BOREL AND BAIRE REDUCIBILITY

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INTRODUCTION

The Borel reducibility theory of Polish equivalence relations, at least in its present form, was initiated in [FS89]. There is now an extensive literature on this topic, including fundamental work on the Glimm-Effros dichotomy in [HKL90], on countable Borel equivalence relations in [DJK94], and on Polish group actions in [BK96]. Current principal contributors include H. Becker, R. Dougherty, L. Harrington, G. Hjorth, S. Jackson, A.S. Kechris, and A. Louveau, R. Sami, and S. Solecki.

A Polish space is a topological space that is separable and completely metrizable. The Borel subsets of a Polish space form the least σ-algebra containing the open subsets. A Borel function from one Polish space to another is a function such that the inverse image of every open set is Borel. Two Polish spaces are Borel isomorphic if and only if there is a one-one onto Borel function from the first onto the second. This is an equivalence relation. Any two uncountable Polish spaces are Borel isomorphic. See [Ke94].

We also consider Baire measurable subsets of a Polish space. A nowhere dense set in a Polish space is a set whose closure contains no nonempty open set. A meager subset of a Polish space is a set which is the countable union of nowhere dense sets. A Baire (measurable) subset of a Polish space is a set whose symmetric difference with some open set is meager. A comeager subset of a Polish space is a subset whose complement is meager. We have the fundamental Baire category theorem: A Polish space is not meager.

A function from one Polish space to another is said to be Baire if and only if the inverse image of every open set is Baire.

A Polish equivalence relation is a pair (X,E), where X is a Polish space and E is an equivalence relation on X. A Borel (analytic) equivalence relation is a Polish equivalence relation (X,E), where E is as a Borel measurable (analytic) subset of X^2.

Let (X,E_1) and (Y,E_2) be Polish equivalence relations. We say that $h:X \rightarrow Y$ is a reduction from (X,E_1) into (Y,E_2) if and only if for all $x,y \in X$, $E_1(x,y) \leftrightarrow E_2(h(x),h(y))$. We say that (X,E1) is Borel reducible to (Y,E_2) if and only if there is a Borel reduction from (X,E_1) to (Y,E_2). We also say that (X,E_1) is Baire reducible to (Y,E_2) if and only if there is a Baire reduction from (X,E_1) to (Y,E_2).

Let STR = (STR,≈) be the analytic equivalence relation of all structures of countable similarity type (in the sense of model theory) whose domain is ω, under isomorphism. Also let BR = (P(ωxω),≈), be the analytic equivalence relation of binary relations on ω under isomorphism. It is well known that BR is not a Borel equivalence relation and BR is Borel isomorphic to STR. It is generally simpler to work with BR.

The condition that a Polish equivalence relation E is Borel reducible to BR has received a considerable amount of attention, and has assumed the name “E admits classification by countable structures.” See, e.g., ????

Let (X,E) be a Polish equivalence relation. We define CS(X,E) as the Polish equivalence relation (X∞,E′), where x E′ y if and only if every term of x is related to a term of y by E and every term of y is related to a term of x by E. (Here NCS is “nonempty countable subsets”). Also if S is a countable family of Polish equivalence relations, then ∑S is the Polish equivalence relation whose domain is the disjoint union of the domains of the elements of S, and whose equivalence relation is the union of the equivalence relations of the elements of S.

Define $T(0)$ to be the dicrete equivalence relation on ω. For countable ordinals α and countable limit ordinals λ, let $T(\alpha+1) = CS(T(\alpha))$, and $T(\lambda) = \sum\{T(\beta): \beta < \lambda\}$.

In section 1, we prove that a Borel equivalence relation is Borel (or Baire) reducible to BR if and only if it is Borel (or Baire) reducible to some $T(\alpha)$.

Let ZD be the Borel equivalence relation on the Polish space $P(\omega)$ given by $ZD(x,y)$ if and only if $x \Delta y$ has zero density. Here we say that $z \subseteq \omega$ has zero density if and only if

$$\lim_{n\to\infty}|z \cap [0,n)|/n = 0.$$

In [FS89], it was claimed that ZD was not Borel reducible into BR, with no indication of proof. In section 2, we give this proof, due to this author. In fact, we prove that ZD is not Baire reducible into BR.

