

5.4. Limited formulas, limited indiscernibles, x-definability, normal form.

We fix $M = (A, <, 0, 1, +, -, \cdot, \uparrow, \log, E, c_1, c_2, \dots)$ as given by Lemma 5.3.18.

DEFINITION 5.4.1. Let $L(E)$ be the first order predicate calculus with equality, using $<, 0, 1, +, -, \cdot, \uparrow, \log, E$, where E is 1-ary. The c 's are not allowed in $L(E)$. We will always write $t \in E$ instead of $E(t)$.

We follow the convention that $\varphi(v_1, \dots, v_k)$ represents a formula of $L(E)$ whose free variables are among v_1, \dots, v_k . This does not require that v_k be free or even appear in φ . Recall that all variables are of the form v_n , where $n \geq 1$.

In this section, we will only be concerned with what we call the E formulas of $L(E)$.

DEFINITION 5.4.2. The E formulas of $L(E)$ are inductively defined as follows.

- i) Every atomic formula of $L(E)$ is an E formula;
- ii) If φ, ψ are E formulas then $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$ are E formulas;
- iii) If φ is an E formula and $n \geq 1$, then $(\exists v_n \in E)(\varphi)$, $(\forall v_n \in E)(\varphi)$ are E formulas.

DEFINITION 5.4.3. We take

$$(\exists v_n \in E)(\varphi), (\forall v_n \in E)(\varphi)$$

to be abbreviations of

$$(\exists v_n)(v_n \in E \wedge \varphi), (\forall v_n)(v_n \in E \rightarrow \varphi).$$

Although general formulas of $L(E)$ will arise in this section, content will be focused on their relativizations, which are E formulas of $L(E)$.

DEFINITION 5.4.4. Let $\varphi(v_1, \dots, v_k)$ be a formula of $L(E)$ and v be a variable not among v_1, \dots, v_k . We let $\varphi(v_1, \dots, v_k)^v$ be the result of bounding all quantifiers in $\varphi(v_1, \dots, v_k)$ to

$$E \cap [0, v].$$

I.e., we replace each quantifier

$(\forall u)$ by $(\forall u \in E \cap [0, v])$
 $(\exists u)$ by $(\exists u \in E \cap [0, v])$.

These bounded quantifiers are (can be) expanded in the well known way to create an actual formula in $L(E)$.

We now define a very important 6-ary relation.

DEFINITION 5.4.5. We define $A(r, n, m, \varphi, a, b)$ if and only if

- i) $r, n, m, a, b \in \mathbb{N} \setminus \{0\}$, $n < m$;
- ii) $\varphi = \varphi(v_1, \dots, v_r)$ is a formula of $L(E)$; i.e., all free variables of φ are among v_1, \dots, v_r ;
- iii) Let $x_1, \dots, x_r \in E \cap [0, c_n]$. Then $\varphi(x_1, \dots, x_r)^{c-n} \leftrightarrow \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b \in E$.

LEMMA 5.4.1. Let $r, n \geq 1$ and $\varphi(v_1, \dots, v_r)$ be a quantifier free formula of L . There exist a, b such that $A(r, n, n+1, \varphi, a, b)$.

Proof: Let r, n, φ be as given. Note that $\varphi^{c-n} = \varphi$.

By Lemma 5.3.18 vii), let $a, b \in \mathbb{N} \setminus \{0\}$ be such that the following holds. Let $n \geq 1$ and $x_1, \dots, x_r \in E \cap [0, c_n]$.

$(\exists y \in E) (y \leq |c_n, x_1, \dots, x_r| \uparrow \uparrow \wedge y \leq |x_1, \dots, x_r| \wedge \varphi(c_n, x_1, \dots, x_r, y))$

\Leftrightarrow
 $(\exists y \in E) (y \leq |c_n, x_1, \dots, x_r| \uparrow \uparrow \wedge \rho(c_n, x_1, \dots, x_r, y))$
 \Leftrightarrow

$\text{aCODE}(c_{n+1}; c_n, x_1, \dots, x_r) + b \in E$.

$\varphi(x_1, \dots, x_r) \Leftrightarrow$
 $\text{aCODE}(c_{n+1}; c_n, x_1, \dots, x_r) + b \in E$.

Hence $A(r, n, n+1, \varphi, a, b)$. QED

Note that in the proof of Lemma 5.4.1, we have written the second equivalent in a form that justifies the application of Lemma 5.3.18 vii). The formula ρ used can be read off easily from the first equivalent. We will be using this style of exposition throughout this section.

LEMMA 5.4.2. Let $r \geq 1$ and φ be $t \in E$, where $t(v_1, \dots, v_r)$ is a term of L . There exist a, b such that the following holds. Let $n \geq 1$. Then $A(r, n, n+1, \varphi, a, b)$.

Proof: Let r, φ, t be as given.

