5.9. ZFC + V = L + $\{(\exists\kappa)(\kappa$ is strongly k-Mahlo$)\}_k$ + TR($\Pi^0{}_1$,L), and 1-Con(SMAH).

We fix a countable model M^+ and $d_1,d_2,\ldots$, as given by Lemma 5.8.37. We will show that M^+ satisfies, for each $k \geq 1$, that "there exists a strongly k-Mahlo cardinal".

In section 4.1, we presented a basic discussion of n-Mahlo cardinals and strongly n-Mahlo cardinals. The formal systems MAH, SMAH, MAH^+, and $SMAH^+$, were introduced in section 4.1 just before Theorem 4.1.7.

Recall the partition relation given by Lemma 4.1.2. Note that Lemma 4.1.2 states this partition relation with an infinite homogenous set. A closely related partition relation was studied in [Sc74], for both infinite and finite homogenous sets. In [Sc74] it is shown that this closely related partition relation with finite homogenous sets produces strongly Mahlo cardinals of finite order, where the order corresponds to the arity of the partition relation.

We give a self contained treatment of the emergence of strongly Mahlo cardinals of finite order from this related partition relation for finite homogenous sets. We have been inspired by [HKS87], which also contains a treatment of essentially the same partition relation, and answers some questions left open in [Sc74]. Our main combinatorial result, in the spirit of [Sc74], is Theorem 5.9.5. This is a theorem of ZFC, and so we use it within M^+.

We then show that this partition relation for finite homogenous sets holds in M^+. As a consequence, M^+ has strongly Mahlo cardinals of every finite order.

DEFINITION 5.9.1. We write $S \subseteq On$ to indicate that S is a set of ordinals.

The only proper class considered in this section is On, which is the class of all ordinals. Hence S must be bounded in On.

DEFINITION 5.9.2. We write sup(S) for the least ordinal that is at least as large as every element of S.

DEFINITION 5.9.3. We write $[S]^k$ for the set of all k element subsets of S. We say that $f:[S]^k \to On$ is regressive if and only if for all $A \in [S\backslash\{0\}]^k$, $f(A) < \min(A)$.

DEFINITION 5.9.4. We say that E is min homogeneous for f if and only if $E \subseteq S$ and for all $A,B \in [E]^k$, if $\min(A) = \min(B)$ then $f(A) = f(B)$.

DEFINITION 5.9.5. We write R(S,k,r) if and only if $S \subseteq On$, $k,r \geq 1$, and for all regressive $f:[S]^k \to On$, there exists min homogenous $E \in [S]^r$ for f.

DEFINITION 5.9.6. We say that $S \subseteq On$ is closed if and only if the sup of every nonempty subset of S lies in S. Thus $\emptyset$ is closed. Note that every nonempty closed S has $\sup(S) \in S$.

DEFINITION 5.9.7. Let $f:[S]^k \to On$. When we write $f(\alpha_1,\ldots,\alpha_k)$, we mean $f(\{\alpha_1,\ldots,\alpha_k\})$, and it is assumed that $\alpha_1 < \ldots < \alpha_k$.

LEMMA 5.9.1. The following is provable in ZFC. Suppose R(S,k,r), where $S \subseteq On\backslash\omega$. Let $n \geq 1$ and $f_1,\ldots,f_n$ each be regressive functions from $[S]^k$ into On. There exists $E \in [S]^r$ which is min homogenous for $f_1,\ldots,f_n$.

Proof: Let $S,k,r,n,f_1,\ldots,f_n$ be as given. Let $H:(\sup(S)+1)^{1+n} \to \sup(S)+1$ be such that

i) For all $\omega \leq \alpha \leq \sup(S)$ and $\beta_1,\ldots,\beta_n \leq \alpha$, $H(\alpha,\beta_1,\ldots,\beta_n) < \alpha$;
ii) For all $\omega \leq \alpha \leq \sup(S)$ and $\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_n \leq \alpha$, $H(\alpha,\beta_1,\ldots,\beta_n) = H(\alpha,\gamma_1,\ldots,\gamma_n) \to (\beta_1 = \gamma_1 \wedge \ldots \wedge \beta_n = \gamma_n)$.

We can find such an H because for all $\alpha \geq \omega$, $|\alpha^n| = |\alpha|$.

