

5.2. From length 3 towers to length n towers.

In this section, we obtain a variant of Lemma 5.1.7 (Lemma 5.2.12) involving length n towers rather than length 3 towers of infinite sets. However, we only assert that the sets in the length n tower have at least r elements, for any $r \geq 1$. Thus we pay a real cost for lengthening the towers.

Because the sets in the tower are finite and not infinite, certain indiscernibility properties of the first set in the tower must now be stated explicitly as additional conditions. See Lemma 5.2.12, iii), viii). These indiscernibility properties can of course be obtained from the usual infinite Ramsey theorem by taking a subset of the infinite $A \subseteq \mathbb{N}$ from Lemma 5.1.7 - but then we would only have a tower of length 3.

We will apply Lemma 5.1.7 with f arising from term assignments. Thus Lemma 5.2.12 uses g and not f.

Recall the definition of the language L (Definition 5.1.8). In order to avoid having to write too many parentheses in terms and formulas of L, we use the following two standard precedence tables.

$$\begin{array}{c} \uparrow \\ \cdot \\ +, - \\ \neg \\ \wedge, \vee \\ \rightarrow, \leftrightarrow \end{array}$$

DEFINITION 5.2.1. Let t be a term of L. We write $\#(t)$ for the maximum of: the subscripts of variables in t, and the number of occurrences of the symbols

$$0, 1, +, -, \cdot, \uparrow, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \log$$

We count log as a single symbol. Note that for all $n \geq 0$, $\{t: \#(t) \leq n\}$ is finite.

DEFINITION 5.2.2. Let φ be a quantifier free formula in L. We write $\#(\varphi)$ for the maximum of: the subscripts of

variables in φ , and the number of occurrences of the symbols

$$01+-\bullet\uparrow()=\langle\neg\wedge\vee\rightarrow\leftrightarrow v_1v_2,\dots,v_r \log$$

in φ . Note that for all $n \geq 0$, $\{\varphi: \#(\varphi) \leq n\}$ is finite.

DEFINITION 5.2.3. For all $r \geq 1$, let $\beta(r)$ be the number of terms t in L with $\#(t) \leq r$. We fix a doubly indexed sequence $t[i,r]$ of terms in L , which is defined if and only if $r \geq 1$ and $1 \leq i \leq \beta(r)$. For each $r \geq 1$, the $t[i,r]$, $1 \leq i \leq \beta(r)$, enumerate the terms t with $\#(t) \leq r$, without repetition.

DEFINITION 5.2.4. For all $r \geq 1$, let $\gamma(r)$ be the number of quantifier free formulas φ in L with $\#(\varphi) \leq r$. We fix a doubly indexed sequence $\varphi[i,r]$ of quantifier free formulas in L , which is defined if and only if $r \geq 1$ and $1 \leq i \leq \gamma(r)$. For each $r \geq 1$, the $\varphi[i,r]$, $1 \leq i \leq \gamma(r)$, enumerate the quantifier free formulas φ with $\#(\varphi) \leq r$, without repetition.

We adhere to the convention of displaying all free variables (and possibly additional variables). Thus $t(v_1, \dots, v_n)$ and $\varphi(v_1, \dots, v_m)$ respectively indicate that all variables in the term t are among the first n variables v_1, \dots, v_n , and all variables in the quantifier free formula φ are among the first m variables v_1, \dots, v_m .

Note that all terms $t[i,r]$ have variables among v_1, \dots, v_r , and all formulas $\varphi[i,r]$ have variables among v_1, \dots, v_r .

We want to be more specific about the enumerations of terms and formulas in Definitions 5.2.3, 5.2.4.

DEFINITION 5.2.5. Let $r \geq 1$. The enumeration $t[1,r], \dots, t[\beta(r),r]$ in Definition 5.2.3 is the enumeration of all terms t of L with $\#(t) \leq r$, ordered first by $\#(t)$, and second by the lexicographic ordering of strings of symbols, where, for specificity, the symbols are ordered by

$$01+-\bullet\uparrow()v_1v_2\dots v_r \log$$

DEFINITION 5.2.6. Let $r \geq 1$. The enumeration $\varphi[1,r], \dots, \varphi[\gamma(r),r]$ in Definition 5.2.4 is the enumeration of all quantifier free formulas φ of L with $\#(\varphi) \leq r$, ordered first by $\#(\varphi)$, and second by the lexicographic

ordering of strings of symbols, where the symbols are ordered by

$$01+-\bullet\uparrow()=\langle\neg\wedge\nu\rightarrow\leftrightarrow\nu_1\nu_2\dots\nu_r \log$$

An important condition concerning the way we have enumerated terms and formulas is the following.

$$\begin{aligned} 1 \leq i \leq \beta(r) \wedge 1 \leq r \leq r' &\rightarrow t[i,r] = t[i,r'] . \\ 1 \leq i \leq \gamma(r) \wedge 1 \leq r \leq r' &\rightarrow \varphi[i,r] = \varphi[i,r'] . \end{aligned}$$

DEFINITION 5.2.7. For $E \subseteq \mathbb{N}$ and $r \geq 1$, we write $\alpha(r,E)$ for the set of values of all terms $t[i,r]$, at assignments f to the variables in t , with $\text{rng}(f) \subseteq E$, including $t[i,r]$ that are closed.

DEFINITION 5.2.8. For $E \subseteq \mathbb{N}$ and integers $p,q \geq 0$, we write $\alpha(r,E;p,q)$ for the set of all nonnegative integers x such that the following holds. There is a term $t[i,r]$ that is not closed, and an assignment f to its variables, with $\text{rng}(f) \subseteq E$, such that x is the value of $t[i,r]$ under f , and $x \in [p\max(\text{rng}(f)), q\max(\text{rng}(f))]$.

Note that for $E \subseteq \mathbb{N}$, $r \geq 1$, $p,q \geq 0$, $\alpha(r,E;p,q) \subseteq [p\min(E), \infty)$.

Here is a version of Lemma 5.1.7, where the role of f is taken up by α . Recall Definition 5.1.12.

LEMMA 5.2.1. Let $r \geq 1$ and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$, $\text{rng}(g) \subseteq 6\mathbb{N}$. There exist infinite $A \subseteq B \subseteq C \subseteq \mathbb{N} \setminus \{0\}$ such that

- i) $6\alpha(r,A^*;1,r) \subseteq B \cup gB$;
- ii) $6\alpha(r,B^*;1,r) \subseteq C \cup gC$;
- iii) $2\alpha(r,A^*;1,r)+1 \subseteq B$;
- iv) $3\alpha(r,A^*;1,r)+1 \subseteq B$;
- v) $2\alpha(r,B^*;1,r)+1 \subseteq C$;
- vi) $3\alpha(r,B^*;1,r)+1 \subseteq C$;
- vii) $C \cap gC = \emptyset$;
- viii) $A \cap \alpha(r,B^*;2,r) = \emptyset$.