Let $p \geq 2$ be such that for all $x_1, \dots, x_r \in A$, $t(x_1, \dots, x_r) \leq \uparrow p(|x_1, \dots, x_r|)$. By Lemma 5.3.18 vii), let $a, b \in \mathbb{N} \setminus \{0\}$ be such that the following holds. Let $n \geq 1$ and $x_1, \dots, x_r \in E \cap [0, c_n]$. Then

$$\begin{aligned} & (\exists y \in E) (y \leq \uparrow p(|c_n, x_1, \dots, x_r|) \wedge y = t(x_1, \dots, x_r)) \\ & \quad \Leftrightarrow \\ & (\exists y \in E) (y \leq \uparrow p(|c_n, x_1, \dots, x_r|) \wedge \rho(c_n, x_1, \dots, x_r, y)) \\ & \quad \Leftrightarrow \\ & \quad a\text{CODE}(c_{n+1}; c_n, x_1, \dots, x_r) + b \in E. \\ & \quad t(x_1, \dots, x_r) \in E \Leftrightarrow \\ & \quad a\text{CODE}(c_{n+1}; c_n, x_1, \dots, x_r) + b \in E. \end{aligned}$$

Hence $A(r, n, n+1, \varphi, a, b)$. QED

LEMMA 5.4.3. Let $A(r, n, m, \varphi, a, b)$. There exist d, e such that $A(r, n, m, \neg\varphi, d, e)$.

Proof: Let $A(r, n, m, \varphi, a, b)$. By Lemma 5.3.18 vi), fix $i, j \in \mathbb{N} \setminus \{0\}$ such that the following holds. Let $x_1 \in \alpha(E; 1, <\infty)$. Then

$$\begin{aligned} & (\exists x_2 \in E) (x_2 \leq x_1 \wedge x_2 = x_1) \Leftrightarrow \\ & \quad ix_1 + j \notin E. \end{aligned}$$

Clearly for all $x_1 \in \alpha(E; 1, <\infty)$,

$$1) \quad x_1 \in E \Leftrightarrow ix_1 + j \notin E.$$

Now let $x_1, \dots, x_r \in E \cap [0, c_n]$. By $A(r, n, m, \varphi, a, b)$,

$$\begin{aligned} & 2) \quad \varphi(x_1, \dots, x_r)^{c-n} \Leftrightarrow \\ & \quad a\text{CODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b \in E. \end{aligned}$$

By Lemma 5.3.18 viii),

$$\text{CODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) \in E.$$

By 1),

$$\begin{aligned} & 3) \quad a(\text{CODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r)) + b \in E \Leftrightarrow \\ & \quad ia(\text{CODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r)) + ib + j \notin E. \end{aligned}$$

By 2), 3),

$$\neg\varphi(x_1, \dots, x_r)^{c-n} \leftrightarrow ia(\text{CODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r)) + ib + j \in E.$$

Therefore $A(r, n, m, \neg\varphi, ia, ib + j)$. QED

LEMMA 5.4.4. Let $a, b, d, e \in \mathbb{N} \setminus \{0\}$. There exist $f, g \in \mathbb{N} \setminus \{0\}$ such that the following holds. Let $w \in \alpha(E; 1, <\infty)$. Then $(aw + b \in E \wedge dw + e \in E) \leftrightarrow fw + g \in E$.

Proof: Let $a, b, d, e \in \mathbb{N} \setminus \{0\}$. Let $p = \max(a, b, d, e)$.

By Lemma 5.3.18 vi), let $f, g \in \mathbb{N} \setminus \{0\}$ such that the following holds. Let $w \in \alpha(E; 1, <\infty)$. Then

$$(\exists y, z \in E) (y, z \leq pw + p \wedge y = aw + b \wedge z = cw + d) \leftrightarrow fw + g \in E.$$

$$(\exists y, z \in E) (y = aw + b \wedge z = cw + d) \leftrightarrow fw + g \in E.$$

$$(aw + b \in E \wedge cw + d \in E) \leftrightarrow fw + g \in E.$$

QED

LEMMA 5.4.5. Let $A(r, n, m, \varphi, a, b)$ and $A(r, n, m, \psi, d, e)$. There exist f, g such that $A(r, n, m, \varphi \wedge \psi, f, g)$.

Proof: Assume $A(r, n, m, \varphi, a, b)$, $A(r, n, m, \psi, d, e)$. Let $x_1, \dots, x_r \in E \cap [0, c_n]$, Then

$$1) \varphi(x_1, \dots, x_r)^{c-n} \leftrightarrow a\text{CODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b \in E.$$

$$\psi(x_1, \dots, x_r)^{c-n} \leftrightarrow d\text{CODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_e) + e \in E.$$

Let f, g be given by Lemma 5.4.4 using a, b, d, e . By Lemma 5.3.18 viii),

$$\text{CODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) \in E.$$

Hence by Lemma 5.4.4,

$$2) a\text{CODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b \in E \wedge$$

$$\begin{aligned} & \text{dCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + e \in E \\ & \quad \Leftrightarrow \\ & \text{fCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + g \in E. \end{aligned}$$

By 1), 2),

$$\begin{aligned} & ((\varphi \wedge \psi)(x_1, \dots, x_r))^{c-n} \\ & \quad \Leftrightarrow \\ & \varphi(x_1, \dots, x_r)^{c-n} \wedge \psi(x_1, \dots, x_r)^{c-n} \\ & \quad \Leftrightarrow \\ & \text{fCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + g \in E. \end{aligned}$$

QED

LEMMA 5.4.6. Let $1 \leq i \leq r$ and $A(r, n, m, \varphi, a, b)$. There exists d, e such that $A(r, n, m+1, (\exists v_i)(\varphi), d, e)$.

