5.7. Transfinite induction, comprehension, indiscernibles, infinity, Π^0_1 correctness.

We now fix $M\# = (D,<,\in,NAT,0,1,+,-,\bullet,\uparrow,\log,d_1,d_2,...)$ as given by Lemma 5.6.18.

While working in M#, we must be cautious.

a. The linear ordering < may not be internally well ordered. In fact, there may not even be a < minimal element above the initial segment given by NAT.
b. We may not have extensionality.

Note that we have lost the internally second order nature of M* as we passed from M* to the present M# in section 5.6. The goal of this section is to recover this internally second order aspect, and gain internal well foundedness of <.

To avoid confusion, we use the three symbols =, ≡, ≈. Here = is the standard identity relation we have been using throughout the book.

DEFINITION 5.7.1. We use ≡ for extensionality equality in the form

$$x \equiv y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y).$$

DEFINITION 5.7.2. We use ≈ as a special symbol in certain contexts.

DEFINITION 5.7.3. We write $x \approx \emptyset$ if and only if x has no elements.

We avoid using $\{x_1,...,x_k\}$ out of context, as there may be more than one set represented in this way.

DEFINITION 5.7.4. Let $k \geq 1$. We write $x \approx \{y_1,...,y_k\}$ if and only if

$$(\forall z)(z \in x \leftrightarrow (z = y_1 \vee ... \vee z = y_k)).$$

LEMMA 5.7.1. Let $k \geq 1$. For all $y_1,...,y_k$ there exists $x \approx \{y_1,...,y_k\}$. Here x is unique up to ≡.

Proof: Let $y = \max(y_1,...,y_k)$. By Lemma 5.6.18 iv),

$(\exists x)(\forall z)(z \in x \leftrightarrow (z \leq y \wedge (z = y_1 \vee \ldots \vee z = y_k)))$.

The last claim is obvious. QED

DEFINITION 5.7.5. We write $x \approx \langle y,z \rangle$ if and only if there exists a,b such that

i) $x \approx \{a,b\}$;
ii) $a \approx \{y\}$;
iii) $b \approx \{y,z\}$.

LEMMA 5.7.2. If $x \approx \langle y,z \rangle \wedge w \in x$, then $w \approx \{y\} \vee w \approx \{y,z\}$. If $x \approx \langle y,z \rangle \wedge x \approx \langle u,v \rangle$, then $y = u \wedge z = v$. For all y,z, there exists $x \approx \langle y,z \rangle$.

Proof: For the first claim, let x,y,z,w be as given. Let a,b be such that $x \approx \{a,b\}$, $a \approx \{y\}$, $b \approx \{y,z\}$. Then $w = a \vee w = b$. Hence $w \approx \{y\} \vee w \approx \{y,z\}$.

For the second claim, let $x \approx \langle y,z \rangle$, $x \approx \langle u,v \rangle$. Let

$x \approx \{a,b\}$, $a \approx \{y\}$, $b \approx \{y,z\}$
$x \approx \{c,d\}$, $c \approx \{u\}$, $d \approx \{u,v\}$.

Then

$(a = c \vee a = d) \wedge (b = c \vee b = d) \wedge (c = a \vee c = b) \wedge (d = a \vee d = b)$.

Since $a = c \vee a = d$, we have $y = u \vee (y = u = v)$. Hence $y = u$.

We have $b \approx \{y,z\}$, $d \approx \{y,v\}$. If $b = d$ then $z = v$. So we can assume $b \neq d$. Hence $b = c$, $d = a$. Therefore $u = y = z$, $y = u = v$.

For the third claim, let y,z. By Lemma 5.7.1, let $a \approx \{y\}$ and $b \approx \{y,z\}$. Let $x \approx \{a,b\}$. Then $x \approx \langle y,z \rangle$. QED

DEFINITION 5.7.6. Let $k \geq 2$. We inductively define $x \approx \langle y_1,\ldots,y_k \rangle$ as follows. $x \approx \langle y_1,\ldots,y_{k+1} \rangle$ if and only if $(\exists z)(x \approx \langle z,y_3,\ldots,y_{k+1} \rangle \wedge z \approx \langle y_1,y_2 \rangle)$. In addition, we define $x \approx \langle y \rangle$ if and only if $x = y$.

LEMMA 5.7.3. Let $k \geq 1$. If $x \approx \langle y_1,\ldots,y_k \rangle$ and $x \approx \langle z_1,\ldots,z_k \rangle$, then $y_1 = z_1 \wedge \ldots \wedge y_k = z_k$. For all $y_1,\ldots,y_k$, there exists x such that $x \approx \langle y_1,\ldots,y_k \rangle$.

Proof: The first claim is by external induction on $k \geq 2$, the case $k = 1$ being trivial. The basis case $k = 2$ is by Lemma 5.7.2. Suppose this is true for a fixed $k \geq 2$. Let $x \approx \langle y_1,\ldots,y_{k+1}\rangle$, $x \approx \langle z_1,\ldots,z_{k+1}\rangle$. Let u,v be such that $x \approx \langle u,y_3,\ldots,y_{k+1}\rangle$, $x \approx \langle v,z_3,\ldots,z_{k+1}\rangle$, $u \approx \langle y_1,y_2\rangle$, $v \approx \langle z_1,z_2\rangle$. By induction hypothesis, $u = v \wedge y_3 = z_3 \wedge \ldots \wedge y_{k+1} = z_{k+1}$. By Lemma 5.7.2, since $u = v$, we have $y_1 = z_1 \wedge y_2 = z_2$.

The second claim is also by external induction on $k \geq 2$, the case $k = 1$ being trivial. The basis case $k = 2$ is by Lemma 5.7.2. Suppose this is true for a fixed $k \geq 2$. Let $y_1,\ldots,y_{k+2}$. By Lemma 5.7.2, let $z \approx \langle y_1,y_2\rangle$. By induction hypothesis, let $x \approx \langle z,y_3,\ldots,y_{k+2}\rangle$. Then $x \approx \langle y_1,\ldots,y_{k+2}\rangle$. QED

DEFINITION 5.7.7. Let $k \geq 1$. We say that R is a k-ary relation if and only if $(\forall x \in R)(\exists y_1,\ldots,y_k)(x \approx \langle y_1,\ldots,y_k\rangle)$. If R is a k-ary relation then we define $R(y_1,\ldots,y_k)$ if and only if

$$(\exists x \in R)(x \approx \langle y_1,\ldots,y_k\rangle).$$

Note that if R is a k-ary relation with $R(y_1,\ldots,y_k)$, then there may be more than one $x \in R$ with $x \approx \langle y_1,\ldots,y_k\rangle$.

We use set abstraction notation with care.

DEFINITION 5.7.8. We write

$$x \approx \{y: \varphi(y)\}$$

if and only if

$$(\forall y)(y \in x \leftrightarrow \varphi(y)\}.$$

If there is such an x, then x is unique up to $\equiv$.

Let R,S be k-ary relations. The notion $R \equiv S$ is usually too strong for our purposes.

DEFINITION 5.7.9. We define $R \equiv' S$ if and only if

$$(\forall x_1,\ldots,x_k)(R(x_1,\ldots,x_k) \leftrightarrow S(x_1,\ldots,x_k)).$$

DEFINITION 5.7.10. We define $R \subseteq' S$ if and only if

$$(\forall x_1,\ldots,x_k)(R(x_1,\ldots,x_k) \rightarrow S(x_1,\ldots,x_k).$$

We now prove comprehension for relations. To do this, we need a bounding lemma.

LEMMA 5.7.4. Let $n,k \geq 1$, and $x_1,\ldots,x_k \leq d_n$. There exists $y \approx \{x_1,\ldots,x_k\}$ such that $y \leq d_{n+1}$. There exists $z \approx \langle x_1,\ldots,x_k\rangle$ such that $z \leq d_{n+1}$.