Proof: Let r,g be as given. We define $f \in \text{ELG} \cap \text{SD} \cap \text{BAF}$ of arity $\beta(r)+12+r$ as follows. Let $x^* =$

$$\begin{aligned} (Y_1, \dots, Y_{\beta(r)}, Z_1, \dots, Z_6, W_1, \dots, W_6, X_1, \dots, X_r) \\ \in \mathbb{N}^{\beta(r)+12+r} . \end{aligned}$$

Let i,j,k be greatest such that

$$\begin{aligned} y_1 &= \dots = y_i \\ z_1 &= \dots = z_j \\ w_1 &= \dots = w_k \end{aligned}$$

respectively.

Define $f(x^*) =$

$$\begin{aligned} & jt[i,r](x_1, \dots, x_r) + k - 1 \text{ if} \\ & |x^*| + 1, 2|x^*| \leq jt[i,r](x_1, \dots, x_r) + k - 1 \leq r|x^*|; \\ & \max(|x^*| + 1, 2|x^*|) \text{ otherwise.} \end{aligned}$$

Clearly $f \in \text{ELG} \cap \text{SD} \cap \text{BAF}$. We claim that for any $D \subseteq \mathbb{N}$, $2 \leq p \leq 6$, and $0 \leq q \leq 5$,

$$\alpha(r, D^*; 2, r) \cup p\alpha(r, D^*; 1, r) + q \subseteq fD.$$

To see this, let $u \in \alpha(r, D^*; 2, r)$, $v \in p\alpha(r, D^*; 1, r) + q$, and write

$$\begin{aligned} u &= t[i,r](x_1, \dots, x_r) \\ v &= pt[i',r](x_1, \dots, x_r) + q \end{aligned}$$

where $x_1, \dots, x_r \in D^*$, $1 \leq i, i' \leq \beta(r)$, $2|x_1, \dots, x_r| \leq u \leq r|x_1, \dots, x_r|$, $|x_1, \dots, x_r| \leq v \leq r|x_1, \dots, x_r|$, and $t[i,r], t[i',r]$ are not closed.

First let $y_1 = \dots = y_i = \min(D)$, $y_{i+1} = \dots = y_{\beta(r)} = |x_1, \dots, x_r|$, $z_1 = w_1 = \min(D)$, $z_2 = \dots = z_6 = w_2 = \dots = w_6 = |x_1, \dots, x_r|$. Then $f(y_1, \dots, y_{\beta(r)}, z_1, \dots, z_6, w_1, \dots, w_6, x_1, \dots, x_r) = u \in fD$.

Now let $y_1 = \dots = y_i = \min(D)$, $y_{i+1} = \dots = y_{\beta(r)} = |x_1, \dots, x_r|$, $z_1 = \dots = z_p = \min(D)$, $z_{p+1} = \dots = z_6 = |x_1, \dots, x_r|$, $w_1 = \dots = w_{q+1} = \min(D)$, $w_{q+2} = \dots = w_6 = |x_1, \dots, x_r|$.

It is obvious that

$$f(y_1, \dots, y_{\beta(r)}, z_1, \dots, z_6, w_1, \dots, w_6, x_1, \dots, x_r) = v \in fD.$$

Now apply Lemma 5.1.7 to f, g to obtain $A, B, C \subseteq \mathbb{N} \setminus \{0\}$ with the properties i)-viii) cited there.

From the demonstrated claim, we have

$$6\alpha(r, A^*; 1, r) \subseteq fA.$$

$$\begin{aligned}
& 6\alpha(r, B^*; 1, r) \subseteq fB. \\
& 2\alpha(r, A^*; 1, r) + 1 \subseteq fA. \\
& 3\alpha(r, A^*; 1, r) + 1 \subseteq fA. \\
& 2\alpha(r, B^*; 1, r) + 1 \subseteq fB. \\
& 3\alpha(r, B^*; 1, r) + 1 \subseteq fB. \\
& \alpha(r, B^*; 2, r) \subseteq fB.
\end{aligned}$$

We now obtain i)-viii) here immediately from the i)-viii) of Lemma 5.1.7. QED

We are now going to define three properties of finite length towers of sets, of increasing strength: r, g -good for aN , r, g -great for aN , and r, g -terrific for aN . The notion of r -good generalizes some properties from Lemma 5.2.1.

DEFINITION 5.2.9. Let $n \geq 3$, $r, a \geq 1$, and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$. We say that (D_1, \dots, D_n) is r, g -good for aN if and only if

- i) $D_1 \subseteq \dots \subseteq D_n \subseteq N \setminus \{0\}$;
- ii) for all $x < y$ from D_1 , $x \uparrow < y$;
- iii) for all $1 \leq i \leq n-1$, $a\alpha(r, D_i^*; 1, r) \subseteq D_{i+1} \cup gD_{i+1}$;
- iv) for all $1 \leq i \leq n-1$, $2\alpha(r, D_i^*; 1, r) + 1 \subseteq D_{i+1}$;
- v) for all $1 \leq i \leq n-1$, $3\alpha(r, D_i^*; 1, r) + 1 \subseteq D_{i+1}$;
- vi) $D_n \cap gD_n = \emptyset$;
- vii) $D_1 \cap \alpha(r, D_2^*; 2, r) = \emptyset$.

The following proves the existence of length 3 towers that are r, g -good for $6N$.

LEMMA 5.2.2. Let $r \geq 1$ and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$, $\text{rng}(g) \subseteq 6N$. There exists (A, B, C) which is r, g -good for $6N$, where A is infinite.

Proof: Let r, g be as given, and let $A, B, C \subseteq N \setminus \{0\}$ be as given by Lemma 5.2.1. Set $D_1 = A$, $D_2 = B$, $D_3 = C$. Obviously i), iii)-vii) hold in the definition of r, g -good for $6N$. However ii) may fail. We can obviously shrink A so that ii) holds, keeping A infinite, and retaining i), iii)-vii). QED

We now want to define certain $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$ so that any r, g -good sequence for N codes up the truth values of existential closures of quantifier free formulas $\varphi[i, r]$, $1 \leq i \leq \gamma(r)$, in a convenient uniform way. This introduces a kind of quantifier elimination.