Let $g:[S]^k \to On$ be defined as follows. $g(x_1,\ldots,x_k) = H(x_1,f_1(x_1,\ldots,x_k),\ldots,f_n(x_1,\ldots,x_k))$.

To see that g is regressive, let $x_1 < \ldots < x_k$ be from S. Then $\omega \leq x_1,\ldots,x_k$, and so

$$f_1(x_1,\ldots,x_k),\ldots,f_n(x_1,\ldots,x_k) < x_1.$$
$$g(x_1,\ldots,x_k) = H(x_1,f_1(x_1,\ldots,x_k),\ldots,f_n(x_1,\ldots,x_k)) < x_1.$$

By $R(S,k,r)$, let $E \in [S]^r$ be min homogenous for g. To see that E is min homogenous for $f_1,\ldots,f_n$, let $V_1,V_2 \subseteq E$ be k element sets with the same minimum, say $\alpha \in E$. Then $\omega \leq \alpha$ and $g(V_1) = g(V_2)$. Hence

$$H(\alpha,f_1(V_1),\ldots,f_n(V_1)) = H(\alpha,f_1(V_2),\ldots,f_n(V_2)).$$

By ii), each $f_i(V_1) = f_i(V_2)$. QED

LEMMA 5.9.2. The following is provable in ZFC. Let S be a closed set of infinite ordinals, none of which are strongly inaccessible cardinals. Then $\neg R(S,3,5)$.

Proof: Let S be as given, and assume $R(S,3,5)$. Then $|S| \geq 5$. We assume that this S has been chosen so that $\max(S) = \alpha$ is least possible. Then

i. S is a closed set of infinite ordinals with $\max(S) = \alpha$.
ii. S contains no strongly inaccessible cardinals.
iii. $R(S,3,5)$.
iv. If S' is a closed set of infinite ordinals containing no strongly inaccessible cardinals, $\max(S') < \alpha$, then $\neg R(S',3,5)$.

In particular,

v. For all $\delta < \alpha$, $\neg R(S \cap \delta+1,3,5)$.

We will obtain a contradiction. Note that α is infinite, but not a strongly inaccessible cardinal. By i) and $|S| \geq 5$, we see that $\alpha > \omega$.

case 1. α is a limit ordinal, but not a regular cardinal. Let $cf(\alpha) = \beta < \alpha$, and let $\{\alpha_\gamma: \gamma < \beta\}$ be a strictly increasing transfinite sequence of ordinals that forms an unbounded subset of α, where $\alpha_0 > \beta$. Note that β is a regular cardinal.

For $\delta < \alpha$, we write $\tau[\delta]$ for the least γ such that $\delta \leq \alpha_\gamma$.

For each $\gamma < \beta$, let $f_\gamma:[S \cap \alpha_\gamma+1]^3 \to On$ be regressive, where there is no min homogenous $E \in [S \cap \alpha_\gamma+1]^5$ for f_γ.

Let $g:[S]^3 \to On$ be defined as follows. $g(x,y,z) = f_{\tau[z]}(x,y,z)$ if $z < \alpha$; 0 otherwise. Note that in the first case, $z < \alpha$, we have $z \leq \alpha_{\tau[z]} < \alpha$, and $x,y,z \in S \cap \alpha_{\tau[z]}+1$. Hence in the first case, $f_{\tau[z]}(x,y,z)$ is defined.

Let $h:[S]^3 \to On$ be defined by $h(x,y,z) = \tau[y]$ if $\tau[y] < x$; 0 otherwise.

Let $h':[S]^3 \to On$ be defined by $h'(x,y,z) = \tau[z]$ if $\tau[z] < x$; 0 otherwise.

Let $J:[S]^3 \to On$ be defined by $J(x,y,z) = 1$ if $z < \alpha$; 0 otherwise.

Let $K:[S]^3 \to On$ be defined by $K(x,y,z) = 1$ if $y < \beta$; 0 otherwise.

Let $T:[S]^3 \to On$ be defined by $T(x,y,z) = 1$ if $z < \beta$; 0 otherwise.

Obviously g,h,h',J,K,T are regressive. By $R(S,3,5)$ and Lemma 5.9.1, let $E \in [S]^5$ be min homogenous for g,h,h',J,K,T.