Proof: Let $k,n,x_1,\ldots,x_k$ be as given. By Lemmas 5.7.1 and 5.7.3,

$$(\exists y)(y \approx \{x_1,\ldots,x_k\}).$$
$$(\exists z)(z \approx \langle x_1,\ldots,x_k\rangle).$$

By Lemma 5.6.18 iii), let $r > n$ be such that

$$(\exists y \leq d_r)(y \approx \{x_1,\ldots,x_k\}).$$
$$(\exists z \leq d_r)(z \approx \langle x_1,\ldots,x_k\rangle).$$

By Lemma 5.6.18 v),

$$(\exists y \leq d_{n+1})(y \approx \{x_1,\ldots,x_k\}).$$
$$(\exists z \leq d_{n+1})(z \approx \langle x_1,\ldots,x_k\rangle).$$

QED

LEMMA 5.7.5. Let $k,n \geq 1$ and $\varphi(v_1,\ldots,v_{k+n})$ be a formula of L#. Let $y_1,\ldots,y_n,z$ be given. There is a k-ary relation R such that $(\forall x_1,\ldots,x_k)(R(x_1,\ldots,x_k) \leftrightarrow (x_1,\ldots,x_k \leq z \wedge \varphi(x_1,\ldots,x_k,y_1,\ldots,y_n)))$.

Proof: Let $k,n,m,\varphi,y_1,\ldots,y_n,z$ be as given. By Lemma 5.6.18 iii), let $r \geq 1$ be such that $y_1,\ldots,y_n,z \leq d_r$. By Lemma 5.6.18 iv), let R be such that

1) $(\forall x)(x \in R \leftrightarrow (x \leq d_{r+1} \wedge (\exists x_1,\ldots,x_k \leq z)$
$(x \approx \langle x_1,\ldots,x_k\rangle \wedge \varphi(x_1,\ldots,x_k,y_1,\ldots,y_n))))$.

Obviously R is a k-ary relation. We claim that

$$(\forall x_1,\ldots,x_k)(R(x_1,\ldots,x_k) \leftrightarrow (x_1,\ldots,x_k \leq z \wedge \varphi(x_1,\ldots,x_k,y_1,\ldots,y_n))).$$

To see this, let $x_1,\ldots,x_k$. First assume $R(x_1,\ldots,x_k)$. Let $x \approx \langle x_1,\ldots,x_k\rangle$, $x \in R$. By 1),

$$x \le d_{r+1} \wedge (\exists x_1^*,\ldots,x_k^* \le z)(x = \langle x_1^*,\ldots,x_k^*\rangle \wedge \varphi(x_1^*,\ldots,x_k^*,y_1,\ldots,y_n)).$$

Let $x_1^*,\ldots,x_k^*$ be as given by this statement. By Lemma 5.7.3, $x_1^* = x_1$, ..., $x_k^* = x_k$. Hence $x_1,\ldots,x_k \le z \wedge \varphi(x_1,\ldots,x_k,y_1,\ldots,y_n)$.

Now assume

$$x_1,\ldots,x_k \le z \wedge \varphi(x_1,\ldots,x_k,y_1,\ldots,y_n).$$

By Lemma 5.7.4, let

$$x \approx \langle x_1,\ldots,x_k\rangle \wedge x \le d_{r+1}.$$

By 1), $x \in R$. Hence $R(x_1,\ldots,x_k)$. QED

LEMMA 5.7.6. If $x \approx \{y_1,\ldots,y_k\}$ then each $y_i < x$. If $x \approx \langle y_1,\ldots,y_k\rangle$, $k \ge 2$, then each $y_i < x$. If $x \approx \langle y_1,\ldots,y_k\rangle$, $k \ge 1$, then each $y_i \le x$. If $R(x_1,\ldots,x_k)$ then each $x_i < R$.

Proof: The first claim is evident from Lemma 5.6.18 ii). The second claim is by external induction on $k \ge 2$. For the basis case $k = 2$, note that if $x \approx \langle y,z\rangle$ then y,z are both elements of elements of x, and apply Lemma 5.6.18 ii). Now assume true for fixed $k \ge 2$. Let $x \approx \langle y_1,\ldots,y_{k+1}\rangle$, and let $z \approx \langle y_1,y_2\rangle$, $x \approx \langle z,y_3,\ldots,y_{k+1}\rangle$, By induction hypothesis, $z,y_3,\ldots,y_{k+1} < x$, and also $y_1,y_2 < x$.

The third claim involves only the new case $k = 1$, where the claim is trivial.

For the final claim, let $R(x_1,\ldots,x_k)$. Let $x \approx \langle x_1,\ldots,x_k\rangle$, $x \in R$. By the second claim and Lemma 5.6.18 iii), $x_1,\ldots,x_k \le x < R$. QED

DEFINITION 5.7.11. A binary relation is defined to be a 2-ary relation. Let R be a binary relation. We "define"

$$\mathrm{dom}(R) \approx \{x: (\exists y)(R(x,y))\}.$$
$$\mathrm{rng}(R) \approx \{x: (\exists y)(R(y,x))\}.$$
$$\mathrm{fld}(R) \approx \{x: (\exists y)(R(x,y) \vee R(y,x)\}.$$

Note that this constitutes a definition of dom(R), rng(R), fld(R) up to $\equiv$.

LEMMA 5.7.7. For all binary relations R, dom(R) and rng(R) and fld(R) exist.

Proof: Let R be a binary relation. By Lemma 5.6.18 iv), let A,B,C be such that

$$(\forall x)(x \in A \leftrightarrow (x \leq R \wedge (\exists y)(R(x,y)))).$$
$$(\forall x)(x \in B \leftrightarrow (x \leq R \wedge (\exists y)(R(y,x)))).$$
$$(\forall x)(x \in C \leftrightarrow (x \leq R \wedge (\exists y)(R(x,y) \vee R(y,x)))).$$

By Lemma 5.7.6,

$$(\forall x)(x \in A \leftrightarrow (\exists y)(R(x,y)).$$
$$(\forall x)(x \in B \leftrightarrow (\exists y)(R(y,x)).$$
$$(\forall x)(x \in C \leftrightarrow (\exists y)(R(x,y) \vee R(y,x))).$$

QED

DEFINITION 5.7.12. A pre well ordering is a binary relation R such that

i) $(\forall x \in \mathrm{fld}(R))(R(x,x))$;
ii) $(\forall x,y,z \in \mathrm{fld}(R))((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$;
iii) $(\forall x,y \in \mathrm{fld}(R))(R(x,y) \vee R(y,x))$;
iv) $(\forall x \subseteq \mathrm{fld}(R))(\neg(x \approx \emptyset) \rightarrow (\exists y \in x)(\forall z \in x)(R(y,z)))$.

Note that R is a pre well ordering if and only if R is reflexive, transitive, connected, and every nonempty subset of its field (or domain) has an R least element.

Note that all pre well orderings are reflexive. Clearly for pre well orderings R, dom(R) ≡ rng(R) ≡ fld(R).

Let R be a reflexive and transitive relation.

DEFINITION 5.7.13. It will be convenient to write R(x,y) as $x \leq_R y$, and write $x =_R y$ for $x \leq_R y \wedge y \leq_R x$. We also define $x \geq_R y \leftrightarrow y \leq_R x$, $x <_R y \leftrightarrow x \leq_R y \wedge \neg y \leq_R x$, $x >_R y \leftrightarrow y <_R x$, and $x \neq_R y \leftrightarrow \neg x =_R y$.

DEFINITION 5.7.14. Let R be a pre well ordering and $x \in \mathrm{fld}(R)$. We "define" the binary relations R|<x by

$$(\forall y,z)(R|{<}x(y,z) \leftrightarrow y \leq_R z <_R x)).$$

Note that R|<x is unique up to ≡′. Also note that by Lemma 5.7.5, R|<x exists. Furthermore, it is easy to see that R|<x is a pre well ordering.

When we write R|<x, we require that x ∈ fld(R).

DEFINITION 5.7.15. Let R,S be pre well orderings. We say that T is an isomorphism from R onto S if and only if

i) T is a binary relation;
ii) dom(T) ≡ dom(R), rng(T) ≡ dom(S);
iii) Let T(x,y), T(z,w). Then $x \leq_R z \leftrightarrow y \leq_S w$;
iv) Let $x =_R u$, $y =_S v$. Then T(x,y) ↔ T(u,v).