DEFINITION 5.2.10. Let $r \geq 1$ and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$, where $\text{rng}(g) \subseteq 24\mathbb{N}$. We define $\tau(g,r) \in \text{ELG} \cap \text{SD} \cap \text{BAF}$ as follows. $\tau(g,r)$ has arity $\gamma(r)+k+r+1$, where k is the arity of g . Let $x^* = (y_1, \dots, y_{\gamma(r)}, z_1, \dots, z_k, x_1, \dots, x_r, w) \in \mathbb{N}^{\gamma(r)+k+r+1}$. Let $i \in [1, \gamma(r)]$ be greatest such that $1 \leq i \leq \gamma(r)$ and $y_1 = \dots = y_i$.

case 1. $|x^*| = w \wedge x_1, \dots, x_r < w \wedge \varphi[i,r](x_1, \dots, x_r)$. Define $\tau(g,r)(x^*) = 24\gamma(r)w + 24i + 6$.

case 2. $|x^*| = |z_1, \dots, z_k| \wedge x_1 = \dots = x_r = w$. Define $\tau(g,r)(x^*) = g(z_1, \dots, z_k)$.

case 3. Otherwise. Define $\tau(g,r)(x^*) = 24|x^*| + 12$.

We now establish some useful coding properties of $\tau(g,r)$.

LEMMA 5.2.3. $\tau(g,r) \in \text{ELG} \cap \text{SD} \cap \text{BAF}$. The values arising out of the above three cases are mutually disjoint, and lie in $6\mathbb{N}$. Let $E \subseteq \mathbb{N}$. For all $w \in E$ and $1 \leq i \leq \gamma(r)$, $24\gamma(r)w + 24i + 6 \in \tau(g,r)E \Leftrightarrow (\exists v_1, \dots, v_r \in E) (v_1, \dots, v_r < w \wedge \varphi[i,r](v_1, \dots, v_r))$. $gE = \tau(g,r)E \cap 24\mathbb{N}$.

Proof: Note that in case 1, $\gamma(r), w \geq 1$, and $24w \leq 24\gamma(r)w + 24i + 6 \leq 100\gamma(r)w$. Hence

$$|x^*| + 1, 24|x^*| \leq \tau(g,r)(x^*) \leq 100\gamma(r)|x^*|.$$

In case 2,

$$\begin{aligned} |x^*| &= |z_1, \dots, z_k| \\ |\tau(g,r)(x^*)| &= |g(z_1, \dots, z_k)|. \end{aligned}$$

In case 3, $|x^*| \geq 1$, and

$$24|x^*| < \tau(g,r)(x^*) \leq 36|x^*|.$$

Therefore $\tau(g,r) \in \text{ELG} \cap \text{SD} \cap \text{BAF}$.

Since $\text{rng}(g) \subseteq 24\mathbb{N}$, the values arising out of the three cases are mutually disjoint. Also note that the w, i used in case 1 can be recovered from any value of $\tau(g,r)$ obtained by case 1. This is because $1 \leq i \leq \gamma(r)$ in case 1.

Let $E \subseteq \mathbb{N}$ and $w \in E$. First suppose $24\gamma(r)w + 24i + 6 \in \tau(g,r)E$. Then $24\gamma(r)w + 24i + 6$ must arise out of case 1, with, say, $x^* \in E^{\gamma(r)+k+r+1}$. Then the w, i used in case 1 must be this w, i .

Hence the x_1, \dots, x_r used in case 1 must be $< w$, and $\varphi[i, r](x_1, \dots, x_r)$.

Conversely, suppose $x_1, \dots, x_r \in E \cap [0, w)$ and $\varphi[i, r](x_1, \dots, x_r)$. Then we can choose $y_1 = \dots = y_i = x_1$ and $y_{i+1} = \dots = y_{\gamma(r)} = z_1 = \dots = z_k = w$. Then case 1 applies, $y_1, \dots, y_{\gamma(r)}, z_1, \dots, z_k, w \in E$, and i is greatest such that $y_1 = \dots = y_i$. Hence $\tau(g, r)(y_1, \dots, y_{\gamma(r)}, z_1, \dots, z_k, x_1, \dots, x_r, w) = 24\gamma(r)w + 24i + 6$.

For the final claim, note that every element of gE arises out of case 2, since we can set $y_1 = \dots = y_{\gamma(r)} = x_1 = \dots = x_r = w = z_1$, taking z_1, \dots, z_k to be arbitrary elements of E . On the other hand, all elements of $\tau(g, r)E$ lying in $24N$ must arise out of case 2, in which case they must lie in gE . QED

DEFINITION 5.2.11. Throughout the book, we will use the logical construction

$$\varphi_1 \leftrightarrow \dots \leftrightarrow \varphi_k$$

for

$$(\varphi_1 \leftrightarrow \varphi_2) \wedge (\varphi_2 \leftrightarrow \varphi_3) \wedge \dots \wedge (\varphi_{k-1} \leftrightarrow \varphi_k).$$

LEMMA 5.2.4. Let $r \geq 1$, $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$, $\text{rng}(g) \subseteq 24N$, and (A, B, C) be $100\gamma(r), \tau(g, r)$ -good for $6N$. Then

i) for all $1 \leq i \leq \gamma(r)$ and $x \in B^*$,

$$(\exists v_1, \dots, v_r \in C) (v_1, \dots, v_r < x \wedge \varphi[i, r](v_1, \dots, v_r)) \leftrightarrow 24\gamma(r)x + 24i + 6 \notin C;$$

ii) for all $1 \leq i \leq \gamma(r)$ and $x \in A^*$,

$$\begin{aligned} (\exists v_1, \dots, v_r \in B) (v_1, \dots, v_r < x \wedge \varphi[i, r](v_1, \dots, v_r)) &\leftrightarrow \\ (\exists v_1, \dots, v_r \in C) (v_1, \dots, v_r < x \wedge \varphi[i, r](v_1, \dots, v_r)) &\leftrightarrow \\ 24\gamma(r)x + 24i + 6 \notin B &\leftrightarrow \\ 24\gamma(r)x + 24i + 6 \notin C. & \end{aligned}$$

iii) (A, B, C) is r, g -good for $24N$.

Proof: Let r, g, A, B, C be as given. For claim i), let $1 \leq i \leq \gamma(r)$, $x \in B^*$. Then $4\gamma(r)x + 4i + 1 \in \alpha(100\gamma(r), B^*; 1, 100\gamma(r))$. To see this, note that $\gamma(r), x \geq 1$, $2x \leq 4\gamma(r)x + 4i + 1 \leq 100\gamma(r)x$. Also $4\gamma(r)x + 4i + 1$ is a term $t(x)$ with $\#(t) \leq 100\gamma(r)$.