Write $E = \{x,y,z,w,u\}_<$. Suppose $u = \alpha$. Then $J(x,y,u) = J(x,y,w) = 0$, and so $w = u = \alpha$, which is impossible. Hence $u < \alpha$.

Now suppose $y < \beta$. Then $K(x,y,z) = K(x,z,u) = 1$, and so $z < \beta$. Hence $T(x,y,z) = T(x,y,u) = 1$. Therefore $u < \beta$. Hence $\tau[b] = 0$ for all $b \in E$.

We now claim that E is min homogenous for f_0. To see this, let $V_1,V_2 \subseteq E$ be 3 element sets with the same min. Since $\tau[\max(V_1)] = \tau[\max(V_2)] = 0$, we see that $g(V_1) = g(V_2) = f_0(V_1) = f_0(V_2)$. This establishes the claim.

Since $y < \beta$, we have $E \subseteq S \cap \alpha_0+1$ (using $\alpha_0 > \beta$). This min homogeneity contradicts the choice of f_0. Hence $y < \beta$ has been refuted.

We have thus shown that $\beta \leq y,z,w,u < \alpha$. Hence $\tau[z],\tau[w],\tau[u] < y$. Since $h'(y,z,w) = h'(y,z,u)$, we have $\tau[w] = \tau[u]$. Since $h(y,z,w) = h(y,w,u)$, we have $\tau[z] = \tau[w]$.

We claim that E is min homogenous for $f_{\tau[u]}$. To see this, let $V_1,V_2 \subseteq E$ be 3 element sets with the same min. Then $\tau[\max(V_1)] = \tau[\max(V_2)] = \tau[u]$. Hence $g(V_1) = g(V_2) = f_{\tau[u]}(V_1) = f_{\tau[u]}(V_2)$. This establishes the claim. This min homogeneity contradicts the choice of $f_{\tau[u]}$.

case 2. α is a regular cardinal or a successor ordinal. In an abuse of notation, we reuse several letters from case 1.

Since $\alpha > \omega$ is not strongly inaccessible, let $\beta < \alpha$, $2^\beta \geq \alpha$. Let $K:\alpha \to \wp(\beta)$ be one-one, where $\wp(\beta)$ is the power set of β. Obviously $\beta \geq \omega$.

Let $f:[S \cap \beta+1]^3 \to On$ be regressive, where there is no min homogenous $E \in [S \cap \beta+1]^5$ for f.

Let $f':[S]^3 \to On$ extend f with the default value 0.

Let $g:[S]^3 \to On$ be defined by $g(x,y,z) = \min(K(y) \Delta K(z))$ if this min is < x; 0 otherwise. Since K is one-one, we are not taking min of the empty set, and so g is well defined.

Let $h:[S]^3 \to On$ be defined by $h(x,y,z) = 1$ if $y \leq \beta$; 0 otherwise.

Let $h':[S]^3 \to On$ be defined by $h'(x,y,z) = 1$ if $z \leq \beta$; 0 otherwise.

Obviously f',g,h,h' are regressive. By R(S,3,5) and Lemma 5.9.1, let $E \in [S]^5$ be min homogenous for f',g,h,h'. Write $E = \{x,y,z,w,u\}_<$. If $y \leq \beta$ then $h(x,y,z) = 1$, and hence $h(x,w,u) = 1$. Therefore $w \leq \beta$. Also $h'(x,y,w) = 1$. Hence $h'(x,y,u) = 1$, and so $u \leq \beta$. Since E is min homogenous for f', clearly E is min homogenous for f (using $u \leq \beta$). This contradicts the choice of f.

So we have established that $y > \beta$. Note that

$$g(y,z,w) = \min(K(z) \Delta K(w))$$
$$g(y,z,u) = \min(K(z) \Delta K(u))$$
$$g(y,w,u) = \min(K(w) \Delta K(u))$$

since K is one-one, and these min's are $< \beta < y$. Therefore

$$g(y,z,w) = g(y,z,u) = g(y,w,u).$$
$$\min(K(z) \Delta K(w)) = \min(K(z) \Delta K(u)) = \min(K(w) \Delta K(u)).$$

This is a contradiction. Hence the Lemma is proved. QED

LEMMA 5.9.3. The following is provable in ZFC. Let $k \geq 0$ and S be a closed set of infinite ordinals, none of which are strongly k-Mahlo cardinals. Then $\neg R(S,k+3,k+5)$.