Proof: Let $1 \leq i \leq r$ and $A(r, n, m, \varphi, a, b)$. Let $x_1, \dots, x_r \in E \cap [0, c_n]$. Then

$$\begin{aligned} & 1) \varphi(x_1, \dots, x_r)^{c-n} \Leftrightarrow \\ & \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b \in E. \end{aligned}$$

By Lemma 5.3.18 vii), let $d, e \in \mathbb{N} \setminus \{0\}$ be such that the following holds, using m for n . Let $x_1, \dots, x_r \in E \cap [0, c_n]$. Then

$$\begin{aligned} & (\exists x_i, w \in E) (x_i, w \leq c_m \uparrow \uparrow \wedge x_i \leq c_n \wedge \\ & w = \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b) \\ & \quad \Leftrightarrow \\ & (\exists z, w \in E) (z, w \leq c_m \uparrow \uparrow \wedge z \leq c_n \wedge \\ & w = \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_r) + b) \\ & \quad \Leftrightarrow \\ & (\exists z, w \in E) (z, w \leq c_m \uparrow \uparrow \wedge \rho(c_n, \dots, c_m, x_1, \dots, x_r, z, w)) \\ & \quad \Leftrightarrow \\ & \text{dCODE}(c_{m+1}; c_n, \dots, c_m, x_1, \dots, x_r) + e \in E. \end{aligned}$$

$$\begin{aligned} & (\exists x_i, w \in E) (x_i \leq c_n \wedge \\ & w = \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b) \\ & \quad \Leftrightarrow \\ & \text{dCODE}(c_{m+1}; c_n, \dots, c_m, x_1, \dots, x_r) + e \in E. \end{aligned}$$

$$\begin{aligned} & 2) (\exists x_i \in E) (x_i \leq c_n \wedge \\ & \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b \in E) \\ & \quad \Leftrightarrow \\ & \text{dCODE}(c_{m+1}; c_n, \dots, c_m, x_1, \dots, x_r) + e \in E. \end{aligned}$$

By 1), 2),

$$(\exists x_i \in E) (x_i \leq c_n \wedge \varphi(x_1, \dots, x_r)^{c-n}) \leftrightarrow \\ dCODE(c_{m+1}; c_n, \dots, c_m, x_1, \dots, x_r) + e \in E.$$

$$(\exists x_i) (\varphi(x_1, \dots, x_r))^{c-n} \\ \leftrightarrow \\ dCODE(c_{m+1}; c_n, \dots, c_m, x_1, \dots, x_r) + e \in E.$$

Hence $A(r, n, m+1, (\exists v_i) (\varphi), d, e)$. QED

LEMMA 5.4.7. Let $m \leq m'$ and $A(r, n, m, \varphi, a, b)$. There exist d, e such that $A(r, n, m', \varphi, d, e)$.

Proof: Let $m < m'$ and $A(r, n, m, \varphi, a, b)$. Let $x_1, \dots, x_r \in E \cap [0, c_n]$. Then

$$1) \varphi(x_1, \dots, x_r)^{c-n} \leftrightarrow \\ aCODE(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b \in E.$$

By Lemma 5.3.18 vii), let $d, e \in \mathbb{N} \setminus \{0\}$ be such that the following holds. Let $x_1, \dots, x_r \in E \cap [0, c_n]$. Then

$$(\exists y \in E) (y \leq c_{m'} \uparrow \uparrow \wedge y = aCODE(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b) \\ \leftrightarrow \\ (\exists y \in E) (y \leq c_{m'} \uparrow \uparrow \wedge \rho(c_n, \dots, c_{m'-1}, x_1, \dots, x_r, y)) \\ \leftrightarrow \\ dCODE(c_{m'}; c_n, \dots, c_{m'-1}, x_1, \dots, x_r) + e \in E.$$

$$(\exists y \in E) (y = aCODE(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b) \leftrightarrow \\ dCODE(c_{m'}; c_n, \dots, c_{m'-1}, x_1, \dots, x_r) + e \in E.$$

$$2) aCODE(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b \in E \leftrightarrow \\ dCODE(c_{m'}; c_n, \dots, c_{m'-1}, x_1, \dots, x_r) + e \in E.$$

By 1), 2),

$$\varphi(x_1, \dots, x_r)^{c-n} \leftrightarrow \\ dCODE(c_{m'}; c_n, \dots, c_{m'-1}, x_1, \dots, x_r) + e \in E.$$

Therefore $A(r, n, m', \varphi, d, e)$. QED

LEMMA 5.4.8. Let $r \leq r'$ and $A(r', n, m, \varphi, a, b)$, where all free variables of φ are among v_1, \dots, v_r . There exist d, e such that $A(r, n, m+1, \varphi, d, e)$.