LEMMA 5.7.8. Let R,S be pre well orderings, and T be an isomorphism from R onto S. Let T(x,y), T(z,w). Then $x <_R z \leftrightarrow y <_S w$, and $x =_R z \leftrightarrow y =_S w$.

Proof: Let R,S,T,x,y,z,w be as given. Suppose $x <_R z$. Then $y \leq_S w$. If $w \leq_S y$ then $z \leq_R x$. Hence $y <_R w$. Suppose $y <_S w$. Then $x \leq_R z$. If $z \leq_R x$ then $w \leq_S y$. Hence $x <_R z$. Suppose $x =_R z$. Then $y \leq_S w$ and $w \leq_S y$. Hence $y =_S w$. Suppose $y =_S w$. Then $x \leq_R z$ and $z \leq_R x$. Hence $x =_R z$. QED

LEMMA 5.7.9. Let R,S be pre well orderings. Let a,b ∈ dom(S). Let T be an isomorphism from R onto S|<a, and T* be an isomorphism from R onto S|<b. Then $a =_S b$ and T ≡′ T*.

Proof: Let R,S,a,b,T,T* be as given. Suppose there exists x ∈ dom(R) such that for some y, ¬(T(x,y) ↔ T*(x,y)). By Lemma 5.6.18 iv), let x be R least with this property.

case 1. (∃y)(T(x,y) ∧ ¬T*(x,y)). Let T(x,y), ¬T*(x,y). Also let T*(x,y*). If $y =_S y^*$ then by clause iv) in the definition of isomorphism, T*(x,y). Hence $\neg y =_S y^*$.

case 1a. $y <_S y^*$. Then $y <_S b$. Let T*(x*,y).

Suppose $x^* <_R x$. If ¬T(x*,y), then we have contradicted the choice of x. Hence T(x*,y). But this contradicts T(x,y) by Lemma 5.7.8.

Suppose $x \leq_R x^*$. By T*(x,y*), T*(x*,y) and Lemma 5.7.8, $y^* \leq_S y$. This is a contradiction.

case 1b. $y^* <_S y$. Then $y^* <_S a$. Let T(x*,y*). By T(x,y) and Lemma 5.7.8, $x^* <_R x$. By the choice of x, since T(x*,y*), we have $T^*(x^*,y^*)$. By Lemma 5.7.8, since $T^*(x,y^*)$, we have $x =_R x^*$. Since $T(x,y)$, by Lemma 5.7.8 we have $y =_S y^*$. This is a contradiction.

case 2. $(\exists y)(\neg T(x,y) \wedge T^*(x,y))$. Let $\neg T(x,y)$, $T^*(x,y)$. This is the same as case 1, interchanging a,b, and T,T^*.

We have now established that $T \equiv' T^*$. If $a <_S b$ then $a \in rng(T^*)$ but $a \notin rng(T)$. This contradicts $T \equiv' T^*$. If $b <_S a$ then $b \in rng(T)$ but $b \notin rng(T^*)$. This also contradicts $T \equiv' T^*$. Therefore $a =_S b$. QED

DEFINITION 5.7.16. Let R,S be pre well orderings. Let T be an isomorphism from R onto S. Let $x \in dom(R)$. We write $T|<x$ for "the" restriction of T to first arguments $u <_R x$. We write $T|\leq x$ for "the" restriction of T to first arguments $u \leq_R x$. Note that $T|<x$, $T|\leq x$ are each unique up to $\equiv'$.

LEMMA 5.7.10. Let R,S be pre well orderings. Let T be an isomorphism from R onto S, and $T(x,y)$. Then $T|<x$ is an isomorphism from $R|<x$ onto $S|<y$.

Proof: Let R,S,T,x,y be as given. It suffices to show that $rng(T|<x) \equiv \{b: b <_S y\}$. Let $b <_S y$. Let $T(a,b)$. By Lemma 5.7.8, $a <_R x$. Hence $b \in rng(T|<x)$. QED

LEMMA 5.7.11. Let R,S be pre well orderings, T be an isomorphism from R onto S, and T^* be an isomorphism from $R|<x$ onto $S|<y$. Then $T^* \equiv' T|<x$ and $T(x,y)$.

Proof: Let R,S,T,T^*,x,y be as given. Let $T(x,y^*)$. By Lemma 5.7.10, $T|<x$ is an isomorphism from $R|<x$ onto $S|<y^*$. By Lemma 5.7.9, $y =_S y^*$ and $T|<x \equiv' T^*$. Hence $T(x,y)$. QED

DEFINITION 5.7.17. Let T be a binary relation. We write T^{-1} for the binary relation given by $T^{-1}(x,y) \leftrightarrow T(y,x)$. By Lemma 5.7.5, T^{-1} exists. Obviously T^{-1} is unique up to $\equiv'$.

LEMMA 5.7.12. Let R,S be pre well orderings, and T be an isomorphism from R onto S. Then T^{-1} is an isomorphism from S onto R.

Proof: Let R,S,T be as given. Obviously $dom(T^{-1}) \equiv dom(S)$ and $rng(T^{-1}) \equiv dom(R)$. Let $T^{-1}(x,y)$, $T^{-1}(z,w)$. Then $T(y,x)$, $T(w,z)$. Hence $y \leq_R w \leftrightarrow x \leq_S z$.

Finally, let $T^{-1}(x,y)$, $x =_R u$, $y =_S v$. Then $T(y,x)$, $T(v,u)$, $T^{-1}(u,v)$. QED

DEFINITION 5.7.18. Let R be a pre well ordering. We can append a new point ∞ on top and form the extended pre well ordering R^+. The canonical way to do this is to use R itself as the new point. This defines R^+ uniquely up to ≡'.

Clearly $R^+|<\infty$ ≡' R.

LEMMA 5.7.13. Let R,S be pre well orderings. Exactly one of the following holds.
1. R,S are isomorphic.
2. R is isomorphic to some S|<y, y ∈ dom(S).
3. Some R|<x, x ∈ dom(R), is isomorphic to S.
In case 2, the y is unique up to $=_S$. In case 3, the x is unique up to $=_R$. In all three cases, the isomorphism is unique up to ≡'.

Proof: We first prove the uniqueness claims. For case 1, let T,T* be isomorphisms from R onto S. Then T,T* are isomorphisms from R onto $S^+|<\infty$. By Lemma 5.7.9, T ≡' T*.

For case 2, Let T be an isomorphism from R onto S|<y, and T* be an isomorphism from R onto S|<y*. Apply Lemma 5.7.9.

For case 3, Let T be an isomorphism from R|<x onto S, and T* be an isomorphism from R|<x* onto S. By Lemma 5.7.12, T^{-1} is an isomorphism from S onto R|<x, and T^{*-1} is an isomorphism from S onto R|<x*. Apply Lemma 5.7.9.

For uniqueness, it remains to show that at most one case applies. Suppose cases 1,2 apply. Let T be an isomorphism from R onto S, and T* be an isomorphism from R onto S|<y. Then T is an isomorphism from R onto $S^+|<\infty$, and T* is an isomorphism from R onto $S^+|<y$. By Lemma 5.7.9, y is ∞, which is a contradiction.

Suppose cases 1,3 hold. Let T be an isomorphism from R onto S, and T* be an isomorphism from R|<x onto S. Then T^{-1} is an isomorphism from S onto $R^+|<\infty$, and T^{*-1} is an isomorphism from S onto $R^+|<x$. By Lemma 5.7.9, x is ∞, which is a contradiction.

Suppose cases 2,3 hold. Let T be an isomorphism from R onto S|<y and T* be an isomorphism from R|<x onto S. By Lemma 5.7.10, T|<x is an isomorphism from R|<x onto S|<z, where

T(x,z). Hence T|<x is an isomorphism from R|<x onto $S^+|<z$. Also T* is an isomorphism from R|<x onto $S^+|<\infty$. Hence by Lemma 5.7.9, z ix ∞. This is a contradiction.