Proof: We proceed by induction on $k \geq 0$. The case $k = 0$ is from Lemma 5.9.2. Suppose this is true for a fixed $k \geq 0$. We want to prove this for k+1.

Assume this is false for k+1, $k \geq 0$. As in Lemma 5.9.2, we minimize max(S). Thus we start with the following assumptions, and derive a contradiction:

i. S is a closed set of infinite ordinals with $\max(S) = \alpha$,
ii. S contains no strongly k+1-Mahlo cardinals.
iii. R(S,k+4,k+6).
iv. If S' is a closed set of infinite ordinals containing no strongly k+1-Mahlo cardinals, $\max(S') < \alpha$, then $\neg R(S',k+4,k+6)$.
v. If S' is a closed set of infinite ordinals containing no strongly k-Mahlo cardinals, then $\neg R(S',k+3,k+5)$.

In particular,

vi. For all $\beta < \alpha$, $\neg R(S \cap \beta+1,k+4,k+6)$.

We will obtain a contradiction. Note that α is infinite but not a strongly k+1-Mahlo cardinal. By iii), $|S| \geq k+6$, and $\alpha > \omega$.

We first prove that α is a limit ordinal. Suppose $\alpha = \beta+1$. Then $S \cap \beta+1 = S \cap \alpha = S\backslash\{\alpha\}$, and so by vi), $\neg R(S\backslash\{\alpha\},k+4,k+6)$.

Let $G:[S\backslash\{\alpha\}]^{k+4} \to On$ be regressive, where there is no min homogenous $E \in [S\backslash\{\alpha\}]^{k+6}$ for G.

Let $G^*:[S]^{k+4} \to On$ extend G with default value 0.

Let $H:[S]^{k+4} \to On$ be defined by $H(x_1,\ldots,x_{k+4}) = 1$ if $x_{k+4} = \alpha$; 0 otherwise.

Obviously G*,H are regressive. By R(S,k+4,k+6) and Lemma 5.9.1, let $E \in [S]^{k+6}$ be min homogenous for G*,H. Write $E = \{u_1,\ldots,u_{k+6}\}_<$.

Suppose $u_{k+6} = \alpha$. Then $H(u_1,\ldots,u_{k+3},u_{k+6}) = 1 = H(u_1,\ldots,u_{k+4})$. Hence $u_{k+4} = u_{k+6} = \alpha$. This is impossible. Hence $u_{k+6} < \alpha$, $\{u_1,\ldots,u_{k+6}\} \subseteq S\backslash\{\alpha\}$. Obviously $\{u_1,\ldots,u_{k+6}\}$ is min homogenous for G. This is a contradiction.

Thus we have shown that α is a limit ordinal $> \omega$.

Since α is not strongly k+1-Mahlo, let A be a closed and unbounded subset of $[\omega,\alpha]$, where $\omega \in A$, and no element of A is a strongly k-Mahlo cardinal.

By assumptions vi,v, for each $\beta < \alpha$, let

i) $f_\beta:[S \cap \beta+1]^{k+4} \to On$ be regressive, where there is no min homogenous $E \in [S \cap \beta+1]^{k+6}$ for f_β.
ii) $g_\beta:[A \cap \beta+1]^{k+3} \to On$ be regressive, where there is no min homogenous $E \in [A]^{k+5}$ for g_β.

For all $x \in [\omega,\alpha)$, let $\beta[x]$ be the greatest $\beta \in A$ such that $\beta \leq x$. Let $\gamma[x]$ be the least $\gamma \in A$ such that $x < \gamma$.

Let $f':[S]^{k+4} \to On$ be defined by $f'(x_1,\ldots,x_{k+4}) = f_{\gamma[x\_k+4]}(x_1,\ldots,x_{k+4})$ if $x_{k+4} < \alpha$; 0 otherwise.

Let $g':[S]^{k+4} \to On$ be defined by $g'(x_1,\ldots,x_{k+4}) = g_{\beta[x\_k+4]}(\beta[x_1],\ldots,\beta[x_{k+3}])$ if $x_{k+4} \in [\omega,\alpha) \wedge \beta[x_1] < \ldots < \beta[x_{k+4}]$; 0 otherwise.