By clauses iii), vi) in the definition of $100\gamma(r), \tau(g, r)$ -good for $6N$, we have

$$\begin{aligned} 24\gamma(r)x+24i+6 &\in C \cup \tau(g, r)C. \\ C \cap \tau(g, r)C &= \emptyset. \end{aligned}$$

By the above and Lemma 5.2.3,

$$\begin{aligned} (\exists v_1, \dots, v_r \in C) (v_1, \dots, v_r < x \wedge \varphi[i, r](v_1, \dots, v_r)) &\leftrightarrow \\ 24\gamma(r)x+24i+6 \in \tau(g, r)C &\leftrightarrow 24\gamma(r)x+24i+6 \notin C. \end{aligned}$$

For claim ii), let $1 \leq i \leq \gamma(r)$ and $x \in A^*$. Then $4\gamma(r)x+4i+1 \in \alpha(100\gamma(r), A^*; 1, 100\gamma(r))$. By clauses iii), iv), vi) in the definition of $100\gamma(r), \tau(g, r)$ -good for $6N$, we have

$$\begin{aligned} 24\gamma(r)x+24i+6 &\in B \cup \tau(g, r)B \\ B \cap \tau(g, r)B &= \emptyset. \end{aligned}$$

By the above and Lemma 5.2.3,

$$\begin{aligned} (\exists v_1, \dots, v_r \in B) (v_1, \dots, v_r < x \wedge \varphi[i, r](v_1, \dots, v_r)) &\leftrightarrow \\ 24\gamma(r)x+24i+6 \in \tau(g, r)B &\leftrightarrow 24\gamma(r)x+24i+6 \notin B. \end{aligned}$$

Hence

$$\begin{aligned} (\exists v_1, \dots, v_r \in C) (v_1, \dots, v_r < x \wedge \varphi[i, r](v_1, \dots, v_r)) &\rightarrow \\ 24\gamma(r)x+24i+6 \notin C &\rightarrow \\ 24\gamma(r)x+24i+6 \notin B &\rightarrow \\ (\exists v_1, \dots, v_r \in B) (v_1, \dots, v_r < x \wedge \varphi[i, r](v_1, \dots, v_r)) &\rightarrow \\ (\exists v_1, \dots, v_r \in C) (v_1, \dots, v_r < x \wedge \varphi[i, r](v_1, \dots, v_r)) & \end{aligned}$$

and so all of the above \rightarrow are also \leftrightarrow .

For claim iii), by the definition of $100\gamma(r), \tau(g, r)$ -good for $6N$, we have

$$\begin{aligned} 6\alpha(100\gamma(r), A^*; 1, 100\gamma(r)) &\subseteq B \cup \tau(g, r)B \\ 6\alpha(100\gamma(r), B^*; 1, 100\gamma(r)) &\subseteq C \cup \tau(g, r)C \\ 2\alpha(100\gamma(r), A^*; 1, 100\gamma(r))+1 &\subseteq B \\ 3\alpha(100\gamma(r), A^*; 1, 100\gamma(r))+1 &\subseteq B \\ 2\alpha(100\gamma(r), B^*; 1, 100\gamma(r))+1 &\subseteq C \\ 3\alpha(100\gamma(r), B^*; 1, 100\gamma(r))+1 &\subseteq C \\ C \cap \tau(g, r)C &= \emptyset \\ A \cap \alpha(100\gamma(r), B^*; 2, 100\gamma(r)) &= \emptyset \\ \text{for all } x < y \text{ from } A, x \uparrow < y. & \end{aligned}$$

By Lemma 5.2.3, $gB = \tau(g,r)B \cap 24N$ and $gC = \tau(g,r)C \cap 24N$. Hence the conditions

$$\begin{aligned}
24\alpha(r,A^*;1,r) &\subseteq B \cup gB \\
24\alpha(r,B^*;1,r) &\subseteq C \cup gC \\
2\alpha(r,A^*;1,r)+1 &\subseteq B \\
3\alpha(r,A^*;1,r)+1 &\subseteq B \\
2\alpha(r,B^*;1,r)+1 &\subseteq C \\
3\alpha(r,B^*;1,r)+1 &\subseteq C \\
C \cap gC &= \emptyset \\
A \cap \alpha(r,B^*;2,r) &= \emptyset \\
\text{for all } x < y \text{ from } A, x \uparrow < y
\end{aligned}$$

follow immediately. Therefore (A,B,C) is r,g -good for $24N$. QED

We now define r,g -great towers, which feature a special form of indiscernibility for terms. We also define r,g -terrific towers, which feature a special form of indiscernibility for quantifier free formulas. We will only use r,g -terrific towers of length 3.

DEFINITION 5.2.12. Let $n \geq 3$, $r,a \geq 1$, and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$. We say that (D_1, \dots, D_n) is r,g -great for aN if and only if

- i) (D_1, \dots, D_n) is r,g -good for aN ;
- ii) Let $1 \leq i \leq \beta(2r)$, $x_1, \dots, x_{2r} \in D_1$, $y_1, \dots, y_r \in \alpha(r, D_2)$, where $(x_1, \dots, x_r), (x_{r+1}, \dots, x_{2r})$ have the same order type and min, and $y_1, \dots, y_r \leq \min(x_1, \dots, x_r)$. Then $t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) \in D_3^* \leftrightarrow t[i, 2r](x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r) \in D_3^*$.

DEFINITION 5.2.13. Let $r,a \geq 1$, and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$. We say that (A,B,C) is r,g -terrific for aN if and only if

- i) (A,B,C) is r,g -great for aN ;
- ii) A is infinite;
- iii) for all $1 \leq i \leq \gamma(r)$,

$$\begin{aligned}
(\exists v_1, \dots, v_r \in B) (\varphi[i, r](v_1, \dots, v_r)) &\leftrightarrow \\
(\exists v_1, \dots, v_r \in C) (\varphi[i, r](v_1, \dots, v_r)). &
\end{aligned}$$

We now derive an essentially well known infinitary combinatorial lemma. E.g., see [Sc74].

LEMMA 5.2.5. Let D be an infinite subset of N and $r \geq 1$. Let $f: N \rightarrow N$, and R_1, \dots, R_s be a finite list of subsets of N^{2r} . There exists an infinite $D' \subseteq D$ such that the following holds. Let $1 \leq i \leq s$, $x_1, \dots, x_{2r} \in D'$, and $y_1, \dots, y_r \in N$, where (x_1, \dots, x_r) and (x_{r+1}, \dots, x_{2r}) have the same order type and \min , and $y_1, \dots, y_r \leq f(\min(x_1, \dots, x_r))$. Then $R_i(x_1, \dots, x_r, y_1, \dots, y_r) \leftrightarrow R_i(x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r)$.