We now show that at least one of 1-3 holds. Consider all isomorphisms from some $R^+|<x$ onto some $S^+|<y$, $x \in dom(R^+)$, $y \in dom(S^+)$. We call these the local isomorphisms.

We claim the following, concerning these local isomorphisms. Let T be an isomorphism from $R^+|<x$ onto $S^+|<y$, and T* be an isomorphism from $R^+|<x^*$ onto $S^+|<y^*$. If $x =_{R+} x^*$ then $y =_{S+} y^*$ and T ≡' T*. If $x <_{R+} x^*$ then $y <_{S+} y^*$ and T ≡' T*|<x. If $x^* <_{R+} x$ then $y^* <_{S+} y$ and T* ≡' T|<x*.

To see this, let T,T*,x,y be as given.

case 1. $x =_{R+} x^*$. Apply Lemma 5.7.9.

case 2. $x^* <_{R+} x$. Suppose $y \leq_{S+} y^*$. Let T(x*,z), $z <_{S+} y$. By Lemma 5.7.10, T|<x* is an isomorphism from $R^+|<x^*$ onto $S^+|<z$. By Lemma 5.7.9, T* ≡' T|<x* and $z =_{S+} y^*$. This is a contradiction. Hence $y^* <_{S+} y$. By Lemma 5.7.10, T|<x* is an isomorphism from $R^+|<x^*$ onto $S^+|<w$, where T(x*,w), $w <_{S+} y$. By Lemma 5.7.9, T* ≡' T|<x*.

case 3. $x <_{R+} x^*$. Symmetric to case 2.

By Lemma 5.7.5, we can form the union T of all of the local isomorphisms, since the underlying arguments are all in $dom(R^+)$ or $dom(S^+)$, both of which are bounded.

By the pairwise compatibility of the local isomorphisms, T obeys conditions iii),iv) in the definition of isomorphism. It is also clear that the domain of T is closed downward in R^+, and the range of T is closed downward in S^+. Hence $dom(T) \approx \{u: u <_{R+} x\}$, $rng(T) \approx \{v: v <_{S+} y\}$, for some $x \in dom(R^+)$, $y \in dom(S^+)$. Hence T is an isomorphism from $R^+|<x$ onto $S^+|<y$.

We now argue by cases.

case 1. x,y are ∞. Then T is an isomorphism from R onto S.

case 2. x is ∞, y ∈ dom(S). Then T is an isomorphism from R onto S|<y*, y* defined below.

case 3. $x \in dom(R)$, y is ∞. Then T is an isomorphism from $R|<x^*$ onto S, x^* defined below.

case 4. $x \in dom(R)$, $y \in dom(S)$. Then T is an isomorphism from $R|<x$ onto $S|<y$. Using Lemma 5.7.5, let T^* be defined by

$$T^*(u,v) \leftrightarrow$$
$$T(u,v) \vee (u =_R x \wedge v =_S y).$$

Then T^* is an isomorphism from $R|<x^*$ onto $S|<y^*$, where x^*,y^* are respective immediate successors of x,y in R^+,S^+. This contradicts the definition of T. QED

LEMMA 5.7.14. Let R,S,S^* be pre well orderings. Let T be an isomorphism from R onto S, and T^* be an isomorphism from S onto S^*. Define $T^{**}(x,y) \leftrightarrow (\exists z)(T(x,z) \wedge T^*(z,y))$, by Lemma 5.7.5. Then T^{**} is an isomorphism from R onto S^*.

Proof: Let R,S,S^*,T,T^*,T^{**} be as given. Note that T^{**} is defined up to $\equiv'$. Obviously $dom(T^{**}) \equiv dom(R)$, $rng(T^{**}) \equiv dom(S^*)$.

Suppose $T^{**}(x,y)$, $T^{**}(x^*,y^*)$. Let $T(x,z)$, $T^*(z,y)$, $T(x^*,w)$, $T^*(w,y^*)$. Then $x \leq_R x^* \leftrightarrow z \leq_S w$, $z \leq_R w \leftrightarrow y \leq_S y^*$. Therefore $x \leq_R x^* \leftrightarrow y \leq_S y^*$.

Suppose $T^{**}(x,y)$, $x =_R u$, $y =_{S'} v$. Let $T(x,z)$, $T^*(z,y)$. Then $T(u,z)$, $T^*(z,v)$. Hence $T^{**}(u,v)$. QED

We introduce the following notation in light of Lemma 5.7.13.

DEFINITION 5.7.19. Let R,S be pre well orderings. We define

$$R =^{**} S \leftrightarrow$$
R,S are pre well orderings and R,S are isomorphic.

$$R <^{**} S \leftrightarrow$$
R,S are pre well orderings and there exists $y \in fld(S)$ such that R and $S|<y$ are isomorphic.

$$R \leq^{**} S \leftrightarrow$$
$$R <^{**} S \vee R =^{**} S.$$

LEMMA 5.7.15. In $<^{**}$, the y is unique up to $=_S$. $<^{**}$ is irreflexive and transitive on pre well orderings. $=^{**}$ is an equivalence relation on pre well orderings. ≤** is reflexive and transitive and connected on pre well orderings. Let R,S,S* be pre well orderings. (R ≤** S ∧ S <** S*) → R <** S*. (R <** S ∧ S ≤** S*) → R <** S*. R <** S ∨ S <** R ∨ R =** S, with exclusive ∨. R ≤** S ∨ S ≤** R. (R ≤** S ∧ S ≤** R) → R =** S.

Proof: We apply Lemmas 5.7.13 and 5.7.14. For the first claim, if R <** S then we are in case 2 of Lemma 5.7.13, and the y is unique up to $=_S$.

For the second claim, <** is irreflexive since R <** R implies that cases 1,2 both hold in Lemma 5.7.13 for R,R. Also, suppose R <** S, S <** S*. Let T be an isomorphism from R onto S|<y, and T* be an isomorphism from S onto S*|<z. By Lemma 5.7.10, Let T** be an isomorphism from S|<y onto S*|<w. By Lemma 5.7.14, there is an isomorphism from R onto S*|<w. Hence R <** S*.

For the third claim, note that R =** R because there is an isomorphism from R onto R by defining T(x,y) ↔ $x =_R y$. Now suppose R =** S, and let T be an isomorphism from R onto S. By Lemma 5.7.12, T^{-1} is an isomorphism from S onto R. Hence S =** R. Finally, suppose R =** S, S =** S*, and let T be an isomorphism from R onto S, T* be an isomorphism from S onto S*. By Lemma 5.7.14, R =** S*.

For the fourth claim, since R =** R, we have R ≤** R. For transitivity, let R ≤** S, S ≤** S*. If R <** S, S <** S*, then by the second claim, R <** S*, and so R ≤** S*. If R =** S, S =** S*, then by Lemma 5.7.14, R =** S*, and so R ≤** S*. The remaining two cases for transitivity follow from the fifth and sixth claims. Connectivity of ≤** is by Lemma 5.7.13.

For the fifth claim, let R ≤** S and S <** S*. By the second claim, we have only to consider the case R =** S. Let S be isomorphic to S*|<y. Since R is isomorphic to S, by the third claim, R is isomorphic to S*|<y. Hence R <** S*.

For the sixth claim, let R <** S and S ≤** S*. By the second claim, we have only to consider the case S =** S*. Let R be isomorphic to S|<y. By Lemma 5.7.10, S|<y is isomorphic to S*|<z, for some z ∈ dom(S*). By the third claim, R is isomorphic to S*|<z. Hence R <** S*.

The seventh and eighth claims are immediate from Lemmas 5.7.12 and 5.7.13.

For the ninth claim, let R ≤** S and S ≤** R. Assume R <** S. By the sixth claim R <** R, which is a contradiction. Assume S <** R. By the sixth claim, S <** S, which is also a contradiction. By the eighth claim, R ≤** S ∨ S ≤** R. Under either disjunct, R =** S. QED

LEMMA 5.7.16. Every nonempty set of pre well orderings has a ≤** least element.