Proof: Let $r, r', n, m, \varphi, a, b$ be as given. By $A(r', n, m, \varphi, a, b)$, for all $x_1, \dots, x_{r'} \in E \cap [0, c_n]$,

$$1) \quad \varphi(x_1, \dots, x_{r'})^{c-n} \leftrightarrow \\ \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_{r'}) + b \in E.$$

Note that $\varphi(x_1, \dots, x_{r'})^{c-n} = \varphi(x_1, \dots, x_r)^{c-n}$. Hence for all $x_1, \dots, x_r \in E \cap [0, c_n]$,

$$2) \quad \varphi(x_1, \dots, x_r)^{c-n} \leftrightarrow \\ \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r, x_r, \dots, x_r) + b \in E.$$

By Lemma 5.3.18 vii), let $d, e \in \mathbb{N} \setminus \{0\}$ be such that the following holds. Let $x_1, \dots, x_r \in E \cap [0, c_n]$. Then

$$(\exists z \in E) (z \leq c_m \uparrow \uparrow \wedge z = \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r, x_r, \dots, x_r)) \\ \leftrightarrow \\ (\exists z \in E) (z \leq c_n \uparrow \uparrow \wedge \rho(c_n, \dots, c_m, x_1, \dots, x_r, z)) \\ \leftrightarrow \\ \text{dCODE}(c_{m+1}; c_n, \dots, c_m, x_1, \dots, x_r) + e \in E.$$

$$3) \quad \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r, x_r, \dots, x_r) + b \in E \leftrightarrow \\ \text{dCODE}(c_{m+1}; c_n, \dots, c_m, x_1, \dots, x_r) + e \in E.$$

By 2), 3), for all $x_1, \dots, x_r \in E \cap [0, c_n]$,

$$\varphi(x_1, \dots, x_r)^{c-n} \leftrightarrow \\ \text{dCODE}(c_{m+1}; c_n, \dots, c_m, x_1, \dots, x_r) + e \in E.$$

Hence $A(r, n, m+1, \varphi, d, e)$. QED

LEMMA 5.4.9. Let $r, n \geq 1$ and $\varphi(x_1, \dots, x_r)$ be a formula of $L(E)$. There exists m, a, b such that $A(r, n, m, \varphi, a, b)$.

Proof: By induction on the complexity of φ . Without loss of generality, we can assume that φ uses only the connectives \neg, \wedge , and only the quantifier \exists . For our purposes, we define $c(\varphi)$ as the total number of occurrences of connectives and quantifiers in φ .

We prove the following by induction on $p \geq 0$. Let $r, n \geq 1$ and $\varphi(v_1, \dots, v_r)$ be a formula of $L(E)$ with $c(\varphi) \leq p$. There exist m, a, b such that $A(r, n, m, \varphi, a, b)$.

We first handle the basis case $p = 0$. Let r, n, φ be as given. Then φ has no connectives and no quantifiers, and so

φ is an atomic formula of $L(E)$. Now use Lemmas 5.4.1 and 5.4.2 with $m = n+1$.

Now assume that the statement holds of $p \geq 0$. Let $r, n \geq 1$ and $\varphi(v_1, \dots, v_r)$ be a formula of $L(E)$ with $c(\varphi) = p+1$.

case 1. $\varphi(v_1, \dots, v_r) = \neg\psi(v_1, \dots, v_r)$. By the induction hypothesis, let $A(r, n, m, \psi, a, b)$. By Lemma 5.4.3, there exist d, e such that $A(r, n, m, \varphi, d, e)$.

case 2. $\varphi(v_1, \dots, v_r) = \psi(v_1, \dots, v_r) \wedge \rho(v_1, \dots, v_r)$. By the induction hypothesis, let $A(r, n, m, \psi, a, b), A(r, n, m', \rho, d, e)$. By Lemma 5.4.7, let $A(r, n, \max(m, m'), \psi, a', b')$, $A(r, n, \max(m, m'), \rho, d', e')$. By Lemma 5.4.5, there exists f, g such that $A(r, n, \max(m, m'), \varphi, f, g)$.

case 3. $\varphi(v_1, \dots, v_r) = (\exists v_i)(\psi)$, $1 \leq i \leq r$. Then we can write $\psi = \psi(v_1, \dots, v_r)$ because ψ has all free variables of φ are among v_1, \dots, v_r . By the induction hypothesis, let $A(r, n, m, \psi, a, b)$. By Lemma 5.4.6, there exist d, e such that $A(r, n, m+1, \varphi, d, e)$.

case 4. $\varphi(v_1, \dots, v_r) = (\exists v_i)(\psi)$, $i > r$. Then ψ has all free variables among v_1, \dots, v_i , and we can write $\psi = \psi(v_1, \dots, v_i)$. By the induction hypothesis, let $A(i, n, m, \psi, a, b)$. By Lemma 5.4.6, let $A(i, n, m+1, \varphi, d, e)$. By Lemma 5.4.8, there exists f, g such that $A(r, n, m+2, \varphi, f, g)$.

QED

We now extend the indiscernibility in Lemma 5.3.18 iv) to formulas.