Proof: Let D, r, R_1, \dots, R_s be as given. Here we write $R_i(z_1, \dots, z_{2r})$ for $(z_1, \dots, z_{2r}) \in R_i$. We will partition the ordered $2r$ tuples from N into finitely many pieces as follows. Let $x_1, \dots, x_{2r} \in N$ be given. We partition (x_1, \dots, x_{2r})

- a. first according to the order type of (x_1, \dots, x_{2r}) .
- b. second according to the set of all $i \in [1, s]$ such that for all $y_1, \dots, y_r \leq f(\min(x_1, \dots, x_{2r}))$, $R_i(x_1, \dots, x_r, y_1, \dots, y_r) \leftrightarrow R_i(x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r)$.

By Ramsey's theorem, let $D' \subseteq D$ be infinite, where any two $(x_1, \dots, x_{2r}) \in D'^{2r}$ with the same order type lie in the same partition.

Let $1 \leq i \leq s$ and μ be the order type of an element of N^r . We say that (x_1, \dots, x_{2r}) is μ -special if and only if

- i) (x_1, \dots, x_r) and (x_{r+1}, \dots, x_{2r}) have order type μ ;
- ii) $\min(x_1, \dots, x_r) = \min(x_{r+1}, \dots, x_{2r})$;
- iii) if $x_{r+j} > \min(x_1, \dots, x_r)$, then $|x_1, \dots, x_r| < x_{r+j}$.

The μ -special tuples are exactly the $2r$ -tuples of some particular order type depending on μ . Hence for each μ, i , we have

1) for all μ -special $(x_1, \dots, x_{2r}) \in D'^{2r}$, we have: for all $y_1, \dots, y_r \leq f(\min(x_1, \dots, x_{2r}))$, $R_i(x_1, \dots, x_r, y_1, \dots, y_r) \leftrightarrow R_i(x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r)$; or

2) for all μ -special $(x_1, \dots, x_{2r}) \in D'^{2r}$, we have: \neg (for all $y_1, \dots, y_r \leq f(\min(x_1, \dots, x_{2r}))$, $R_i(x_1, \dots, x_r, y_1, \dots, y_r) \leftrightarrow R_i(x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r)$).

Suppose 2) holds for μ . Let $\alpha_1, \alpha_2, \dots$ be elements of N^r where each $2r$ -tuple (α_j, α_{j+1}) is μ -special. For each $j < k$ from $[1, \infty)$, let $h(j, k)$ be some counterexample (y_1, \dots, y_r) given by 2) for $(x_1, \dots, x_{2r}) = (\alpha_j, \alpha_k)$.

Obviously h is bounded by $f(\min(\alpha_1))$. By Ramsey's theorem, h is constant on the $j < k$ drawn from some infinite subset of N . But $h(j,k) = h(j,p) = h(k,p)$ is obviously impossible for $j < k < p$. We conclude that 2) fails. Hence 1) holds for μ .

We have thus shown that for all μ, i , 1) holds. To complete the argument, let $1 \leq i \leq s$, $x_1, \dots, x_{2r} \in D'$, and $y_1, \dots, y_r \in N$, where (x_1, \dots, x_r) and (x_{r+1}, \dots, x_{2r}) have the same order type and \min , and $y_1, \dots, y_r \leq f(\min(x_1, \dots, x_r))$. Let the order type of (x_1, \dots, x_r) be μ . Choose $x_1', \dots, x_r' \in D'$ such that $(x_1, \dots, x_r, x_1', \dots, x_r')$ and $(x_{r+1}, \dots, x_{2r}, x_1', \dots, x_r')$ are μ -special. By 1),

$$\begin{aligned} R_i(x_1, \dots, x_r, y_1, \dots, y_r) &\leftrightarrow \\ R_i(x_1', \dots, x_r', y_1, \dots, y_r) &. \\ R_i(x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r) &\leftrightarrow \\ R_i(x_1', \dots, x_r', y_1, \dots, y_r) &. \end{aligned}$$

Hence

$$\begin{aligned} R_i(x_1, \dots, x_r, y_1, \dots, y_r) &\leftrightarrow \\ R_i(x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r) &. \end{aligned}$$

as required. QED

We now prove the existence of r, g -terrific towers.

LEMMA 5.2.6. Let $r \geq 1$ and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$, where $\text{rng}(g) \subseteq 24N$. There exists (A, B, C) which is r, g -terrific for $24N$.

Proof: Let r, g be as given. By Lemma 5.2.2, there exists (A, B, C) which is $100\gamma(r), \tau(g, r)$ -good for $6N$, where A is infinite. By Lemma 5.2.4, (A, B, C) is r, g -good for $24N$, and satisfies clauses i) and ii) in Lemma 5.2.4.

For all $1 \leq i \leq \beta(2r)$, let $R_i \subseteq N^{2r}$ be given by

$$\begin{aligned} R_i(x_1, \dots, x_{2r}) &\leftrightarrow \\ t[i, 2r](x_1, \dots, x_{2r}) &\in C^*. \end{aligned}$$

Apply Lemma 5.2.5 to these R_i with $D = A$ to obtain $A' \subseteq A$, A' infinite, such that (A', B, C) is r, g -great for $24N$.

To see that (A', B, C) is r, g -terrific for $24N$, we need only verify clause iii) in that definition. Since (A, B, C) satisfies clause ii) in Lemma 5.2.4, we have that for all $1 \leq i \leq \gamma(r)$ and $x \in A^*$,

$$(\exists v_1, \dots, v_r \in B) (v_1, \dots, v_r < x \wedge \varphi[i, r](v_1, \dots, v_r)) \leftrightarrow (\exists v_1, \dots, v_r \in C) (v_1, \dots, v_r < x \wedge \varphi[i, r](v_1, \dots, v_r)).$$

Since A^* is infinite, we have

$$(\exists v_1, \dots, v_r \in B) (\varphi[i, r](v_1, \dots, v_r)) \leftrightarrow (\exists v_1, \dots, v_r \in C) (\varphi[i, r](v_1, \dots, v_r)).$$

QED

We remark that, using Lemma 5.2.5, we can obtain ii) in the definition of r, g -great with $\alpha(r, D_2)$ replaced by N . However, if we formulated r, g -greatness in such a strong form, we would not be able to push down from C to B in Lemma 5.2.8.