Proof: Let A be a nonempty set of pre well orderings, and fix S ∈ A. We can assume that there exists R ∈ A such that R <** S, for otherwise, S is a ≤** minimal element of A.

By Lemma 5.7.5, define

B ≈ {y ∈ dom(S): (∃R ∈ A)(T =** S|<y)}.

Let y be an S least element of B. Let R ∈ A be isomorphic to S|<y.

We claim that R is a ≤** least element of A. To see this, by trichotomy, let R* <** R, R* ∈ A. Then R* <** S|<y, since R is isomorphic to S|<y.

Let R* be isomorphic to (S|<y)|<z, $z <_S y$. Then R* is isomorphic to S|<z, $z <_S y$. This contradicts the choice of y. QED

DEFINITION 5.7.20. For x,y ∈ D, we define x <# y if and only

there exists a pre well ordering S ≤ y such that for every pre well ordering R ≤ x, R <** S.

We caution the reader that the ≤ in the above definition is not to be confused with ≤**. It is from the < of D in the structure M#. In particular, x,y generally will not be pre well orderings. Thus here we are treating R,S as points.

DEFINITION 5.7.21. We define x ≤# y if and only if

for all pre well orderings R ≤ x there exists a pre well ordering S ≤ y such that R ≤** S.

LEMMA 5.7.17. $<\#$ is an irreflexive and transitive relation on D. $\leq\#$ is a reflexive and transitive relation on D. Let $x,y \in D$. $x \leq\# y \vee y <\# x$. $x <\# y \rightarrow x \leq\# y$. $(x \leq\# y \wedge y <\# z) \rightarrow x <\# z$. $(x <\# y \wedge y \leq\# z) \rightarrow x <\# z$. $x \leq y \rightarrow x \leq\# y$. $x <\# y \rightarrow x < y$. $x \leq\# y \leftrightarrow \neg y <\# x$. $x <\# y \leftrightarrow \neg y \leq\# x$.

Proof: For the first claim, $<\#$ is irreflexive since $<**$ is irreflexive. Suppose $x <\# y$ and $y <\# z$. Let $S \leq y$ be a pre well ordering such that for all pre well orderings $R \leq x$, $R <** S$. Let $S* \leq z$ be a pre well ordering such that for all pre well orderings $R \leq y$, $R <** S*$. Then $S <** S*$. Hence for all pre well orderings $R \leq x$, $R <** S <** S*$. Hence for all pre well orderings $R \leq x$, $R <** S*$, by the transitivity of $<**$. Since $S* \leq z$, we have $x \leq\# z$.

For the second claim, $x \leq\# x$ since $\leq**$ on pre well orderings is reflexive. Suppose $x \leq\# y$ and $y \leq\# z$. Let $R \leq x$. Let $S \leq y$, $R \leq** S$. Let $S* \leq z$, $S \leq** S*$. By the transitivity of $\leq**$, $R \leq** S*$.

For the third claim, let $\neg(x \leq\# y)$. Let $R \leq x$ be a pre well ordering such that for all pre well orderings $S \leq y$, we have $\neg R \leq** S$. We claim that $y <\# x$. To see this, let $S \leq y$ be a pre well ordering. Then $\neg R \leq** S$. By Lemma 5.7.15, $S <** R$.

For the fourth claim, let $x <\# y$. Let $S \leq y$ be a pre well ordering such that for all pre well orderings $R \leq x$, $R <** S$. Let $R \leq x$ be a pre well ordering. Then $R \leq** S$. Hence $x \leq\# y$.

For the fifth claim, let $x \leq\# y$ and $y <\# z$. Let $S \leq z$ be a pre well ordering such that for all pre well orderings $R \leq y$, $R <** S$. Let $R \leq x$ be a pre well ordering. Let $S* \leq y$ be a pre well ordering such that $R \leq** S*$. Then $S* <** S$. By Lemma 5.7.15, $R <** S$. We have verified that $x <\# z$.

For the sixth claim, let $x <\# y$ and $y \leq\# z$. Let $S \leq y$ be a pre well ordering such that for all pre well orderings $R \leq x$, $R <** S$. Let $S* \leq z$ be a pre well ordering such that $S \leq** S*$. By Lemma 5.7.15, for all pre well orderings $R \leq x$, $R <** S*$. Hence $x <\# z$.

The seventh claim is obvious.

For the eight claim, let $x <\# y$. Let $S \leq y$ be a pre well ordering, where for all pre well orderings $R \leq x$, we have R

<** S. If y ≤ x then S ≤ x, and so S <** S. This is a contradiction. Hence x < y.

For the ninth claim, the converse is the first claim. Suppose x ≤# y ∧ y <# x. By the third claim, x <# x, which is impossible.

For the tenth claim, the converse is the first claim. Suppose x <# y ∧ y ≤# x. By the third claim, y <# y, which is impossible. QED

We now define x =# y if and only if x ≤# y ∧ y ≤# x.

LEMMA 5.7.18. =# is an equivalence relation on D. Let x,y ∈ D. x ≤# y ↔ (x <# y ∨ x =# y). x <# y ∨ y <# x ∨ x =# y, with exclusive ∨.

Proof: For the first claim, reflexivity and symmetry are obvious, by Lemma 5.7.17. Let x =# y and y =# z. Then x ≤# y and y ≤# z. Hence x ≤# z. Also z ≤# y and y ≤# x. Hence z ≤# x. Therefore x =# z.

For the second claim, let x,y ∈ D. By Lemma 5.7.17, x ≤# y ∨ y <# x. By the first claim, x <#y ∨ y <# x or x =# y.

To see that the ∨ is exclusive, suppose x <# y, y <# x. By Lemma 5.7.17, x <# x, which is a contradiction. Suppose x <# y, x =# y. By Lemma 5.7.17, x <# x, which is a contradiction. Suppose y <# x, x =# y. By Lemma 5.7.17, y <# y, which is a contradiction. QED

DEFINITION 5.7.22. We say that S is x-critical if and only if

i) S is a pre well ordering;
ii) for all pre well orderings R ≤ x, R <** S;
iii) for all y ∈ dom(S), S|<y is ≤** some pre well ordering R ≤ x.

LEMMA 5.7.19. Assume (∀y ∈ x)(y is a pre well ordering). Then there exists a pre well ordering S such that (∀R ∈ x)(R ≤** S) ∧ (∀u ∈ dom(S))(∃R ∈ x)(S|<u <** R).

Proof: Let x be as given. Let x < d_r, r ≥ 1. By Lemma 5.7.20 iv), define

$$E \approx \{y \leq d_{r+1}:$$

$(\exists R,z)(R \in x \wedge y$ is an $R|{<}z)\}$.

By Lemma 5.7.5, we define

$$S(u,v) \leftrightarrow$$
$$u,v \in E \wedge u \leq^{**} v.$$

Then S is uniquely defined up to ≡′. By Lemmas 5.7.15, 5.7.16, S is a pre well ordering.

Let $R \in x$ and $z \in \mathrm{dom}(R)$. By Lemma 5.6.18 iv),

$$(\exists y)(y \text{ is an } R|{<}z).$$

By Lemma 5.6.18 iii), let $p \geq r+1$ be such that

$$(\exists y < d_p)(y \text{ is an } R|{<}z).$$

By Lemma 5.7.20 v),

$$(\exists y < d_{r+1})(y \text{ is an } R|{<}z).$$

Hence every $R|{<}z$, $R \in x$, is isomorphic to an element of E.

We claim that we can define an isomorphism T_R from any given $R \in x$, onto S or a proper initial segment of S, as follows. T_R relates each $z \in \mathrm{dom}(R)$ to the elements of E that are isomorphic to $R|{<}z$. Note that each $z \in \mathrm{dom}(R)$ gets related by T_R to something; i.e., all of the $R|{<}z$ lying in E.

