defined, $i \geq 0$. Let n_{i+1} be greater than all elements of $f(A \cup fA)$, thus far. Finally, let $A = \{n_0, n_1, \dots\}$. QED

LEMMA 2.5.14. Let $f \in \text{EVSD}$ and let $X \subseteq \mathbb{N}$, where $\min(X)$ is sufficiently large. There exists a unique A such that $A \subseteq X \subseteq A \cup fA$. If X is infinite then A is infinite.

Proof: Let f, X be as given. Then $|x| \geq \min(X) \rightarrow f(x) > |x|$. We can use Lemma 2.4.3 with \mathbb{N} replaced by $[\min(X), \infty)$. Let $A \subseteq X \cap [\min(X), \infty) \subseteq A \cup fA$.

For uniqueness, suppose $A \subseteq X \subseteq A \cup fA$, $A' \subseteq X \subseteq A' \cup fA'$, and let $n = \min(A \Delta A')$. Since $f \in \text{SD}$, clearly $n \in fA \leftrightarrow n \in fA'$. This is a contradiction. QED

As in section 2.4, we start with the 9 elementary inclusions in A, B, fA, fB, \subseteq .

EBRT in A, B, fA, fB, \subseteq on $(\text{ELG}, \text{INF}), (\text{EVSD}, \text{INF})$.

$$A \cap fA = \emptyset.$$

$$B \cup fB = \mathbb{N}.$$

$$\begin{aligned}
B &\subseteq A \cup fB. \\
fB &\subseteq B \cup fA. \\
A &\subseteq fB. \\
B \cap fB &\subseteq A \cup fA. \\
fA &\subseteq B. \\
A \cap fB &\subseteq fA. \\
B \cap fA &\subseteq A.
\end{aligned}$$

Our classification amounts to a determination of the subsets S of the above nine inclusions for which

$$\begin{aligned}
(\forall f \in \text{ELG}) (\exists A \subseteq B \text{ from INF}) (S) \\
(\forall f \in \text{EVSD}) (\exists A \subseteq B \text{ from INF}) (S)
\end{aligned}$$

holds, where S is interpreted conjunctively.

EBRT in A, B, fA, fB, \subseteq on $(\text{ELG}, \text{INF}), (\text{EGS} \cap \text{SD}, \text{INF})$.*

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$$\begin{aligned}
A \cap fA &= \emptyset. \\
B \cup fB &= N. \\
fA &\subseteq B. \\
A &\subseteq fB. \\
B &\subseteq A \cup fB. \\
fB &\subseteq B \cup fA. \\
A \cap fB &\subseteq fA. \\
B \cap fA &\subseteq A. \\
B \cap fB &\subseteq A \cup fA.
\end{aligned}$$

LIST 1.

$$\begin{aligned}
A \cap fA &= \emptyset: \\
B \cup fB &= N. \\
fA &\subseteq B. \\
A &\subseteq fB. \\
B &\subseteq A \cup fB. \\
fB &\subseteq B \cup fA. \\
A \cap fB &\subseteq fA. \quad A \cap fB = \emptyset. \\
B \cap fA &\subseteq A. \quad B \cap fA = \emptyset. \\
B \cap fB &\subseteq A \cup fA.
\end{aligned}$$

LIST 1*.

6

$$\begin{aligned}
A \cap fA &= \emptyset: \\
B \cap fA &= \emptyset. \\
A \cap fB &= \emptyset.
\end{aligned}$$

$fA \subseteq B.$
 $A \subseteq fB.$
 $B \cup fB = N.$
 $B \subseteq A \cup fB.$
 $fB \subseteq B \cup fA.$
 $B \cap fB \subseteq A \cup fA.$

LIST 1.1.