The authors of [BK96] were not aware of this claim from [FS89]. The book [BK96] heavily featured the following problem that our result immediately solved:

is there a Borel equivalence relation that is Borel reducible to an orbit equivalence relation but not Borel reducible to BR?

Here an orbit equivalence relation is an equivalence relation arising from the continuous action of a Polish group on a Polish space. This problem is first raised at 3.5.4 of [BK96]. It is then motivated in section 8.2 and essentially raised again at 8.2.4.

For more information on orbit equivalence relations, see [Hj?].

1. Reducibility into BR and $T(\alpha)$

The Borel equivalence relations $T(\alpha)$, $\alpha < \omega_1$, form a particularly natural hierarchy, but some other hierarchies of Borel equivalence relations appear more often in the literature and are technically easier to work with.

One of these hierarchies is in terms of HC = the hereditarily countable sets. Define HC(0) = $\emptyset$, HC(α+1) = the set of all countable subsets of HC(α), HC(λ) = the union of all HC(β), $\beta < \lambda$. The elements of HC have what is commonly referred to as codes in P(ω). For countable ordinals α, we let HC(α) also denote the Borel equivalence relation on codes of elements of HC(α) according to whether they code the same element of HC(α). We assume familiarity with the machinery of codes for elements of HC (see, e.g., ???).

We quote the following result from the folklore.

LEMMA 1.1. For countable ordinals α, T(α) and HC(ω+α) are Borel reducible into each other.

We now prove that any Borel equivalence relation X = (X,E) that is Borel reducible to BR = (BR,$\approx$) is Borel reducible to some T(α), $\alpha < \omega_1$. We use a streamlined version of the technology introduced in [Sc65].

We use FS(ωxω) for the set of all finite sequences from ω. Let A,B $\subseteq$ ωxω. For countable ordinals α, we define relations R_α(A,B) on FS(ω) by transfinite induction.

R_0(A,B)(x,y) if and only if x,y have the same length, and for all i,j, A(x_i,x_j) $\leftrightarrow$ B(y_i,y_j).

R_α(A,B)(x,y) if and only if x,y have the same length, and for all z $\in$ FS(ω) there exists w $\in$ FS(ω) such that for all $\beta < \alpha$, R_β(A,B)(xz,yz), and for all z $\in$ FS(ω) there exists w $\in$ FS(ω) such that for all $\beta < \alpha$, R_β(A,B)(yz,xz).

The following is standard.

LEMMA 1.2. Let A,B $\subseteq$ ωxω. There exists $\alpha < \omega_1$ such that R_α(A,B) = $R_{\alpha+1}$(A,B), in which case for all $\alpha \leq \beta < \omega_1$, R_α(A,B) = R_β(A,B). A $\approx$ B if and only if for all $\alpha < \omega_1$, R_α(A,B)(< >,< >).

We can think of the R_α(A,B) as comparison relations between A and B that become more and more refined as α increases.

For countable ordinals α and $x \in FS(\omega)$, we define $f_0(A,x) = \langle lth(x),\{\langle i,j\rangle: A(x_i,x_j)\}\rangle$, $f_\alpha(A,x) = \langle lth(x),\{f_\beta(A,y): \beta < \alpha$ and y extends $x\}\rangle$.

The following is also standard.

LEMMA 1.3. Let $A,B \subseteq \omega x\omega$, $\alpha < \omega_1$, and $x,y \in FS(\omega)$. Then $R_\alpha(A,B)(x,y)$ if and only if $f_\alpha(A)(x) = f_\alpha(B)(y)$. f_α is a Borel function from $P(\omega)$ into codes for $HC(\omega+\alpha+1)$.