Let $h:[S]^{k+4} \to On$ be defined by $h(x_1,\ldots,x_{k+4}) = 1$ if $x_{k+4} = \alpha$; 0 otherwise.

For $1 \leq i \leq k+3$, let $J_i:[S]^{k+4} \to On$ be defined by

$$J_i(x_1,\ldots,x_{k+4}) = 1 \text{ if } \beta[x_i] < \beta[x_{i+1}]; \text{ 0 otherwise.}$$

Obviously $f',g',h,J_1,\ldots,J_{k+3}$ are regressive. By R(S,k+4,k+6) and Lemma 5.9.1, let $E \in [S]^{k+6}$ be min homogenous for $f',g',h,J_1,\ldots,J_{k+3}$. Write $E = \{u_1,\ldots,u_{k+6}\}_<$. Obviously, u_1 is infinite, and so $\beta[u_1]$ is defined.

Suppose $u_{k+6} = \alpha$. Then $h(u_1,\ldots,u_{k+3},u_{k+6}) = h(u_1,\ldots,u_{k+3},u_{k+5}) = 1$, and so $u_{k+5} = \alpha$. This is impossible. Hence $u_{k+6} < \alpha$.

Suppose $\beta[u_i] = \beta[u_{i+1}] < \beta[u_{i+2}]$, for some $1 \leq i \leq k+2$. Then $J_i(u_1,\ldots,u_{k+4}) = 0 \wedge J_i(u_1,\ldots,u_i,u_{i+2},\ldots,u_{k+5}) = 1$. This is a contradiction.

Suppose $\beta[u_i] < \beta[u_{i+1}] = \beta[u_{i+2}]$, for some $2 \leq i \leq k+2$. Then $J_i(u_1,\ldots,u_{k+4}) = 1 \wedge J_i(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_{k+5}) = 0$. This is also a contradiction.

We claim that

$$1)\ \beta[u_2] = \ldots = \beta[u_{k+4}] \vee \beta[u_1] < \ldots < \beta[u_{k+4}].$$

To see this, suppose $\neg(\beta[u_i] < \beta[u_{i+1}])$, $1 \leq i \leq k+3$. Then $\beta[u_i] = \beta[u_{i+1}]$. Hence $\beta[u_1] = \ldots = \beta[u_i] = \beta[u_{i+1}] = \ldots = \beta[u_{k+4}]$.

Under the first disjunct of 1), $J_{k+3}(u_1,\ldots,u_{k+4}) = 0 = J_{k+3}(u_1,\ldots,u_{k+2},u_{k+4},u_{k+5}) = J_{k+3}(u_1,\ldots,u_{k+2},u_{k+5},u_{k+6})$. Hence $\beta[u_{k+4}] = \beta[u_{k+5}] = \beta[u_{k+6}]$.

Under the second disjunct of 1), $J_{k+3}(u_1,\ldots,u_{k+4}) = 1 = J_{k+3}(u_1,\ldots,u_{k+2},u_{k+4},u_{k+5}) = J_{k+3}(u_1,\ldots,u_{k+2},u_{k+5},u_{k+6})$. Hence $\beta[u_{k+4}] < \beta[u_{k+5}] < \beta[u_{k+6}]$.

We have thus shown that

$$2)\ \beta[u_2] = \ldots = \beta[u_{k+6}] \vee \beta[u_1] < \ldots < \beta[u_{k+6}].$$

case 1. $\beta[u_2] = \ldots = \beta[u_{k+6}]$. We claim that E is min homogenous for $f_{\gamma[u\_k+6]}$. To see this, let $V_1,V_2 \subseteq E$ be k+4 element sets with the same min. Then $\beta[\max(V_1)] = \beta[\max(V_2)] = \beta[u_{k+6}]$, $\gamma[\max(V_1)] = \gamma[\max(V_2)] = \gamma[u_{k+6}]$, $f'(V_1) = f'(V_2)$, and $u_{k+6} < \alpha$. Hence $f_{\gamma[u\_k+6]}(V_1) = f_{\gamma[u\_k+6]}(V_2)$. This establishes the claim. This contradicts the choice of $f_{\gamma[u\_k+6]}$.