$A \cap fA = \emptyset:$ Redundant.
 $B \cap fA = \emptyset:$
 $A \cap fB = \emptyset.$
 $fA \subseteq B.$ No.
 $A \subseteq fB.$
 $B \cup fB = N.$
 $B \subseteq A \cup fB.$
 $fB \subseteq B \cup fA.$
 $B \cap fB \subseteq A \cup fA.$ $B \cap fB \subseteq A.$

LIST 1.1.*

4

$B \cap fA = \emptyset:$
 $A \cap fB = \emptyset.$
 $A \subseteq fB.$
 $B \cup fB = N.$
 $B \subseteq A \cup fB.$
 $fB \subseteq B \cup fA.$
 $B \cap fB \subseteq A.$

LIST 1.1.1.

$B \cap fA = \emptyset:$
 $A \cap fB = \emptyset:$
 $A \subseteq fB.$ No.
 $B \cup fB = N.$
 $B \subseteq A \cup fB.$
 $fB \subseteq B \cup fA.$
 $B \cap fB \subseteq A.$ $B \cap fB = \emptyset.$

LIST 1.1.1.*

2

$B \cap fA = \emptyset:$
 $A \cap fB = \emptyset:$
 $B \cup fB = N.$
 $B \subseteq A \cup fB.$

$$fB \subseteq B \cup fA.$$

$$B \cap fB = \emptyset.$$

LIST 1.1.1.1.

$$B \cap fA = \emptyset:$$

$$A \cap fB = \emptyset:$$

$$B \cup fB = N:$$

$$B \subseteq A \cup fB. \quad A \cup fB = N.$$

$$fB \subseteq B \cup fA. \quad B \cup fA = N. \quad \text{No. Lemma 2.5.8.}$$

$$B \cap fB = \emptyset. \quad \text{No. Lemma 2.5.10.}$$

LIST 1.1.1.1.*

0

$$B \cap fA = \emptyset:$$

$$A \cap fB = \emptyset:$$

$$B \cup fB = N:$$

$$A \cup fB = N.$$

Entirely RCA_0 correct. Lemma 2.5.5.

LIST 1.1.1.2.

$$B \cap fA = \emptyset:$$

$$A \cap fB = \emptyset:$$

$$B \subseteq A \cup fB:$$

$$fB \subseteq B \cup fA.$$

$$B \cap fB = \emptyset.$$

Entirely RCA_0 correct. Set $A \cap fA = \emptyset$, $B = A$.

LIST 1.1.2.

$$B \cap fA = \emptyset:$$

$$A \subseteq fB:$$

$$B \cup fB = N.$$

$$B \subseteq A \cup fB. \quad B \subseteq fB. \quad \text{No.}$$

$$fB \subseteq B \cup fA.$$

$$B \cap fB \subseteq A.$$

LIST 1.1.2.*

2

$$B \cap fA = \emptyset:$$

$$A \subseteq fB:$$

$$B \cup fB = N.$$

$$fB \subseteq B \cup fA.$$

$$B \cap fB \subseteq A.$$

LIST 1.1.2.1.

$$B \cap fA = \emptyset:$$

$$A \subseteq fB:$$

$$B \cup fB = N:$$

$$fB \subseteq B \cup fA. \quad B \cup fA = N.$$

$$B \cap fB \subseteq A. \quad \text{No. Lemma 2.5.10.}$$

LIST 1.1.2.1.*
0

$$B \cap fA = \emptyset:$$

$$A \subseteq fB:$$

$$B \cup fB = N:$$

$$B \cup fA = N.$$

Entirely RCA_0 correct. Lemma 2.5.9.

LIST 1.1.2.2.

$$B \cap fA = \emptyset:$$

$$A \subseteq fB:$$

$$fB \subseteq B \cup fA:$$

$$B \cap fB \subseteq A.$$

Entirely RCA_0 correct. Lemma 2.5.12.

LIST 1.1.3.

$$B \cap fA = \emptyset:$$

$$B \cup fB = N:$$

$$B \subseteq A \cup fB. \quad A \cup fB = N.$$

$$fB \subseteq B \cup fA. \quad B \cup fA = N.$$

$$B \cap fB \subseteq A. \quad \text{No. Lemma 2.5.10.}$$

LIST 1.1.3.*
0

$$B \cap fA = \emptyset:$$

$$B \cup fB = N:$$

$$A \cup fB = N.$$

$$B \cup fA = N.$$

Entirely RCA_0 correct. Lemma 2.5.4.