We need to consider possibly nonstandard comparison relations. For this purpose, we define a comparison relation between A,B as a system (J,<,R), where

1. $J \subseteq \omega$;
2. (J,<) is a linearly ordered set with a least element 0.
3. $R \subseteq JxFS(\omega)xFS(\omega)$.
4. R(0,x,y) if and only if x,y have the same length, and for all i,j, $A(x_i,x_j) \leftrightarrow B(y_i,y_j)$.
5. R(t,x,y) if and only if x,y have the same length, and for all $z \in FS(\omega)$ there exists $w \in FS(\omega)$ such that for all $t' < t$, R(t′,xz,yw), and for all $z \in FS(\omega)$ there exists $w \in FS(\omega)$ such that for all $t' < t$, R(t′,yw,xz).

LEMMA 1.4. Let $A,B \subseteq \omega x\omega$ and (J,<,R) be a comparison relation between A,B such that for all $t \in J$, $R(t,\langle\ \rangle,\langle\ \rangle)$. If (J,<) is not well founded then $A \approx B$.

Proof: Let A,B,J,<,R be as given, and let $t_1 > t_2 > \ldots$ lie in J. We can inductively build two infinite sequences $x_1,x_2,\ldots$ and $y_1,y_2,\ldots$ which are enumerations without repetitions of A and B, respectively, such that for all $n \geq 1$, $R(t_n,\langle x_1,\ldots,x_n\rangle,\langle y_1,\ldots,y_n\rangle)$. The required isomorphism maps each x_i to y_i.

THEOREM 1.5. Every Borel equivalence relation that is Borel reducible to BR is Borel reducible into some $T(\alpha)$, α countable.

Proof: Let (X,E) be a Borel equivalence relation that is Borel reducible to BR by the function $H:X \to P(\omega x\omega)$. Let Y be the set of all (J,<,R,u,v) such that

i) u,v ∈ X;
ii) ¬E(u,v);
iii) (J,<,R) is a comparison relation between H(u) and H(v);
iv) for all t ∈ J, R(t,< >,< >).

Suppose (J,<,R,u,v) ∈ Y. Then H(u) and H(v) are not isomorphic, and so by Lemma 1.4, (J,<) is well founded. In fact, the order types of the (J,<) for which (J,<,R,u,v) ∈ Y are exactly the ordinals 1+α for which R_α(A,B)(< >,< >), which are also the ordinals 1+α for which f_α(H(u))(< >) = f_α(H(v))(< >).

Note that the set of all (J,<) such that (J,<,R,u,v) ∈ Y for some R,u,v, with ¬E(u,v) is an analytic set of well orderings. Hence by boundedness, let $\alpha < \omega_1$ be greater than the order type of all of these well orderings.

Let u,v ∈ X. If ¬E(u,v) then ¬R_α(H(u),H(v))(< >,< >). On the other hand, by Lemma 1.2, if E(u,v) then H(u) ≈ H(v), and so R_α(H(u),H(v))(< >,< >). By Lemma 1.3, for all u,v ∈ X, E(u,v) if and only if R_α(H(u),H(v))(< >,< >) if and only if f_α(H(u))(< >) = f_α(H(v))(< >). By Lemma 1.3, this provides a Borel reduction from E into HC(ω+α+1). The Theorem now follows by Lemma 1.1.

Note that we have avoided using Lemma 1.2, which is not provable in ATR_0. We have instead used Lemma 1.4, which is provable in ATR_0. Also note that the boundedness of analytic sets of well orderings is provable in ATR_0 (see [Si99], p. 199).

COROLLARY 1.6. The following is provable in ATR_0. Every coanalytic equivalence relation that is Borel reducible into BR is a Borel equivalence relation that is Borel reducible into some T(α), α countable. The same holds with "Borel" replaced by "Baire."

Proof: In the proof of Theorem 1.5, the crucial set of reducts (J,<) is still analytic. Hence E(u,v) if and only if f_α(H(u))(< >) = f_α(H(v))(< >), and so E is Borel and Borel reducible into $T(\omega+\alpha)$ as before. The final claim is an easy consequence that is left to the reader.

2. Zero Density Equivalence Relation

In this section we prove that the equivalence relation ZD on $P(\omega)$ is not classifiable by countable structures; i.e., is not Borel reducible to BR. In fact, we show that ZD is not Baire reducible to BR.