case 2. $\beta[u_1] < \ldots < \beta[u_{k+6}]$. We claim that $\beta[u_1],\beta[u_2],\ldots,\beta[u_{k+5}]\}$ is min homogenous for $g_{\beta[u\_k+6]}$. To see this, let $V_1,V_2 \subseteq \{\beta[u_1],\beta[u_2],\ldots,\beta[u_{k+5}]\}$ be k+3 element subsets with the same min. Then $g'(V_1 \cup \{\beta[u_{k+6}]\}) = g'(V_2 \cup \{\beta[u_{k+6}]\}) = g_{\beta[u\_k+6]}(V_1) = g_{\beta[u\_k+6]}(V_2)$, using $u_{k+6} < \alpha$. This establishes the claim. Note that $\{\beta[u_1],\beta[u_2],\ldots,\beta[u_{k+5}]\} \subseteq A \cap \beta[u_{k+6}]+1$, $u_{k+6} < \alpha$, $\beta[u_{k+6}] < \alpha$. But this contradicts the choice of $g_{\beta[u\_k+6]}$.

We have derived the required contradiction, and the Lemma has been proved. QED

LEMMA 5.9.4. The following is provable in ZFC. For all integers $k \geq 0$ and ordinals α, if $R(\alpha+1\backslash\omega,k+3,k+5)$ then there is a strongly k-Mahlo cardinal $\leq \alpha$.

Proof: Let $k \geq 0$ and $R(\alpha+1\backslash\omega,k+3,k+5)$. Note that $S = \alpha+1\backslash\omega$ is a closed set of infinite ordinals. By Lemma 5.9.3, if none of them are strongly k-Mahlo cardinals, then $\neg R(S,k+3,k+5)$. Hence $\alpha+1\backslash\omega$ contains a strongly k-Mahlo cardinal. Therefore there is a strongly k-Mahlo cardinal $\leq \alpha$. QED

We will not need the following result, which is of independent interest.

THEOREM 5.9.5. The following is provable in ZFC. Let $k < \omega$ and α be an ordinal. Then $R(\alpha\backslash\omega,k+3,k+5)$ if and only if there is a strongly k-Mahlo cardinal $\leq \alpha$.

Proof: Let $R(\alpha\backslash\omega,k+3,k+5)$. It is immediate that $R(\alpha+1\backslash\omega,k+3,k+5)$. By Lemma 5.9.4, there is a strongly k-Mahlo cardinal $\leq \alpha$.

Now let $\kappa \leq \alpha$ be strongly k-Mahlo. It follows easily from [Sc74] that $R(\kappa,k+3,k+5)$. Hence $R(\alpha,k+3,k+5)$. QED

We now return to the model M^+ of ZFC + V = L + $TR(\Pi^0_1,L)$ given by Lemma 5.8.37.

LEMMA 5.9.6. Let $k,r \geq 1$ be standard integers. Then $R(d_{r+2}+1\backslash\omega,k,r)$ holds in M^+.

Proof: Let k be as given. We argue in M^+. By Lemma 5.8.37, M^+ satisfies ZFC + V = L.

Suppose $R(d_{r+2}+1\backslash\omega,k,r)$ fails in M^+. We can choose $f:[d_{r+2}+1\backslash\omega]^k \to On$ to be least in the constructible hierarchy such that f is regressive and there is no $E \in [d_{r+2}+1\backslash\omega]^r$ that is min homogenous for f. Note that f is M^+ definable from d_{r+2}.

We claim that $\{d_2,\ldots,d_{r+1}\}$ is min homogenous for f. To see this, let $2 \leq i_1 < \ldots < i_k \leq r+1$, and $2 \leq j_1 < \ldots < j_k \leq r+1$, where $i_1 = j_1$. By Lemma 5.8.37 ii), for all $\alpha \leq d_{i\_1}$,

$$f(d_{i\_1},\ldots,d_{i\_k}) = \alpha \leftrightarrow f(d_{j\_1},\ldots,d_{j\_k}) = \alpha.$$