LEMMA 5.4.10. Let $r \geq 1$ and $\varphi(v_1, \dots, v_{2r})$ be a formula in $L(E)$. Let $1 \leq i_1, \dots, i_{2r} < n$, where (i_1, \dots, i_r) and (i_{r+1}, \dots, i_{2r}) have the same order type and the same min. Let $Y_1, \dots, Y_r \in E$, $y_1, \dots, y_r \leq \min(c_{i_1}, \dots, c_{i_r})$. Then $\varphi(c_{i_1}, \dots, c_{i_r}, Y_1, \dots, Y_r)^{c-n} \leftrightarrow \varphi(c_{i_{r+1}}, \dots, c_{i_{2r}}, Y_1, \dots, Y_r)^{c-n}$.

Proof: Let $r, \varphi, i_1, \dots, i_{2r}$ be as given. Let $n > i_1, \dots, i_{2r}$. By Lemma 5.4.9, let m, a, b be such that the following holds. For all $x_1, \dots, x_{2r} \in E \cap [0, c_n]$,

$$\varphi(x_1, \dots, x_{2r})^{c-n} \leftrightarrow \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_{2r}) + b \in E.$$

Let y_1, \dots, y_r be as given. Then

$$1) \quad \varphi(c_{i_1}, \dots, c_{i_r}, y_1, \dots, y_r)^{c_n} \leftrightarrow \\ \text{aCODE}(c_m; c_n, \dots, c_{m-1}, c_{i_1}, \dots, c_{i_r}, y_1, \dots, y_r) + b \in E.$$

$$\varphi(c_{i_{r+1}}, \dots, c_{i_{2r}}, y_1, \dots, y_r)^{c_n} \leftrightarrow \\ \text{aCODE}(c_m; c_n, \dots, c_{m-1}, c_{i_{r+1}}, \dots, c_{i_{2r}}, y_1, \dots, y_r) + b \in E.$$

By Lemma 5.3.18 ix),

$$2) \quad \text{aCODE}(c_m; c_n, \dots, c_{m-1}, c_{i_1}, \dots, c_{i_r}, y_1, \dots, y_r) + b \in E \leftrightarrow \\ \text{aCODE}(c_m; c_n, \dots, c_{m-1}, c_{i_{r+1}}, \dots, c_{i_{2r}}, y_1, \dots, y_r) + b \in E.$$

By 1), 2),

$$\varphi(c_{i_1}, \dots, c_{i_r}, y_1, \dots, y_r)^{c_n} \leftrightarrow \\ \varphi(c_{i_{r+1}}, \dots, c_{i_{2r}}, y_1, \dots, y_r)^{c_n}.$$

QED

LEMMA 5.4.11. Let $k, n \geq 1$ and $\varphi(v_1, \dots, v_k)$ be a formula of $L(E)$. There exist $m, a, b \in \mathbb{N} \setminus \{0\}$, $n < m$, and $y_n, \dots, y_m \in E \cap [0, c_{n+1}]$ such that for all $x_1, \dots, x_k \in E \cap [0, c_n]$,

$$\varphi(x_1, \dots, x_k)^{c_n} \leftrightarrow \\ \text{aCODE}(y_m; y_n, \dots, y_{m-1}, x_1, \dots, x_k) + b \in E.$$

Proof: Let k, n, φ be as given. By Lemma 5.4.9, there exist $m, a, b \in \mathbb{N} \setminus \{0\}$ such that

$$1) \quad (\forall x_1, \dots, x_k \in E \cap [0, c_n]) (\varphi(x_1, \dots, x_r)^{c_n} \\ \leftrightarrow \\ \text{aCODE}(c_m; c_n, \dots, c_{m-1}, x_1, \dots, x_r) + b \in E).$$

$$2) \quad (\exists y_n, \dots, y_m \in E) (\forall x_1, \dots, x_k \in E \cap [0, c_n]) (\varphi(x_1, \dots, x_r)^{c_n} \\ \leftrightarrow \\ \text{aCODE}(y_m; y_n, \dots, y_{m-1}, x_1, \dots, x_r) + b \in E).$$

$$3) \quad (\exists y_n, \dots, y_m \in E) (\forall x_1, \dots, x_k \in E \cap [0, c_n]) (\varphi(x_1, \dots, x_r)^{c_n} \\ \leftrightarrow \\ (\exists z \in E) (z = \text{aCODE}(y_m; y_n, \dots, y_{m-1}, x_1, \dots, x_r) + b)).$$

By Lemma 5.3.18 iii), choose t so large that

$$4) \quad (\exists y_n, \dots, y_m \in E \cap [0, c_t]) (\forall x_1, \dots, x_k \in E \cap \\ [0, c_n]) (\varphi(x_1, \dots, x_r)^{c_n}$$

$$\Leftrightarrow$$

$$(\exists z \in E \cap [0, c_{t+1}]) (z = \text{aCODE}(y_m; y_n, \dots, y_{m-1}, x_1, \dots, x_r) + b)).$$