We use HC for the set of all hereditarily countable sets. Let ZFC\P be the usual axioms of ZFC with the power set axiom deleted. We assume familiarity with Cohen forcing over M, where M is a countable transitive model of ZFC\P and the forcing conditions are finite sequences of 0’s and 1’s. Here $p(i) = 0$ asserts that the generic object - which is a subset of ω - does not contain i, and $p(i) = 1$ indicates that the generic object contains i. In particular, if x is Cohen generic over M then M[x] is the least transitive model containing $M \cup \{x\}$ satisfying ZFC\P.

LEMMA 2.1. Let M be a countable transitive model of ZFC\P. Suppose x,y are Cohen generic over M, and $M[x] \cap M[y] \cap P(M) = M$. Then $M[x] \cap M[y] = M$.

Proof: Let x,y be as given. We argue by transfinite induction. Suppose all elements of $M[x] \cap M[y]$ of rank $\beta < \alpha$ lie in M. Let $z \in M[x] \cap M[y]$ be of rank α. Then every element of z lies in $M[x] \cap M[y]$ and is of rank $< \alpha$. Hence $z \subseteq M$, and so $z \in M$. (Here we have used the Cohen genericity of x,y only for the well definedness of M[x] and M[y]).

LEMMA 2.2. Let M be a countable transitive model of ZFC\P. Let p,q be Cohen conditions of length n, and $\varphi = \varphi(x)$, $\psi = \psi(x)$ be forcing statements over M such that p does not decide φ. Then we can find Cohen conditions p' extending p and q' extending q, of the same length, where (p' forces φ and q' forces $\neg\psi$) or (p' forces $\neg\varphi$ and q' forces ψ), where p'\p and q'\q differ at at most one position.

Proof: Let M,p,q,φ,ψ be as given. First assume that for all extensions p^* of p and q^* of q, where $p^*\backslash p = q^*\backslash q$, we have p^* forces $\neg\varphi$ if and only if q^* forces $\neg\psi$.

Let p_1,p_2 be two extensions of p of the same length which force φ in the opposite way, say, with p_1 forcing φ and p_2 forcing $\neg\varphi$. Then we can find p_1^*,p_2^* extending p, of this same length such that p_1^* forces φ and p_2^* does not force φ, and where $p_1^*\backslash p$ and $p_2^*\backslash p$ differ at exactly one position. (To see this, successively change p_1 into p_2 by changing p_1 at exactly one position at a time to agree with p_2. At some point, there is going to be a change in whether p_1, as successively modified, forces φ.)

By further extension, we may assume that p_1^*,p_2^* extend p, p_1^* forces φ, p_2^* forces $\neg\varphi$, and are of the same length and differ at exactly one position. Now consider $p_2^*\backslash p \cup q$. By assumption, $p_2^*\backslash p \cup q$ forces $\neg\psi$. This establishes the conclusion of the Lemma.

Now let p^* be an extension of p and q^* be an extension of q, where $p^*\backslash p = q^*\backslash q$, and where p^* forces $\neg\varphi$ and q^* does not force $\neg\psi$. By further extension with a common new part, we may assume that q^* forces ψ. This also establishes the conclusion of the Lemma.

Finally, let p^* be an extension of p and q^* be an extension of q, where $p^*\backslash p = q^*\backslash q$, and where p^* does not force $\neg\varphi$ and q^* forces $\neg\psi$. By further extension with a common new part, we may assume that p^* forces φ. This also establishes the conclusion of the Lemma.

LEMMA 2.3. Let M be a countable transitive model of ZFC\P. There exist $x,y \subseteq \omega$ such that
i) ZD(x,y);
2) x,y are Cohen generic over M;
3) $M[x] \cap M[y] = M$.

Proof: By Lemma 2.1, it suffices to prove the Lemma with 3) replaced by $M[x] \cap M[y] \cap P(M) = M$.

Construct x,y by a pair of sequential sequences of conditions. At the n-th stage we will have two conditions of the same length. We make extensions of p,q, respectively, so that the lengths are extended by at least n positions, and where the new parts of the conditions differ at at most one position. This will guarantee that ZD(x,y) holds.

We diagonalize over forcing statements and forcing terms. At the even stages 2n we will simply extend both conditions in the same way (by 2n new places) so that each decides the n-th forcing statement (possibly in opposite ways).