LIST 1.1.4.

$B \cap fA = \emptyset$:
 $B \subseteq A \cup fB$:
 $fB \subseteq B \cup fA$.
 $B \cap fB \subseteq A$.

Entirely RCA_0 correct. Set $A \cap fA = \emptyset$, $B = A$.

LIST 1.2.

$A \cap fA = \emptyset$: Redundant.
 $A \cap fB = \emptyset$:
 $fA \subseteq B$.
 $A \subseteq fB$. No.
 $B \cup fB = N$.
 $B \subseteq A \cup fB$.
 $fB \subseteq B \cup fA$.
 $B \cap fB \subseteq A \cup fA$. $B \cap fB \subseteq fA$.

LIST 1.2.*

3

$A \cap fB = \emptyset$:
 $fA \subseteq B$.
 $B \cup fB = N$.
 $B \subseteq A \cup fB$.
 $fB \subseteq B \cup fA$.
 $B \cap fB \subseteq fA$.

LIST 1.2.1.

$A \cap fB = \emptyset$:
 $fA \subseteq B$:
 $B \cup fB = N$.
 $B \subseteq A \cup fB$.
 $fB \subseteq B \cup fA$. $fB \subseteq B$. No. Lemma 2.4.4.
 $B \cap fB \subseteq fA$.

LIST 1.2.1.*

2

$A \cap fB = \emptyset$:
 $fA \subseteq B$:
 $B \cup fB = N$.
 $B \subseteq A \cup fB$.

$$B \cap fB \subseteq fA.$$

LIST 1.2.1.1.

$$A \cap fB = \emptyset:$$

$$fA \subseteq B:$$

$$B \cup fB = N:$$

$$B \subseteq A \cup fB.$$

$$B \cap fB \subseteq fA. \text{ No. Lemma 2.5.10.}$$

LIST 1.2.1.1.*

0

$$A \cap fB = \emptyset:$$

$$fA \subseteq B:$$

$$B \cup fB = N:$$

$$B \subseteq A \cup fB.$$

Entirely RCA_0 correct. See Lemma 2.5.11.

LIST 1.2.1.2.

0

$$A \cap fB = \emptyset:$$

$$fA \subseteq B:$$

$$B \subseteq A \cup fB:$$

$$B \cap fB \subseteq fA.$$

Entirely RCA_0 correct. Let A be given by Lemma 2.5.13. Set $B = A \cup fA$.

LIST 1.2.2.

$$A \cap fB = \emptyset:$$

$$B \cup fB = N:$$

$$B \subseteq A \cup fB. \quad A \cup fB = N.$$

$$fB \subseteq B \cup fA. \text{ No. Lemma 2.5.7.}$$

$$B \cap fB \subseteq fA. \text{ No. Lemma 2.5.10.}$$

LEMMA 1.2.2.*

0

$$A \cap fB = \emptyset:$$

$$B \cup fB = N:$$

$$A \cup fB = N.$$

Entirely RCA_0 correct. Lemma 2.5.5.

LIST 1.2.3.

$A \cap fB = \emptyset$:
 $B \subseteq A \cup fB$:
 $fB \subseteq B \cup fA$.
 $B \cap fB \subseteq fA$.

Entirely RCA_0 correct. Set $A \cap fA = \emptyset$, $B = A$.

LIST 1.3.

$A \cap fA = \emptyset$:
 $fA \subseteq B$:
 $A \subseteq fB$.
 $B \cup fB = N$.
 $B \subseteq A \cup fB$.
 $fB \subseteq B \cup fA$.
 $B \cap fB \subseteq A \cup fA$.

LIST 1.3.*
3

$A \cap fA = \emptyset$:
 $fA \subseteq B$:
 $A \subseteq fB$.
 $B \cup fB = N$.
 $B \subseteq A \cup fB$.
 $fB \subseteq B$.
 $B \cap fB \subseteq A \cup fA$.

LIST 1.3.1.