By Lemma 5.4.10,

$$5) (\exists y_n, \dots, y_m \in E \cap [0, c_{n+1}]) (\forall x_1, \dots, x_k \in E \cap [0, c_n]) (\varphi(x_1, \dots, x_r)^{c-n} \Leftrightarrow (\exists z \in E \cap [0, c_{n+2}]) (z = \text{aCODE}(y_m; y_n, \dots, y_{m-1}, x_1, \dots, x_r) + b)).$$

$$6) (\exists y_n, \dots, y_m \in E \cap [0, c_{n+1}]) (\forall x_1, \dots, x_k \in E \cap [0, c_n]) (\varphi(x_1, \dots, x_r)^{c-n} \Leftrightarrow (\exists z \in E) (z = \text{aCODE}(y_m; y_n, \dots, y_{m-1}, x_1, \dots, x_r) + b)).$$

$$7) (\exists y_n, \dots, y_m \in E \cap [0, c_{n+1}]) (\forall x_1, \dots, x_k \in E \cap [0, c_n]) (\varphi(x_1, \dots, x_r)^{c-n} \Leftrightarrow \text{aCODE}(y_m; y_n, \dots, y_{m-1}, x_1, \dots, x_r) + b \in E).$$

QED

Recall that $\alpha(E)$ is the set of all values of terms in L at arguments from E (Definition 5.3.3).

LEMMA 5.4.12. $\alpha(E) = E - E$. Let $n \geq 1$. $\alpha(E \cap [0, c_n]) \subseteq (E \cap [0, c_{n+1}]) - (E \cap [0, c_{n+1}])$.

Proof: Since $E - E \subseteq \alpha(E)$, it suffices to prove $\alpha(E) \subseteq E - E$. Let $t(v_1, \dots, v_k)$ be a term in L , and let $x_1, \dots, x_k \in E$. Let n be such that $x_1, \dots, x_k < c_n$.

Note that by Lemma 5.3.18 iv), v),

$$\begin{aligned} t(x_1, \dots, x_k) &< c_{n+1}. \\ 2c_{n+1} + t(x_1, \dots, x_k), 3c_{n+1} + t(x_1, \dots, x_k) &\in \alpha(E; 1, < \infty). \\ 3(2c_{n+1} + t(x_1, \dots, x_k)) + 1, 2(3c_{n+1} + t(x_1, \dots, x_k)) + 1 &\in E. \\ 6c_{n+1} + 3t(x_1, \dots, x_k) + 1, 6c_{n+1} + 2t(x_1, \dots, x_k) + 1 &\in E. \\ (6c_{n+1} + 3t(x_1, \dots, x_k) + 1) - (6c_{n+1} + 2t(x_1, \dots, x_k) + 1) &= \\ t(x_1, \dots, x_k) &\in E - E. \end{aligned}$$

Thus we have written $t(x_1, \dots, x_k)$ as the difference between two elements of E . This establishes the first claim.

For the second claim, let $n \geq 1$. Let $t(v_1, \dots, v_k)$ be a term in L . By the proof of the first claim,

$$1) (\forall x_1, \dots, x_k \in E \cap [0, c_n]) (\exists y, z \in E \cap [0, c_{n+2}]) (t(x_1, \dots, x_k) = y - z).$$

By Lemma 5.4.10,

$$2) (\forall x_1, \dots, x_k \in E \cap [0, c_n]) (\exists y, z \in E \cap [0, c_{n+1}]) \\ (t(x_1, \dots, x_k) = y - z).$$

QED

LEMMA 5.4.13. Let $k, r \geq 1$ and $x_1, \dots, x_k, y_1, \dots, y_r \in A$. Then $P(y_1, \dots, y_r, x_1, \dots, x_k) = P(P(y_1, \dots, y_r), x_1, \dots, x_k)$.

Proof: Recall the definition of P in section 5.3, right after the proof of Lemma 5.3.10. We prove the following by induction on $r \geq 1$:

$$\text{for all } k \geq 1 \text{ and } x_1, \dots, x_k, y_1, \dots, y_r \in A, \\ P(y_1, \dots, y_r, x_1, \dots, x_k) = P(P(y_1, \dots, y_r), x_1, \dots, x_k).$$

For the basis case $r = 1$, this asserts that for all $k \geq 1$

$$P(y_1, x_1, \dots, x_k) = P(P(y_1), x_1, \dots, x_k)$$

which follows from $P(y_1) = y_1$.

Fix $r \geq 1$. Suppose that for all $k \geq 1$ and $x_1, \dots, x_k, y_1, \dots, y_r$,

$$1) P(y_1, \dots, y_r, x_1, \dots, x_k) = \\ P(P(y_1, \dots, y_r), x_1, \dots, x_k).$$

We want to verify that for all $k \geq 1$ and $x_1, \dots, x_k, y_1, \dots, y_{r+1}$,

$$P(y_1, \dots, y_{r+1}, x_1, \dots, x_k) = \\ P(P(y_1, \dots, y_{r+1}), x_1, \dots, x_k).$$

Let $k \geq 1$ and $x_1, \dots, x_k, y_1, \dots, y_{r+1} \in A$. By the induction hypothesis 1) using $k = k+1$,

$$2) P(y_1, \dots, y_{r+1}, x_1, \dots, x_k) = \\ P(P(y_1, \dots, y_r), y_{r+1}, x_1, \dots, x_k).$$