To verify the claim, we first show that $\mathrm{rng}(T_R)$ is closed downward under $\leq^{**}$ in E. Fix $T_R(z,w)$. Let w^* be an S least element of E, $w^* <^{**} w$, which is not in $\mathrm{rng}(T_R)$. Then T_R must act as an isomorphism from some proper initial segment J of $R|{<}z$ onto $S|{<}w^*$. We can assume $J \in E$ (by taking an isomorphic copy). Hence $T_R(J,w^*)$, contradicting that $w^* \notin \mathrm{rng}(T_R)$.

Since $\mathrm{rng}(T_R)$ is closed downward under $\leq^{**}$ in E, we see that $\mathrm{rng}(T_R) \equiv E$, or $\mathrm{rng}(T_R) \equiv S|{<}v$, for some $v \in E$. From the definition of T_R, T_R is an isomorphism from R onto S or a proper initial segment of S. Hence $R \leq^{**} S$.

Now let $u \in \mathrm{dom}(S)$. Then u is some $R|{<}z$, $R \in x$. Therefore $u <^{**} R$, for some $R \in x$. QED

LEMMA 5.7.20. Assume (∀y ∈ x)(y is a pre well ordering). Then there exists a pre well ordering S such that (∀R ∈ x)(R <** S) ∧ (∀R <** S)(∃y ∈ x)(R ≤** y).

Proof: Let x be as given.

case 1. x has a ≤** greatest element R. Set S ≡ R^+.

case 2. Otherwise. Set S to be as provided by Lemma 5.7.19 applied to x.

QED

LEMMA 5.7.21. For all x, there exists an x-critical S. If S is x-critical then x < S.

Proof: Let x be given. By Lemma 5.6.18 iv), define

x* ≈ {R: R ≤ x ∧ R is a pre well ordering}.

Let S be as provided by Lemma 5.7.20. Then S is x-critical.

Now let S be x-critical. If S ≤ x then S <* S, which is impossible by ii) in the definition of x-critical. QED

LEMMA 5.7.22. For all x, all x-critical S are isomorphic. For all x,y, x <# y if and only if (∃R,S)(R is x-critical ∧ S is y-critical ∧ R <** S).

Proof: Let R,S be x-critical. Suppose R <** S, and let R =** S|<y. By clause iii) in the definition of x-critical, let S|<y ≤** R* ≤ x, R* a pre well ordering. By clause ii) in the definition of R is x-critical, R* <** R. Hence R ≤** R* <** R. This is a contradiction. Hence ¬(R <** S). By symmetry, we also obtain ¬(S <** R). Hence R,S are isomorphic.

For the second claim, let x,y ∈ D. First assume x <# y. Let R be x-critical and S be y-critical. Let S* ≤ y be a pre well ordering such that for all pre well orderings R* ≤ x, we have R* <** S*.

We claim that R ≤** S*. To see this, suppose S* <** R, and let S* be isomorphic to R|<z. Since R is x-critical, let R|<z ≤** R* ≤ x, where R* is a pre well ordering. Then S* ≤** R*. Since R* ≤ x, we have R* <** S*, which is a contradiction. Thus R ≤** S*.

Note that $S^* <^{**} S$ since $S^* \leq y$ and S is y-critical. Hence $R <^{**} S$.

For the converse, assume R is x-critical, S is y-critical, and $R <^{**} S$. Let R be isomorphic to $S|<z$. Since S is y-critical, let $S|<z \leq^{**} R^* \leq y$, where R^* is a pre well ordering. Then $R \leq^{**} R^* \leq y$.

We claim that for all pre well orderings $S^* \leq x$, $S^* <^{**} R^*$. To see this, let $S^* \leq x$ be a pre well ordering. Since R is x-critical, $S^* <^{**} R \leq^{**} R^* \leq y$.

We have shown that $x <\# y$ using $R^* \leq y$, as required. QED

LEMMA 5.7.23. Let $n \geq 1$. For all $x \leq d_n$ there exists x-critical $S < d_{n+1}$. $d_n <\# d_{n+1}$.

Proof: Let $n \geq 1$ and $x \leq d_n$. By Lemmas 5.7.21 and 5.6.18 ii), there exists $m > n$ such that the following holds.

$$(\exists S < d_m)(S \text{ is x-critical}).$$

By Lemma 5.6.18 v),

$$(\exists S < d_{n+1})(S \text{ is x-critical}).$$

For the second claim, by the first claim let $R < d_{n+1}$, where R is d_n-critical. Let S be d_{n+1}-critical. Then $R <^{**} S$. By Lemma 5.7.22, $d_n <\# d_{n+1}$. QED

LEMMA 5.7.24. If $y \in x$ then x has a $<\#$ least element. Every first order property with parameters that holds of some x, holds of a $<\#$ least x. 0 is a $<\#$ least element.

Proof: Let $y \in x$. By Lemma 5.6.18 ii), let $n \geq 1$ be such that $x \leq d_n$. By Lemma 5.7.23, for each $y \in x$ there exists a y-critical $S < d_{n+1}$. By Lemma 5.6.18 iv), we can define

$$B \approx \{S < d_{n+1} : (\exists y \in x)(S \text{ is y-critical})\}$$

uniquely up to $\equiv$.

By Lemma 5.7.16, let S be a $<^{**}$ least element of B. Let S be y-critical, $y \in x$. We claim that y is a $<\#$ minimal element of x. Suppose $z <\# y$, $z \in x$. By Lemma 5.7.23, let R be z-critical, $R \in B$. By the choice of S, $S \leq^{**} R$. By Lemma

5.7.22, let R*,S* be such that R* is z-critical, S* is y-critical, and R* <** S*. By the first claim of Lemma 5.7.22, R <** S. This is a contradiction.

For the second claim, let $\varphi(y)$. By Lemma 5.6.18 ii), let y < d_n. By Lemma 5.6.18 iv), let x ≈ {y < d_{n+1}: $\varphi(y)$}. By the first claim, let y be a <# minimal element of x. Suppose $\varphi(z)$, z <# y. Since z ∉ x, we have z ≥ d_{n+1}. Since z <# y, we have z < y (Lemma 5.7.17). This contradicts y < d_{n+1} ∧ z ≥ d_{n+1}.

The third claim follows immediately from the last claim of Lemma 5.7.17. QED

LEMMA 5.7.25. If x ≤ y then x ≤# y. If x ≤ y ≤ z and x =# z, then x =# y =# z.

Proof: The first claim is trivial.

For the second claim, let x ≤ y ≤ z, x =# z. Using the first claim and Lemmas 5.7.17, 5.7.18, x ≤# y ≤# z ≤# x. Hence x =# y =# z. QED

From Lemma 5.7.25, we obtain a picture of what <# looks like.

LEMMA 5.7.26. =# is an equivalence relation on D whose equivalence classes are nonempty intervals in D (not necessarily with endpoints). These are called the intervals of =#. x <# y if and only if the interval of =# in which x lies is entirely below the interval of =# in which y lies. There is no highest interval for =#. The d's lie in different intervals of =#, each entirely higher than the previous.

Proof: For the first claim, =# is an equivalence relation by Lemma 5.7.18. Suppose x < y, x =# y. By Lemma 5.7.25, any x < z < y has x =# z =# y. So the equivalence classes under =# are intervals in <.

For the second claim, let x <# y. Let z lie in the same interval of =# as x. Let w lie in the same interval of =# as y. Then x =* z, y =* w. By Lemma 5.7.18, z <# w. By Lemma 5.7.17, z < w.

Conversely, assume the interval of =# in which x lies is entirely below the interval of =# in which y lies. Then ¬(x

=# y). By Lemma 5.7.18, x <# y ∨ y <# x. The later implies y < x, which is impossible. Hence x <# y.

For the final claim, by Lemma 5.7.23, each d_n <# d_{n+1}. By the second claim, the intervals of =# in which d_n lies is entirely below the interval of =# in which d_{n+1} lies. QED

Recall the component NAT in the structure M#.