$A \cap fA = \emptyset$:
 $fA \subseteq B$:
 $A \subseteq fB$:
 $B \cup fB = N$.
 $B \subseteq A \cup fB$. $B \subseteq fB$. No. Lemma 2.4.5.
 $fB \subseteq B$.
 $B \cap fB \subseteq A \cup fA$.

LIST 1.3.1.*
2

$A \cap fA = \emptyset$:
 $fA \subseteq B$:
 $A \subseteq fB$:

$B \cup fB = N.$
 $fB \subseteq B.$
 $B \cap fB \subseteq A \cup fA.$

LIST 1.3.1.1.

$A \cap fA = \emptyset:$
 $fA \subseteq B:$
 $A \subseteq fB:$
 $B \cup fB = N:$
 $fB \subseteq B.$
 $B \cap fB \subseteq A \cup fA.$ No. Lemma 2.5.10.

LIST 1.3.1.1.*
0

$A \cap fA = \emptyset:$
 $fA \subseteq B:$
 $A \subseteq fB:$
 $B \cup fB = N:$
 $fB \subseteq B.$

Entirely RCA_0 correct. Let A be given by Lemma 2.4.3 with $A \subseteq fN \subseteq A \cup fA$. Set $B = N$.

LIST 1.3.1.2.

$A \cap fA = \emptyset:$
 $fA \subseteq B:$
 $A \subseteq fB:$
 $fB \subseteq B:$
 $B \cap fB \subseteq A \cup fA.$

Entirely RCA_0 correct. Let $B = [n, \infty)$, n sufficiently large. By Lemma 2.5.14, let $A \subseteq fB \subseteq A \cup fA$.

LIST 1.3.2.

$A \cap fA = \emptyset:$
 $fA \subseteq B:$
 $B \cup fB = N:$
 $B \subseteq A \cup fB.$ $A \cup fB = N.$
 $fB \subseteq B.$ $B = N.$
 $B \cap fB \subseteq A \cup fA.$ No. Lemma 2.5.10.

LIST 1.3.2.*
0

$A \cap fA = \emptyset$:
 $fA \subseteq B$:
 $B \cup fB = N$:
 $A \cup fB = N$.
 $B = N$.

Entirely RCA_0 correct. Set $A = N \setminus fN$, $B = N$.

LIST 1.3.3.

$A \cap fA = \emptyset$:
 $fA \subseteq B$:
 $B \subseteq A \cup fB$:
 $fB \subseteq B$.
 $B \cap fB \subseteq A \cup fA$.

Entirely RCA_0 correct. Let $B = [n, \infty)$ for n sufficiently large. Let $A \subseteq B \subseteq A \cup fA$, by Lemma 2.5.14.

LIST 1.4.

$A \cap fA = \emptyset$:
 $A \subseteq fB$:
 $B \cup fB = N$.
 $B \subseteq A \cup fB$. $B \subseteq fB$. No. Lemma 2.4.5.
 $fB \subseteq B \cup fA$.
 $B \cap fB \subseteq A \cup fA$.

LIST 1.4.*
2

$A \cap fA = \emptyset$:
 $A \subseteq fB$:
 $B \cup fB = N$.
 $fB \subseteq B \cup fA$.
 $B \cap fB \subseteq A \cup fA$.

LIST 1.4.1.

$A \cap fA = \emptyset$:
 $A \subseteq fB$:
 $B \cup fB = N$:
 $fB \subseteq B \cup fA$.
 $B \cap fB \subseteq A \cup fA$. No. Lemma 2.5.10.

LIST 1.4.1.*

0

$$A \cap fA = \emptyset:$$

$$A \subseteq fB:$$

$$B \cup fB = N:$$

$$fB \subseteq B \cup fA.$$

Entirely RCA_0 correct. Let $A \subseteq fN \subseteq A \cup fA$ be given by Lemma 2.4.3. Set $B = N$.

LIST 1.4.2.

$$A \cap fA = \emptyset:$$

$$A \subseteq fB:$$

$$fB \subseteq B \cup fA.$$

$$B \cap fB \subseteq A \cup fA.$$

Entirely RCA_0 correct. Lemma 2.5.12.