At the odd stages 2n+1, we wish to extend, say, p and q, and we work on the i-th forcing term s and the j-th forcing term t, where $\langle i,j \rangle$ has index n. Here s,t are forcing terms representing elements of the generic extension.

First extend p,q to p',q' of the same length so that at least 2n+1 new places are created, where the new places are identical, and where p' decides if s represents a subset of M that lies outside M, and q' decides if t represents a subset of M that lies outside M.

If p',q' don't both force these statements, then we are done with this odd stage. So we may assume that p',q' force these respective statements. Therefore there must exist $v \in M$ such that p' does not decide $v \in s$. Now apply Lemma 2.2 to φ = '$v \in s$', and ψ = '$v \in t$', to make the final extensions for this odd stage.

In the limit, this infinite length construction results in two Cohen generic x,y such that ZD(x,y). Suppose $u \in M[x] \cap M[y] \cap P(M)$, where u is outside M. Represent u by two forcing terms, s(x) and t(y). Then in the construction, at some odd stage we worked on s,t. We must have gotten past the first part of that stage, because of u. So $v \in s$ and $v \in t$ are forced by respective initial segments of x,y in opposite ways. This is the desired contradiction, since s(x) = t(y) is true.

LEMMA 2.4. There exists a map $h:P(\omega x\omega) \to HC$ and a formula of set theory $\varphi(x,y)$ with only the free variables shown such that the following holds.
1) for all $x,y \subseteq \omega x\omega$, $BR(x,y) \leftrightarrow h(x) = h(y)$;

2) ZFC\P proves $(\forall x \subseteq \omega x \omega)(\exists! y \in HC)(\varphi(x,y))$;
3) for all countable transitive models M of ZFC\P, the graph of φ in M is the same as the graph of h restricted to M.

Proof: Let h(x) to be the Scott sentence defined in the usual way. We are relying on the technology of Scott sentences from [Sc65].

We fix h and a definition of h according to Lemma 2.4 for the remainder of the proof.

THEOREM 2.5. ZD is not Baire reducible to BR.

Proof: We will assume that $F:P(\omega) \to P(\omega x \omega)$ is a Baire reduction of ZD to BR and obtain a contradiction. Let S be a comeager Borel subset of $P(\omega)$ such that F|S is a Borel function. Let M be a countable transitive model of ZFC\P which contains a Borel code for F|S. Then every $x \subseteq \omega$ that is Cohen generic over M lies in S.

We work with the function $G:P(\omega) \to HC$ given by G(x) = h(F(x)). We can apply forcing to G because of Lemma 2.4.

We claim that for all $x \subseteq \omega$, if x is Cohen generic over M then $G(x) \notin M$. To see this, let x be Cohen generic over M and $G(x) = b \in M$. Then for all $y \subseteq \omega$ that is Cohen generic over M with a certain finite initial segment, G(y) = b. Choose $y \subseteq \omega$ so that $x \Delta y$ is coinfinite and y has that certain finite initial segment. Then x,y lie in S and $\neg$ZD(x,y), and so $G(x) \neq G(y)$. This contradiction establishes the claim.

By Lemma 2.3, fix $x,y \subseteq \omega$ that are Cohen generic over S such that ZD(x,y) and $M[x] \cap M[y] = M$. Since $G(x),G(y) \notin M$, we see that $G(x) \neq G(y)$, contradicting ZD(x,y).

COROLLARY 2.6. It is provable in ATR_0 that ZD is not Baire reducible to BR.

Proof: We indicate that the modifications of the proof of Theorem 2.5 that are necessary. Note that we have used the existence of Scott sentences, which requires hyperjumps, and therefore lies outside ATR_0. So we now need to invoke Theorem

1.5, which is itself provable in ATR_0 by Corollary 1.6. Thus it suffices to show that ZD is not Baire reducible to any $T(\alpha)$, $\alpha < \omega_1$. But now we no longer need to use Scott sentences, and can instead work with $M = L_\lambda(u)$, where $u \subseteq \omega$ is a Borel code for F|S, and λ is an additive countable limit ordinal greater than the Borel rank of F|S.

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