LEMMA 5.7.27. There is a binary relation RNAT (recursively defined natural numbers) such that
i) dom(RNAT) ≈ {x: NAT(x)};
ii) (∀y)(RNAT(0,y) ↔ y is a <# least element);
iii) (∀x)(NAT(x) → (∀w)(RNAT(x+1,w) ↔ (∃z)(RNAT(x,z) ∧ w is an immediate successor of z in <#)));
iv) RNAT < d_2.
Any two RNAT's (even without iv)) are ≡'. If NAT(x) then {y: RNAT(x,y)} forms an equivalence class under =#.

Proof: We will use the following facts. The set of all <# minimal elements exists and is nonempty. For all y, the set of all immediate successors of y in <# exists and is nonempty. These follow from Lemmas 5.7.24, 5.7.26, and 5.6.18 iv).

DEFINITION 5.7.23. We say that a binary relation R is x-special if and only if

i) NAT(x);
ii) dom(R) ≈ {y: y ≤ x};
iii) (∀y)(R(0,y) ↔ y is a <# minimal element);
iv) (∀y ≤ x)(∀w)(R(y+1,w) ↔ (∃z)(R(y,z) ∧ w is an immediate successor of z in <#)).

We claim that for all x with NAT(x), there exists an x-special R. This is proved by induction, which is supported by Lemma 5.6.18 iv), vi), vii), and Lemma 5.7.5. The basis case x = 0 is immediate.

For the induction case, let R be x-special. By Lemma 5.7.5, define

S(y,w) ↔ R(y,w) ∨ (y = x+1 ∧
(∃z)(R(x,z) ∧ w is an immediate successor of z in <#)).

uniquely up to ≡'. We claim that S is x+1-special. It is clear that dom(S) ≈ {y: y ≤ x+1} since dom(R) ≈ {y: y ≤ x} and we can find immediate successors in <#. Also the conditions

$$(\forall y)(S(0,y) \leftrightarrow y \text{ is a } <\# \text{ minimal element}).$$
$$(\forall y \leq x)(\forall w)(S(y+1,w) \leftrightarrow$$
$$(\exists z)(R(y,z) \wedge w \text{ is an immediate successor of } z \text{ in } <\#)).$$

are inherited from R. To see that

$$(\forall w)(S(x+1,w) \leftrightarrow$$
$$(\exists z)(S(x,z) \wedge w \text{ is an immediate successor of } z \text{ in } <\#))$$

we need to know that {z: R(x,z)} forms an equivalence class under =#. This is proved by induction on x from 0 through x.

We have thus shown that there exists an x-special R for all x with NAT(x). Another induction on NAT shows that

$$\text{1) } NAT(x) \wedge NAT(y) \wedge x \leq y \wedge R \text{ is x-special} \wedge$$
$$S \text{ is y-special} \wedge z \leq x \rightarrow$$
$$R(z,w) \leftrightarrow S(z,w).$$

We also claim that

$$NAT(x) \rightarrow$$
$$\text{there exists an x-special } R < d_2.$$

To see this, let NAT(x). By Lemma 5.6.18 iii), let n > 1 be so large that

$$(\exists y < d_n)(y \text{ is x-special}).$$

By Lemma 5.6.18 vi), $x < d_1$. Hence by Lemma 5.6.18 v),

$$(\exists y < d_2)(y \text{ is x-special}).$$

Because of this d_2 bound, we an apply Lemma 5.7.5 to form a union RNAT of the x-special relations with NAT(x), uniquely up to ≡'. Claims i)-iii) are easily verified using 1). Thus we have

$$(\exists R)(R \text{ is an RNAT} \wedge R \text{ obeys clauses i)-iii)}).$$

Hence by Lemma 5.6.18 v),

$$(\exists R < d_2)(R \text{ is an RNAT} \wedge R \text{ obeys clauses i)-iii)}).$$

(∃R)(R obeys clauses i)-iv)).

The remaining claims can be proved from properties i)-iii) by induction. QED

DEFINITION 5.7.24. We fix the RNAT of Lemma 5.7.27, which is unique up to ≡′.

The limit point provided by the next Lemma will be used to interpret ω.

LEMMA 5.7.28. There is a <# least limit point of <#. I.e., there exists x such that
i) (∃y)(y <# x);
ii) (∀y <# x)(∃z <# x)(y <# z);
iii) for all x* with properties i),ii), x ≤# x*.
All <# least limit points of <# are =#, and < d_2.

Proof: We say that z is an ω if and only if z is a <# least limit point of <#; i.e., z obeys i)-iii).

By an obvious induction, if NAT(x) then {z: (∃y ≤ x)(RNAT(y,z))} forms an initial segment of <#. Therefore rng(RNAT) forms an initial segment of <#. Since RNAT < d_2, rng(RNAT) ⊆ [0,d_2)). According to Lemma 5.7.24, let z be <# least such that (∀x ∈ rng(RNAT))(x <# z).

It is clear that z obeys claims i),ii). Suppose x* has properties i),ii). By an obvious induction, we see that (∀y ∈ rng(RNAT))(y <# x*). Hence z ≤# x*. Thus we have verified claim iii) for z. I.e., z is an ω.

Suppose z,z* are ω's. By iii), z ≤# z*, z* ≤# z. Hence z =# z*.

By Lemma 5.6.18 iii), let n > 1 be such that

"there exists an ω < d_n".

Hence By Lemma 5.6.18 v),

"there exists an ω < d_2".

Finally, we establish that every ω is < d2. Suppose

"there exists an ω > d_2".

By Lemma 5.6.18 v),

"there exists an $\omega > d_3$".

Hence the ω's form an interval, with an element $< d_2$ and an element $> d_3$. Hence $d_2 =\# d_3$. This contradicts Lemma 5.7.26. QED

We are now prepared to define the system M^.

DEFINITION 5.7.25. M^ = $(C,<,0,1,+,-,\bullet,\uparrow,\log,\omega,c_1,c_2,\ldots,Y_1,Y_2,\ldots)$, where the following components are defined below.

i) $(C,<)$ is a linear ordering;
ii) $c_1,c_2,\ldots$ are elements of C;
iii) for $k \geq 1$, Y_k is a set of k-ary relations on C;
iv) $0,1,\omega$ are elements of C;
v) $+,-,\bullet$ are binary functions from C into C;
vi) $\uparrow$,log are unary functions from C into C.

DEFINITION 5.7.26. For $x \in D$, we write [x] for the equivalence class of x under =#. Recall from Lemma 5.7.26 that each [x] is a nonempty interval in (D,<).

DEFINITION 5.7.27. We define $C = \{[x]; x \in D\}$. We define $[x] < [y] \leftrightarrow x <\# y$. For all $n \geq 1$, we define $c_n = [d_{n+1}]$.

DEFINITION 5.7.28. Let $k \geq 1$. We define Y_k to be the set of all k-ary relations R on C, where there exists a k-ary relation S on D, internal to M#, (i.e., given by a point in D), such that for all $x_1,\ldots,x_k \in C$,

$$R(x_1,\ldots,x_k) \leftrightarrow (\exists y_1,\ldots,y_k \in D)(y_1 \in x_1 \wedge \ldots \wedge y_k \in x_k \wedge S(y_1,\ldots,y_k)).$$

Since k-ary relations S on D are required to be bounded in D, by Lemma 5.7.26 every $R \in Y_k$ is bounded in C.

DEFINITION 5.7.29. By Lemma 5.7.28, we define the ω of M^ to be [z], where z is an ω of M#, as defined in the first line of the proof of Lemma 5.7.28.

DEFINITION 5.7.30. Define the following function f externally. For each $x \in D$ such that NAT(x), let $f(x) = \{y: \text{RNAT}(x,y)\}$. Note that by Lemma 5.7.27, $f(x) \in C$. Note that the relation y ∈ f(x) is internal to M#. Also by Lemma 5.7.28 and an internal induction argument, f is one-one.

DEFINITION 5.7.31. We define 0 to be f(0) = [0], and 1 to be f(1).