LIST 1.5.

$$A \cap fA = \emptyset:$$

$$B \cup fB = N:$$

$$B \subseteq A \cup fB.$$

$$fB \subseteq B \cup fA.$$

$$B \cap fB \subseteq A \cup fA. \text{ No. Lemma 2.5.10.}$$

LIST 1.5.*

0

$$A \cap fA = \emptyset:$$

$$B \cup fB = N:$$

$$B \subseteq A \cup fB.$$

$$fB \subseteq B \cup fA.$$

Entirely RCA_0 correct. Lemma 2.5.4.

LIST 1.6.

$$A \cap fA = \emptyset:$$

$$B \subseteq A \cup fB:$$

$$fB \subseteq B \cup fA.$$

$$B \cap fB \subseteq A \cup fA.$$

Entirely RCA_0 correct. Let $A \cap fA = \emptyset$, $B = A$.

LIST 2.

$B \cup fB = N$:
 $fA \subseteq B$.
 $A \subseteq fB$.
 $B \subseteq A \cup fB$. $A \cup fB = N$.
 $fB \subseteq B \cup fA$. $B \cup fA = N$.
 $A \cap fB \subseteq fA$.
 $B \cap fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

LIST 2.*
 # 3

$B \cup fB = N$:
 $fA \subseteq B$.
 $A \subseteq fB$.
 $A \cup fB = N$.
 $B \cup fA = N$.
 $A \cap fB \subseteq fA$.
 $B \cap fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

LIST 2.1.

$B \cup fB = N$:
 $fA \subseteq B$:
 $A \subseteq fB$.
 $A \cup fB = N$.
 $B \cup fA = N$. $B = N$.
 $A \cap fB \subseteq fA$.
 $B \cap fA \subseteq A$. $fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

LIST 2.1.*
 # 2

$B \cup fB = N$:
 $fA \subseteq B$:
 $A \subseteq fB$.
 $A \cup fB = N$.
 $B = N$.
 $A \cap fB \subseteq fA$.
 $fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

LIST 2.1.1.

$B \cup fB = N$:
 $fA \subseteq B$:
 $A \subseteq fB$:
 $A \cup fB = N$. $fB = N$. No. Lemma 2.4.5.
 $B = N$.
 $A \cap fB \subseteq fA$. $A \subseteq fA$. No. Lemma 2.4.5.
 $fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

LIST 2.1.1.*

0

$B \cup fB = N$:
 $fA \subseteq B$:
 $A \subseteq fB$:
 $B = N$.
 $fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

Entirely RCA_0 correct. Set $A = fN$, $B = N$.

LIST 2.1.2.

$B \cup fB = N$:
 $fA \subseteq B$:
 $A \cup fB = N$.
 $B = N$.
 $A \cap fB \subseteq fA$.
 $B \cap fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

Entirely RCA_0 correct. Set $A = B = N$.

LIST 2.2.

$B \cup fB = N$:
 $A \subseteq fB$:
 $A \cup fB = N$. Yes.
 $B \cup fA = N$.
 $A \cap fB \subseteq fA$. $A \subseteq fA$. No. Lemma 2.4.5.
 $B \cap fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

LIST 2.2.*

0

$B \cup fB = N$:

$A \subseteq fB$:
 $fB \subseteq B \cup fA$. $B \cup fA = N$.
 $B \cap fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

Entirely RCA_0 correct. Set $A = fN$, $B = N$.

LIST 2.3.

$B \cup fB = N$:
 $A \cup fB = N$.
 $B \cup fA = N$.
 $A \cap fB \subseteq fA$.
 $B \cap fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

Entirely RCA_0 correct. Set $A = B = N$.

LIST 3.

$fA \subseteq B$:
 $A \subseteq fB$.
 $B \subseteq A \cup fB$.
 $fB \subseteq B \cup fA$.
 $A \cap fB \subseteq fA$.
 $B \cap fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

LIST 3*.