By the definition of P ,

$$3) P(P(y_1, \dots, y_r), y_{r+1}, x_1, \dots, x_k) = \\ P(P(P(y_1, \dots, y_r), y_{r+1}), x_1, \dots, x_k).$$

By the induction hypothesis 1) using $k = 1$,

$$4) P(P(y_1, \dots, y_r), y_{r+1}) = P(y_1, \dots, y_{r+1}).$$

By 2), 3), 4),

$$P(y_1, \dots, y_{r+1}, x_1, \dots, x_k) = P(P(y_1, \dots, y_{r+1}), x_1, \dots, x_k)$$

as required. QED

LEMMA 5.4.14. Let $k, n, r \geq 1$, and $\varphi(v_1, \dots, v_{r+k})$ be a formula of $L(E)$. Let $y_1, \dots, y_r \in E \cap [0, c_n]$. There exist $d, e, f, g, h, i, j, p \in E \cap [0, c_{n+1}]$ such that for all $x_1, \dots, x_k \in E \cap [0, c_n]$,

$$\varphi(y_1, \dots, y_r, x_1, \dots, x_k)^{c-n} \leftrightarrow (d-e) \text{CODE}(f-g; h-i, x_1, \dots, x_k) + (j-p) \in E.$$

Proof: Let k, n, r, φ be as given. By Lemma 5.4.11, let $m, a, b \in N \setminus \{0\}$, $n < m$, and $z_n, \dots, z_m \in E \cap [0, c_{n+1}]$, be such that for all $y_1, \dots, y_r, x_1, \dots, x_k \in E \cap [0, c_n]$,

$$\varphi(y_1, \dots, y_r, x_1, \dots, x_k)^{c-n} \leftrightarrow a \text{CODE}(z_m; z_n, \dots, z_{m-1}, y_1, \dots, y_r, x_1, \dots, x_k) + b \in E.$$

By the definition of CODE introduced right after the proof of Lemma 5.3.10,

$$1) \varphi(y_1, \dots, y_r, x_1, \dots, x_k)^{c-n} \leftrightarrow a(8((\log(z_m)) \uparrow + P(z_n, \dots, z_{m-1}, y_1, \dots, y_r, x_1, \dots, x_k)) + 1) + b \in E.$$

Now fix y_1, \dots, y_r as given. These are in addition to the already fixed $z_n, \dots, z_m \in E$.

By Lemma 5.4.13 and 1), for all $x_1, \dots, x_k \in E \cap [0, c_n]$,

$$\varphi(y_1, \dots, y_r, x_1, \dots, x_k)^{c-n} \leftrightarrow a(8((\log(z_m)) \uparrow + P(P(z_n, \dots, z_{m-1}, y_1, \dots, y_r), x_1, \dots, x_k)) + 1) + b \in E.$$

$$2) \varphi(y_1, \dots, y_r, x_1, \dots, x_k)^{c-n} \leftrightarrow a \text{CODE}(z_m; P(z_n, \dots, z_{m-1}, y_1, \dots, y_r), x_1, \dots, x_k) + b \in E.$$

By Lemma 5.4.12, let $a = d-e$, $z_m = f-g$, $P(z_n, \dots, z_{m-1}, y_1, \dots, y_r) = h-i$, $b = j-p$, where $d, e, f, g, h, i, j, p \in E \cap$

$[0, c_{n+2}]$. Make these substitutions for $a, z_m, P(z_n, \dots, z_{m-1}, y_1, \dots, y_r), b$, respectively, in 2).

The Lemma requires that $d, e, f, g, h, i, j, p \in E \cap [0, c_{n+1}]$, and we only have $d, e, f, g, h, i, j, p \in E \cap [0, c_{n+2}]$. However, we can apply Lemma 5.4.10 in the obvious way to reduce to $E \cap [0, c_{n+1}]$. QED

LEMMA 5.4.15. For all $k \geq 1$ there exists a term $t(v_1, \dots, v_{k+8})$ of $L(E)$ such that the following holds. Let $n, r \geq 1$, $\varphi(v_1, \dots, v_{r+k})$ be a formula of $L(E)$, and $y_1, \dots, y_r \in E \cap [0, c_n]$. There exists $w_1, \dots, w_8 \in E \cap [0, c_{n+1}]$ such that for all $x_1, \dots, x_k \in E \cap [0, c_n]$,

$$\begin{aligned} \varphi(y_1, \dots, y_r, x_1, \dots, x_k) \stackrel{c_n}{\leftrightarrow} \\ t(x_1, \dots, x_k, w_1, \dots, w_8) \in E. \end{aligned}$$

Proof: Let $k \geq 1$. Let $t(v_1, \dots, v_{k+8})$ be the obvious term of $L(E)$ such that for all $x_1, \dots, x_k, w_1, \dots, w_8 \in A$,

$$\begin{aligned} t(x_1, \dots, x_k, w_1, \dots, w_8) = \\ (w_1 - w_2) \text{CODE}(w_3 - w_4; w_5 - w_6, x_1, \dots, x_k) + (w_7 - w_8). \end{aligned}$$