LEMMA 5.2.7. For all $n \geq 3$ and $k, p, r \geq 1$, there exists $m \geq 1$ such that the following holds. Let $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$ be k -ary, $a \geq 1$, and (D_1, \dots, D_n) be r, g -great for aN , $|D_1| = p$. There exists (D_1', \dots, D_n') which is r, g -great for aN , where $D_1' = D_1$, each $D_i' \subseteq D_i$, and $|D_n'| \leq m$.

Proof: Let n, k, p, r, a be as given. Let g, D_1, \dots, D_n also be as given. We will construct the required D_1', \dots, D_n' by induction on $1 \leq j \leq n$, in such a way that there is an obvious bound on the cardinality of each D_{j+1}' that depends only on j, k, p, r and not on a, n, g, D_1, \dots, D_n .

Suppose $D_1 = D_1' \subseteq \dots \subseteq D_j'$ have been defined, $1 \leq j < n$, such that $(\forall i \in [1, j]) (D_i' \subseteq D_i)$. We now construct $D_{j+1}' \subseteq D_{j+1}$.

First throw all elements of D_j' into D_{j+1}' , and also $\min(D_{j+1})$ into D_{j+1}' . Then for each $x \in \alpha(r, D_j'; 1, r)$, throw x into D_{j+1}' if $x \in D_{j+1}$; otherwise find a k -tuple y from D_{j+1} such that $g(y) = x$ and throw y_1, \dots, y_k into D_{j+1}' . Next, throw all elements of $2\alpha(r, D_j'; 1, r)+1$, $3\alpha(r, D_j'; 1, r)+1$, into D_{j+1}' . Note that these elements are in D_{j+1} , because (D_1, \dots, D_n) is r, g -good.

Finally, if $j = 2$ then let $1 \leq i \leq \beta(2r)$, $x_1, \dots, x_r \in D_1$, and $y_1, \dots, y_r \in \alpha(r, D_2')$, $y_1, \dots, y_r \leq \min(x_1, \dots, x_r)$. If $t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) \in D_3^*$, then throw $t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r)$ in D_3' . Otherwise, take no action.

It is clear that (D_1', \dots, D_n') is r, g -good for aN . We have to verify clause ii) in the definition of r, g -great for aN .

Let $1 \leq i \leq \beta(2r)$, $x_1, \dots, x_r \in D_1$, $y_1, \dots, y_r \in \alpha(r, D_2')$, where $y_1, \dots, y_r \leq \min(x_1, \dots, x_r)$. We claim that

$$\begin{aligned} t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) \in D_3'^* &\leftrightarrow \\ t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) \in D_3^* &. \end{aligned}$$

The forward direction is immediate. For the reverse direction, first note that $\min(D_3) = \min(D_3')$ by construction. If the right side holds, then $t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r)$ has been thrown into D_3' , and since $t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) > \min(D_3) = \min(D_3')$, the left side follows.

Now let $1 \leq i \leq \beta(2r)$, $x_1, \dots, x_{2r} \in D_1$, $y_1, \dots, y_r \in \alpha(r, D_2')$, where (x_1, \dots, x_r) and (x_{r+1}, \dots, x_{2r}) have the same order type and \min , and $y_1, \dots, y_r \leq \min(x_1, \dots, x_r)$. We must verify that

$$\begin{aligned} t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) \in D_3'^* &\leftrightarrow \\ t[i, 2r](x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r) \in D_3'^* &. \end{aligned}$$

By the above, this is equivalent to

$$\begin{aligned} t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) \in D_3^* &\leftrightarrow \\ t[i, 2r](x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r) \in D_3^* &. \end{aligned}$$

which follows from the hypothesis on (D_1, \dots, D_n) - in particular, from ii) in the definition of r, g -great.

It is clear that we can write m as a specific iterated exponential in n, k, p, r . QED

We show that, at the cost of increasing r to much larger s , we can guarantee that for any s, g -terrific tower (A, B, C) , any r, g -great tower contained in C can be shrunk to an r, g -great tower contained in B .

LEMMA 5.2.8. Let $n \geq 3$, $k, p, r \geq 1$, and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$ be k -ary. There exists $s \geq 1$ such that the following holds. Let (A, B, C) be s, g -terrific for $24N$. Let (D_1, \dots, D_n) be r, g -great for $24N$, $|D_1| = p$, and $D_n \subseteq C$. Then some (D_1', \dots, D_n') is r, g -great for $24N$, where $|D_1'| = p$ and $D_n' \subseteq B$ is finite.

Proof: Let n, k, p, r, g be as given. Let $m \geq 1$ be given by Lemma 5.2.7, with $a = 24$, which depends only on n, k, p, r .

Let $s \gg n, k, p, r, m$ and the presentation of g . (Some specific iterated exponential in n, k, p, r, m , and the size of the presentation of g , will suffice). Let (A, B, C) be s, g -terrific for $24N$. Let (D_1, \dots, D_n) be r, g -great for $24N$, $|D_1| = p$, and $D_n \subseteq C$.

By Lemma 5.2.7, the following statement is true:

*) there exists (D_1, \dots, D_n) which is r, g -great for $24N$, where $|D_1| = p$ and $D_n = \{x_1, \dots, x_m\} \subseteq C$.

We claim that *) asserts the existence of $x_1, \dots, x_m \in C$ such that a quantifier free formula $\varphi(x_1, \dots, x_m)$ in L holds. This crucially depends on the fact that $g \in \text{BAF}$. The actual formula depends on n, k, p, r , and the function g . By the choice of s , write $\varphi = \varphi[i, s]$, where $1 \leq i \leq \gamma(s)$.

To see this, $\varphi(x_1, \dots, x_m)$ asserts that x_1, \dots, x_m can be arranged into sets $D_1 \subseteq \dots \subseteq D_n = \{x_1, \dots, x_m\}$, where (D_1, \dots, D_n) is r, g -great for $24N$. We have to put clauses i), ii) in Definition 5.2.12, with $a = 24$, in quantifier free form.

Each arrangement of x_1, \dots, x_m into sets $D_1 \subseteq \dots \subseteq D_n = \{x_1, \dots, x_m\}$ is given by a double sequence x_{ij} , $1 \leq i \leq n$, $1 \leq j \leq m$, where the x_{ij} are among the variables x_1, \dots, x_m . So we disjunct over the finitely many such double sequences of variables.