2

$fA \subseteq B$:
 $A \subseteq fB$.
 $B \subseteq A \cup fB$.
 $fB \subseteq B \cup fA$.
 $A \cap fB \subseteq fA$.
 $B \cap fA \subseteq A$.
 $B \cap fB \subseteq A \cup fA$.

LIST 3.1.

$fA \subseteq B$:
 $A \subseteq fB$:
 $B \subseteq A \cup fB$. $B \subseteq fB$. No. Lemma 2.4.5.
 $fB \subseteq B \cup fA$.
 $A \cap fB \subseteq fA$. $A \subseteq fA$. No. Lemma 2.4.5.
 $B \cap fA \subseteq A$.

$$B \cap fB \subseteq A \cup fA.$$

LIST 3.1.*

0

$$fA \subseteq B:$$

$$A \subseteq fB:$$

$$fB \subseteq B \cup fA.$$

$$B \cap fA \subseteq A.$$

$$B \cap fB \subseteq A \cup fA.$$

Entirely RCA_0 correct. Set $A = fN$, $B = N$.

LIST 3.2.

$$fA \subseteq B:$$

$$B \subseteq A \cup fB:$$

$$fB \subseteq B \cup fA.$$

$$A \cap fB \subseteq fA.$$

$$B \cap fA \subseteq A.$$

$$B \cap fB \subseteq A \cup fA.$$

Entirely RCA_0 correct. Set $A = B = fN$.

LIST 4.

$$A \subseteq fB:$$

$$B \subseteq A \cup fB. B \subseteq fB. \text{No. Lemma 2.4.5.}$$

$$fB \subseteq B \cup fA.$$

$$A \cap fB \subseteq fA. A \subseteq fA. \text{No. Lemma 2.4.5.}$$

$$B \cap fA \subseteq A.$$

$$B \cap fB \subseteq A \cup fA.$$

LIST 4.*

0

$$A \subseteq fB:$$

$$fB \subseteq B \cup fA.$$

$$B \cap fA \subseteq A.$$

$$B \cap fB \subseteq A \cup fA.$$

Entirely RCA_0 correct. Set $A = fN$, $B = N$.

LIST 5.

$$B \subseteq A \cup fB:$$

$$fB \subseteq B \cup fA.$$

$A \cap fB \subseteq fA.$
 $B \cap fA \subseteq A.$
 $B \cap fB \subseteq A \cup fA.$

Entirely RCA_0 correct. Set $A = B = N.$

THEOREM 2.5.15. EBRT in A, B, fA, fB, \subseteq on $(ELG, INF),$
 $(EVSD, INF)$ have the same correct formats. EBRT in
 A, B, fA, fB, \subseteq on (ELG, INF) and $(EVSD, INF)$ are RCA_0 secure.

Proof: We have presented an RCA_0 classification of EBRT in
 A, B, fA, fB, \subseteq on $(ELG, INF), (EVSD, INF)$ in the sense of the
tree methodology of section 2.1. All of the documentation
works equally well on (ELG, INF) and $(EVSD, INF).$ We have
stayed within $RCA_0.$ QED

THEOREM 2.5.16. There are at most 26 maximal α correct α
formats, where α is EBRT in A, B, fA, fB, \subseteq on $(ELG, INF),$
 $(EVSD, INF).$

Proof: Here is the list of numerical labels of terminal
vertices in the RCA_0 classification of EBRT in A, B, fA, fB, \subseteq
on $(ELG, INF), (EVSD, INF)$ given above:

1.1.1.1.*
1.1.1.2.
1.1.2.1.*
1.1.2.2.
1.1.3.*
1.1.3.
1.2.1.1.*
1.2.1.2.
1.2.2.*
1.2.3.
1.3.1.1.*
1.3.1.2.
1.3.2.*
1.3.3.
1.4.1.*
1.4.2.
1.5.*
1.6.
2.1.1.*
2.1.2.
2.2.*
2.3.
3.1.*
3.2.

4.*

5.

The count is 26. Apply Theorem 2.1.5. QED