DEFINITION 5.7.32. For x,y such that NAT(x),NAT(y), we define

$$f(x)+f(y) = f(x+y).$$
$$f(x)-f(y) = f(x-y).$$
$$f(x)\bullet f(y) = f(x\bullet y).$$
$$f(x)\uparrow = f(x\uparrow).$$
$$\log(f(x)) = f(\log(x)).$$

Here the operations on the left side are in M^, and the operations on the right side are in M#. Note that the above definitions of +,-,•,log on rng(f) are internal to M#.

DEFINITION 5.7.33. Let u,v ∈ C, where ¬(u,v ∈ rng(f)). We define

$$u+v = u-v = u\bullet v = u\uparrow = \log(u) = [0].$$

We now define the language L^ suitable for M^, without the c's.

DEFINITION 5.7.34. L^ is based on the following primitives.

i) The binary relation symbol <;
ii) The constant symbols 0,1,ω;
iii) The unary function symbols ↑,log;
iv) The binary function symbols +,-,•;
v) The first order variables v_n, $n \geq 1$;
vi) The second order variables B^n_m, $n,m \geq 1$;

In addition, we use ∀,∃,¬,∧,∨,→,↔,=. Commas and parentheses are also used. "B" indicates "bounded set".

DEFINITION 5.7.35. The first order terms of L^ are inductively defined as follows.

i) The first order variables v_n, $n \geq 1$ are first order terms of L^;
ii) The constant symbols 0,1,ω are first order terms of L^;
iii) If s,t are first order terms of L^ then s+t, s-t, s•t, t↑, log(t) are first order terms of L^.

DEFINITION 5.7.36. The atomic formulas of L^ are of the form

$$s = t$$
$$s < t$$
$$B^n_m(t_1,...,t_n)$$

where $s,t,t_1,...,t_n$ are first order terms and $n \geq 1$. The formulas of L^ are built up from the atomic formulas of L^ in the usual way using the connectives and quantifiers.

Note that there is no epsilon relation in L^.

The first order quantifiers range over C. The second order quantifiers B^n_k range over Y_n.

LEMMA 5.7.29. Let $k \geq 1$ and $R \subseteq C^k$ be M^ definable (with first and second order parameters allowed). Then $\{(x_1,...,x_k): R([x_1],...,[x_k])\}$ is M# definable (with parameters allowed). If R is M^ definable without parameters, then $\{(x_1,...,x_k): R([x_1],...,[x_k])\}$ is M# definable without parameters.

Proof: The construction of M^ takes place in M#, where equality in M^ is given by the equivalence relation =# in M#. Note that =# is defined in M# without parameters. The $<,0,1,\omega$ of M^ are also defined without parameters.

Let $k \geq 1$. The relations in Y_k are each coded by arbitrary internal k ary relations R in M#, where the application relation "the relation coded by R holds at points $x_1,...,x_k$" is defined in M# without parameters.

Using these considerations, it is straightforward to convert M^ definitions to M# definitions. QED

LEMMA 5.7.30. There exists a structure M^ = $(C,<,0,1,+,-,\bullet,\uparrow,\log,\omega,c_1,c_2,...,Y_1,Y_2,...)$ such that the following holds.
i) $(C,<)$ is a linear ordering;
ii) ω is the least limit point of $(C,<)$;
iii) $(\{x: x < \omega\},<,0,1,+,-,\bullet,\uparrow,\log)$ satisfies $TR(\Pi^0_1,L)$;
iv) For all $x,y \in C$, $\neg(x < \omega \wedge y < \omega) \rightarrow x+y = x\bullet y = x-y = x\uparrow = \log(x) = 0$;
v) The c_n, $n \geq 1$, form a strictly increasing sequence of elements of C, all $> \omega$, with no upper bound in C;

vi) For all $k \geq 1$, Y_k is a set of k-ary relations on C whose field is bounded above;
vii) Let $k \geq 1$, and φ be a formula of L^ in which the k-ary second order variable B^k_n is not free, and the variables B^m_r range over Y_r. Then $(\exists B^k_n \in Y_k)(\forall x_1,...,x_k)(B^k_n(x_1,...,x_k) \leftrightarrow (x_1,...,x_k \leq y \wedge \varphi))$;
viii) Every nonempty M^ definable subset of C has a < least element;
ix) Let $r \geq 1$ and $\varphi(v_1,...,v_{2r})$ be a formula of L^. Let $1 \leq i_1,...,i_{2r}$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type and the same min. Let $y_1,...,y_r \in C$, $y_1,...,y_r \leq \min(c_{i\_1},...,c_{i\_r})$. Then $\varphi(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \leftrightarrow \varphi(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r)$.

Proof: We show that the M^ we have constructed obeys these properties. Claim i) is by construction, since <# is irreflexive, transitive, and has trichotomy. Claim ii) is by the definition of ω (see Definition 5.7.29).

For claim iii), note that the f used in the construction of M^ defines an isomorphism from the ({x: NAT(x)},0,1,+,-,•,↑,log) of M# onto the ({x: x < ω},<,0,1,+,-,•,↑,log) of M^. Now apply Lemma 5.6.18 viii).

Claim iv) is by construction.

For claim v), for all $n \geq 1$, $c_n = [d_{n+1}]$. By Lemma 5.7.26, the c_n's are strictly increasing. Let $[x] \in C$. By Lemma 5.6.18 iii), let $x < d_{m+1}$, in M#. By Lemma 5.7.17, $\neg(d_{m+1}$ <# $x)$. Therefore x ≤# d_{m+1}. Hence $[x] \leq [d_{m+1}] = c_m$. Hence the c_n's have no upper bound in C. By Lemma 5.7.27, any ω of M# is <# d2 in M#. Hence $\omega < c_1$ in M^.

Claim vi) is by construction. This uses that there is no <# greatest point in M# (Lemma 5.7.26).

For claim vii), it suffices to show that every M^ definable relation R on C whose field is bounded above (≤ on C) lies in Y_k. By Lemma 5.7.29, the k-ary relation S on D given by

$$S(y_1,...,y_k) \leftrightarrow R([y_1],...,[y_k])$$

is M# definable. Since the field of R is bounded above (≤ on C), the field of S is bounded above (< on D). This uses that < on C has no greatest element (Lemma 5.7.26). Hence we can take S to be internal to M#; i.e., given by a point in D. Therefore $R \in Y_k$.

For claim viii), let R be a nonempty M^ definable subset of C. By Lemma 5.7.29, S ≈ {y: [y] ∈ R} is nonempty and M# definable. By Lemma 5.7.24, let y be a <# least element of S.

We claim that in M^, [y] is the < least element of R. To see this, let [z] ∈ R, [z] < [y]. Then z <# y and z ∈ S, which contradicts the choice of y.

For claim ix), let $\varphi(x_1,...,x_{2r}),i_1,...,i_{2r},y_1,...,y_r$ be as given. Let $i = \min(i_1,...,i_r)$. Since $y_1,...,y_r \leq c_i = [d_{i+1}]$, every element of the equivalence classes $y_1,...,y_r$ is ≤# d_{i+1}. Hence we can write $y_1 = [z_1],...,y_r = [z_r]$, where $z_1,...,z_r \leq d_{i+1}$.

By Lemma 5.7.29, the 2r-ary relation S on D given by

$$S(w_1,...,w_{2r}) \leftrightarrow$$
$$\varphi([w_1],...,[w_{2r}]) \text{ holds in M\^{}}$$

is definable in M# without parameters.

Note that $\min(i_1+1,...,i_{2r}+1) = i+1$. Hence by Lemma 5.6.18 v), we have

$$S(d_{i\_1+1},...,d_{i\_r+1},z_1,...,z_r) \leftrightarrow$$
$$S(d_{i\_r+1+1},...,d_{i\_2r+1},z_1,...,z_r).$$

Hence in M^,

$$\varphi(c_{i\_1},...,c_{i\_r},[z_1],...,[z_r]) \leftrightarrow$$
$$\varphi(c_{i\_r+1},...,c_{i\_2r},[z_1],...,[z_r]).$$

$$\varphi(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \leftrightarrow$$
$$\varphi(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r).$$

QED