According to Definition 5.2.12, we assert

i. $(\{x_{11}, \dots, x_{1m}\}, \dots, \{x_{n1}, \dots, x_{nm}\})$ is r, g -good for $24N$.
 ii. Let $1 \leq i \leq \beta[2r]$, $x_1, \dots, x_{2r} \in \{x_{11}, \dots, x_{1m}\}$, $y_1, \dots, y_r \in \alpha(r, \{x_{21}, \dots, x_{2m}\})$, where (x_1, \dots, x_r) , (x_{r+1}, \dots, x_{2r}) have the same order type and \min , and $y_1, \dots, y_r \leq \min(x_1, \dots, x_r)$. Then

$$\begin{aligned} t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) \in \{x_{31}, \dots, x_{3m}\} \setminus \{0\} &\leftrightarrow \\ t[i, 2r](x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r) \in \{x_{31}, \dots, x_{3m}\} \setminus \{0\}. & \end{aligned}$$

It is clear that ii) is given by a quantifier free formula in L .

As for i), it asserts

i'. $\{x_{11}, \dots, x_{1m}\} \subseteq \dots \subseteq \{x_{n1}, \dots, x_{nm}\} \subseteq N \setminus \{0\}$.
 ii'. $x_{1i} < x_{1j} \rightarrow x_{1i} \uparrow < x_{1j}$.

- iii'. For all $1 \leq i \leq n-1$, $24\alpha(r, \{x_{i1}, \dots, x_{im}\} \setminus \{0\}; 1, r) \subseteq \{x_{i+1,1}, \dots, x_{i+1,m}\} \cup g\{x_{i+1,1}, \dots, x_{i+1,m}\}$.
- iv'. For all $1 \leq i \leq n-1$, $2\alpha(r, \{x_{i1}, \dots, x_{im}\} \setminus \{0\}; 1, r) + 1 \subseteq \{x_{i+1,1}, \dots, x_{i+1,m}\}$;
- v'. Same as iv' with 2 replaced by 3.
- vi'. $\{x_{n1}, \dots, x_{nm}\} \cap g\{x_{n1}, \dots, x_{nm}\} = \emptyset$.
- vii'. $\{x_{11}, \dots, x_{1m}\} \cap \alpha(r, \{x_{21}, \dots, x_{2m}\}; 2, r) = \emptyset$.

It is now clear that i) is also given by a quantifier free formula.

By Lemma 5.2.7, we have

$$(\exists v_1, \dots, v_m \in C) (\varphi[i, s](v_1, \dots, v_m)).$$

By clause iii) in the definition of s, g -terrific for $24N$,

$$(\exists v_1, \dots, v_m \in B) (\varphi[i, s](v_1, \dots, v_m)).$$

Hence

$$(\exists v_1, \dots, v_m \in B) (\exists D_1, \dots, D_n) ((D_1, \dots, D_n) \text{ is } r, g\text{-great for } 24N \wedge |D_1| = p \wedge D_n = \{v_1, \dots, v_m\}).$$

I.e., some (D_1', \dots, D_n') is r, g -great for $24N$, where $|D_1'| = p$ and $D_n' \subseteq B$ has at most m elements. QED

DEFINITION 5.2.14. Let $s(n, k, p, r, g)$ be an s given by Lemma 5.2.8.

LEMMA 5.2.9. Let $n \geq 3$, $k, p, r \geq 1$, and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$ be k -ary. There exists $t \geq 1$ such that the following holds. Let (A, B, C) be t, g -terrific for $24N$. Then some (D_1, \dots, D_n) is r, g -great for $24N$, where $|D_1| = p$ and $D_n \subseteq B$ is finite.

Proof: Let n, k, p, r, g be as given. Let $t = \max\{s(q, k, p, r, g) : 3 \leq q \leq n\}$. Let (A, B, C) be t, g -terrific for $24N$. We prove by induction on $3 \leq q \leq n$ that some (D_1, \dots, D_q) is r, g -great for $24N$, where $|D_1| = p$ and $D_n \subseteq B$ is finite.

For the basis case $q = 3$, apply Lemma 5.2.8 to (D_1, D_2, D_3) , where D_1 is any subset of A of cardinality p , and $D_2 = B$, $D_3 = C$. Note that $t \geq s(3, k, p, r, g)$.

Let $3 \leq q < n$ and (D_1, \dots, D_q) be r, g -great for $24N$, where $|D_1| = p$ and $D_q \subseteq B$ is finite.

We claim that (D_1, \dots, D_q, C) is r, g -great for $24N$.

We first verify that (D_1, \dots, D_q, C) is r, g -good for $24N$. In light of the fact that (D_1, \dots, D_q) is r, g -good for $24N$ and $q \geq 3$, it suffices to show that

$$\begin{aligned} 24\alpha(r, D_q^*; 2, r) &\subseteq C \cup gC \\ 2\alpha(r, D_q^*; 2, r) + 1 &\subseteq C \\ 3\alpha(r, D_q^*; 2, r) + 1 &\subseteq C \\ C \cap gC &= \emptyset. \end{aligned}$$

These are immediate from $D_q \subseteq B$ and that (A, B, C) is r, g -good for $24N$.

Clause ii) in the definition of (D_1, \dots, D_q, C) is immediate since $q \geq 3$ and (D_1, \dots, D_q) is r, g -great for $24N$.

Now apply Lemma 5.2.8 to (D_1, \dots, D_q, C) to obtain (D_1', \dots, D_{q+1}') that is r, g -great for $24N$, where $|D_1| = p$ and $D_{q+1}' \subseteq B$ is finite. Note that $t \geq s(q+1, k, p, r, g)$. QED

LEMMA 5.2.10. Let $n \geq 3$, $p, r \geq 1$, and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$, where $\text{rng}(g) \subseteq 24N$. There exists (D_1, \dots, D_n) which is r, g -great for $24N$, where $|D_1| = p$ and D_n is finite.

Proof: Let n, p, r, g be as given. Let g be k -ary. Let t be given by Lemma 5.2.9. By Lemma 5.2.6, let (A, B, C) be t, g -terrific for $24N$. By Lemma 5.2.9, let (D_1, \dots, D_n) be r, g -great for $24N$, where $|D_1| = p$ and D_n is finite. QED

LEMMA 5.2.11. Let $r \geq 3$ and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$, where $\text{rng}(g) \subseteq 24N$. There exists (D_1, \dots, D_r) such that

- i) $D_1 \subseteq \dots \subseteq D_r \subseteq N \setminus \{0\}$;
- ii) $|D_1| = r$ and D_r is finite;
- iii) for all $x < y$ from D_1 , $x \uparrow < y$;
- iv) for all $1 \leq i \leq r-1$, $24\alpha(r, D_i^*; 1, r) \subseteq D_{i+1} \cup gD_{i+1}$;
- v) for all $1 \leq i \leq r-1$, $2\alpha(r, D_i^*; 1, r) + 1, 3\alpha(r, D_i^*; 1, r) + 1 \subseteq D_{i+1}$;
- vi) $D_r \cap gD_r = \emptyset$;
- vii) $D_1 \cap \alpha(r, D_2^*; 2, r) = \emptyset$;
- viii) Let $1 \leq i \leq \beta(2r)$, $x_1, \dots, x_{2r} \in D_1$, $y_1, \dots, y_r \in \alpha(r, D_2)$, where (x_1, \dots, x_r) and (x_{r+1}, \dots, x_{2r}) have the same order type and \min , and $y_1, \dots, y_r \leq \min(x_1, \dots, x_r)$. Then $t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) \in D_3^* \Leftrightarrow t[i, 2r](x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r) \in D_3^*$.