Let $n, r, \varphi, y_1, \dots, y_r$ be as given. By Lemma 5.4.14, there exist $w_1, \dots, w_8 \in E \cap [0, c_{n+1}]$ such that for all $x_1, \dots, x_k \in E \cap [0, c_n]$,

$$\begin{aligned} \varphi(y_1, \dots, y_r, x_1, \dots, x_k) \stackrel{c_n}{\leftrightarrow} \\ (w_1 - w_2) \text{CODE}(w_3 - w_4; w_5 - w_6, x_1, \dots, x_k) + (w_7 - w_8) \in E. \end{aligned}$$

QED

DEFINITION 5.4.6. Let $k \geq 1$ and $x \in E$. An x -definable k -ary relation is a relation R of the form

$$R = \{(x_1, \dots, x_k) \in E^k \cap [0, x]^k : \varphi(x_1, \dots, x_p)^x\}$$

where $p \geq k$, $\varphi(v_1, \dots, v_p)$ is a formula of $L(E)$, and $x_{k+1}, \dots, x_p \in E \cap [0, x]$.

LEMMA 5.4.16. Let $k, n \geq 1$ and R be a c_n -definable k -ary relation. There exists $y_1, \dots, y_8 \in E \cap [0, c_{n+1}]$ such that $R = \{(x_1, \dots, x_k) \in E^k \cap [0, c_n]^k : t_k(x_1, \dots, x_k, y_1, \dots, y_8) \in E\}$. Here t_k is the term of $L(E)$ given by Lemma 5.4.15.

Proof: Let $n \geq 1$ and R be a c_n -definable k -ary relation. Write

$$R = \{(x_1, \dots, x_k) \in E^k \cap [0, c_n]^k : \varphi(x_1, \dots, x_p)^{c-n}\}$$

where $p \geq k \geq 1$, $\varphi(v_1, \dots, v_p)$ is a formula of $L(E)$, and $x_{k+1}, \dots, x_p \in E \cap [0, c_n]$.

We now apply Lemma 5.4.15 to the formula

$$\varphi'(v_1, \dots, v_p) = \varphi(v_{p-k+1}, \dots, v_p, v_1, \dots, v_{p-k}).$$

We use the present x_{k+1}, \dots, x_p for the parameters y_1, \dots, y_r in Lemma 5.4.15.

Let $y_1, \dots, y_8 \in E \cap [0, c_{n+1}]$, where for all $x_1, \dots, x_k \in E \cap [0, c_n]$,

$$\begin{aligned} \varphi'(x_{k+1}, \dots, x_p, x_1, \dots, x_k)^{c-n} &\Leftrightarrow \varphi(x_1, \dots, x_p)^{c-n} \Leftrightarrow \\ R(x_1, \dots, x_k) &\Leftrightarrow t_k(x_1, \dots, x_k, z_1, \dots, z_8) \in E. \end{aligned}$$

QED

Below, the new features over Lemma 5.3.18 are items vi) and vii).

LEMMA 5.4.17. There exists a countable structure $M = (A, <, 0, 1, +, -, \cdot, \uparrow, \log, E, c_1, c_2, \dots)$, and terms t_1, t_2, \dots of L , where for all i , t_i has variables among v_1, \dots, v_{i+8} , such that the following holds.

i) $(A, <, 0, 1, +, -, \cdot, \uparrow, \log)$ satisfies $TR(\Pi_1^0, L)$;

ii) $E \subseteq A \setminus \{0\}$;

iii) The c_n , $n \geq 1$, form a strictly increasing sequence of nonstandard elements in $E \setminus \alpha(E; 2, < \infty)$ with no upper bound in A ;

iv) Let $r, n \geq 1$ and $t(v_1, \dots, v_r)$ be a term of L , and $x_1, \dots, x_r \leq c_n$. Then $t(x_1, \dots, x_r) < c_{n+1}$;

v) $2\alpha(E; 1, < \infty) + 1, 3\alpha(E; 1, < \infty) + 1 \subseteq E$;

vi) Let $k, n \geq 1$ and R be a c_n -definable k -ary relation.

There exists $y_1, \dots, y_8 \in E \cap [0, c_{n+1}]$ such that $R = \{(x_1, \dots, x_k) \in E^k \cap [0, c_n]^k : t_k(x_1, \dots, x_k, y_1, \dots, y_8) \in E\}$;

vii) Let $r \geq 1$ and $\varphi(v_1, \dots, v_{2r})$ be a formula of $L(E)$. Let $1 \leq i_1, \dots, i_{2r} < n$, where (i_1, \dots, i_r) and (i_{r+1}, \dots, i_{2r}) have the same order type and the same min. Let $y_1, \dots, y_r \in E$, $Y_1, \dots, Y_r \leq \min(c_{i_1}, \dots, c_{i_r})$. Then $\varphi(c_{i_1}, \dots, c_{i_r}, Y_1, \dots, Y_r)^{c-n} \Leftrightarrow \varphi(c_{i_{r+1}}, \dots, c_{i_{2r}}, Y_1, \dots, Y_r)^{c-n}$.

Proof: The t 's are given by Lemma 5.4.15. i)-v) are from Lemma 5.3.18 i)-v). vi) is by Lemma 5.4.16. vii) is by Lemma 5.4.10. QED