Proof: Immediate from Lemma 5.2.10 and the definition of r, g -great for $24N$, setting n, p, r there to be r here. QED

We now eliminate the use of the D_i^* .

LEMMA 5.2.12. Let $r \geq 3$ and $g \in \text{ELG} \cap \text{SD} \cap \text{BAF}$, where $\text{rng}(g) \subseteq 48N$. There exists (D_1, \dots, D_r) such that

- i) $D_1 \subseteq \dots \subseteq D_r \subseteq N \setminus \{0\}$;
- ii) $|D_1| = r$ and D_r is finite;
- iii) for all $x < y$ from D_1 , $x \uparrow < y$;
- iv) for all $1 \leq i \leq r-1$, $48\alpha(r, D_i; 1, r) \subseteq D_{i+1} \cup gD_{i+1}$;
- v) for all $1 \leq i \leq r-1$, $2\alpha(r, D_i; 1, r)+1, 3\alpha(r, D_i; 1, r)+1 \subseteq D_{i+1}$;
- vi) $D_r \cap gD_r = \emptyset$;
- vii) $D_1 \cap \alpha(r, D_2; 2, r) = \emptyset$;
- viii) Let $1 \leq i \leq \beta(2r)$, $x_1, \dots, x_{2r} \in D_1$, $y_1, \dots, y_r \in \alpha(r, D_2)$, where (x_1, \dots, x_r) and (x_{r+1}, \dots, x_{2r}) have the same order type and \min , and $y_1, \dots, y_r \leq \min(x_1, \dots, x_r)$. Then $t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) \in D_3 \leftrightarrow t[i, 2r](x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r) \in D_3$.

Proof: Let r, g be as given. Let $g: N^k \rightarrow 48N$.

Define $g': N^{k+1} \rightarrow 24N$ by $g'(x_1, \dots, x_{k+1}) = g(x_1, \dots, x_k)$ if $x_{k+1} < x_1, \dots, x_k$; $48|x_1, \dots, x_{k+1}|+24$ otherwise.

Note that $\text{rng}(g') \subseteq 24N$, and $g' \in \text{ELG} \cap \text{SD} \cap \text{BAF}$. Let $D_1, \dots, D_n \subseteq N$ be given by Lemma 5.2.11 applied to $r+1, g'$. In particular, $|D_1| = r+1$.

We now verify that D_1^*, \dots, D_r^* is as required.

For claim i), since $D_1 \subseteq \dots \subseteq D_r$, we have $\min(D_1) \geq \dots \geq \min(D_r)$. We claim that $D_1^* \subseteq \dots \subseteq D_r^*$. To see this, let $n \in D_i^*$. Then $n \in D_{i+1}$, $n > \min(D_i) \geq \min(D_{i+1})$, $n \in D_{i+1}^*$.

For claim ii), since $|D_1| = r+1$, we have $|D_1^*| = r$. since D_r is finite, D_r^* is finite.

Claim iii) is immediate from iii) of Lemma 5.2.11.

For claim iv), let $1 \leq i \leq r-1$, $x \in 48\alpha(r, D_i^*; 1, r)$. Then $x > \min(D_i) \geq \min(D_{i+1})$. By Lemma 5.2.11 iv), $x \in D_{i+1} \cup g'D_{i+1}$. If $x \in D_{i+1}$ then $x \in D_{i+1}^*$. If $x \in g'D_{i+1}$ then $x \in g(D_{i+1}^*)$, because x must arise from the first clause in the definition of g' .

For claim v), let $1 \leq i \leq r-1$, $x \in 2\alpha(r, D_i^*; 1, r)+1 \cup 3\alpha(r, D_i^*; 1, r)+1$. Then $x > \min(D_i) \geq \min(D_{i+1})$. By Lemma 5.2.11 v), $x \in D_{i+1}$. Hence $x \in D_{i+1}^*$.

For vi), we have $D_r \cap g'D_r = \emptyset$. Since $g(D_r^*) \subseteq g'(D_r)$, we have $D_r^* \cap g(D_r^*) = \emptyset$.

Claim vii) is the same as vii) of Lemma 5.2.11.

For claim viii), let $1 \leq i \leq \beta(2r)$. Let $1 \leq i' \leq \beta(2r+2)$ be such that $t[i', 2r+2]$ is the result of replacing the variables v_{r+1}, \dots, v_{2r} in $t[i, 2r]$ with the variables v_{r+2}, \dots, v_{2r+1} .

Let $x_1, \dots, x_{2r} \in D_1^*$, $y_1, \dots, y_r \in \alpha(r, D_2^*)$, where (x_1, \dots, x_r) and (x_{r+1}, \dots, x_{2r}) have the same order type and \min , and $y_1, \dots, y_r \leq \min(x_1, \dots, x_r)$. Clearly

$$t[i, 2r](x_1, \dots, x_r, y_1, \dots, y_r) = t[i', 2r+2](x_1, \dots, x_r, x_r, y_1, \dots, y_r, y_r).$$

$$t[i, 2r](x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r) = t[i', 2r+2](x_{r+1}, \dots, x_{2r}, x_{2r}, y_1, \dots, y_r, y_r).$$

By Lemma 5.2.11 viii),

$$t[i', 2r+2](x_1, \dots, x_r, x_r, y_1, \dots, y_r, y_r) \in D_3^* \Leftrightarrow t[i', 2r+2](x_{r+1}, \dots, x_{2r}, x_{2r}, y_1, \dots, y_r, y_r) \in D_3^*.$$

$$t[i, r](x_1, \dots, x_r, y_1, \dots, y_r) \in D_3^* \Leftrightarrow t[i, r](x_{r+1}, \dots, x_{2r}, y_1, \dots, y_r) \in D_3^*.$$

QED