Since f is regressive, choose $\alpha = f(d_{i\_1},\ldots,d_{i\_k}) < d_{i\_1}$. By Lemma 5.8.37 ii),

$$f(d_{i\_1},\ldots,d_{i\_k}) = \alpha \leftrightarrow f(d_{j\_1},\ldots,d_{j\_k}) = \alpha.$$
$$f(d_{j\_1},\ldots,d_{j\_k}) = \alpha = f(d_{i\_1},\ldots,d_{i\_k}).$$

Note that by Lemma 5.8.37, $d_2 > \omega$. Hence $\{d_2,...,d_{r+1}\} \subseteq d_{r+2}+1\backslash\omega$ is min homogenous for f. But this contradicts the choice of f. QED

LEMMA 5.9.7. Let $k \geq 0$ be a standard integer. Then "there exists a strongly k-Mahlo cardinal" holds in M^+. As a consequence, ZFC + V = L + {there exists a strongly k-Mahlo cardinal}$_k$ + TR(Π^0_1,L) is consistent.

Proof: Immediate from Lemmas 5.8.37, 5.9.4, and 5.9.6. QED

LEMMA 5.9.8. ZFC proves that Proposition C implies 1-Con(SMAH).

Proof: We argue in ZFC + Proposition C. Now the entire reversal from section 5.1 through Lemma 5.9.7 was conducted within ZFC. So M^+ is available, and we know that SMAH holds in M^+. Let SMAH prove φ, where φ is a Σ^0_1 sentence of L. Since SMAH holds in M^+, so does φ. If φ is false then $\neg\varphi \in$ TR(Π^0_1,L), in which case $\neg\varphi$ holds in M^+. This contradicts that φ holds in M^+. Hence φ is true. (Here the outermost $\neg$ in $\neg\varphi$ is pushed inside). QED

THEOREM 5.9.9. None of Propositions A,B,C are provable in SMAH, provided MAH is consistent. They are provable in MAH^+. These claims are provable in RCA_0.

Proof: Suppose Proposition C is provable in SMAH. By Lemma 5.9.8, SMAH proves the consistency of SMAH. By Gödel's second incompleteness theorem, SMAH is inconsistent. By the last claim of Theorem 4.1.7, it follows that MAH is inconsistent. Both Propositions A,B each imply Proposition C over RCA_0 (see Lemma 4.2.1).

The second claim is by Theorem 4.2.26. These claims are provable in RCA_0 since RCA_0 can recognize proofs, and prove the Gödel second incompleteness theorem. QED

We now provide more refined information.

Recall the formal system ACA' from Definition 1.4.1.

LEMMA 5.9.10. The derivation of 1-Con(SMAH) from Proposition C, in sections 5.1-5.9, can be formalized in ACA'. I.e., ACA' proves that each of Propositions A,B,C implies 1-Con(SMAH).

Proof: Most of the development lies within RCA_0. But since we are stuck using ACA' already in section 5.2, we will use the stronger fragment ACA_0 of ACA' instead of RCA_0 for the discussion. We regard Proposition C, which is readily formalized in ACA_0 (or even RCA_0), as the hypothesis, which we take as implicit in the section by section analysis below.

section 5.1. All within ACA_0.

section 5.2. All within ACA_0 except Lemma 5.2.5. Lemma 5.2.5 is a sharp form of the usual Ramsey theorem on N. This is provable in ACA'. In fact, it is provably equivalent to ACA' over RCA_0. Hence Lemma 5.2.12 is provable in ACA'.

section 5.3. All within ACA_0, from Lemma 5.2.12. In the proof of Lemma 5.3.3, we apply the compactness theorem to a set T of sentences that is Π^0_1. T has bounded quantifier complexity, and the proof that every finite subset of T has a model, and the proof that every finite subset of T has a model can be formulated and proved in ACA_0. The application of compactness to obtain a model M of T can be formalized in ACA_0. In fact, we obtain a model M of T with a satisfaction relation, within ACA_0. In the proof, we then adjust M by taking an initial segment. This construction can also be formalized in ACA_0. However, we lose the satisfaction relation within ACA_0, and cannot recover it even within ACA'. Nevertheless, we retain a satisfaction relation for all formulas whose quantifiers are bounded in the adjusted M, since this restricted satisfaction relation is obtained from the satisfaction relation for the original unadjusted M in ACA_0. The statement of Lemma 5.3.18 has bounded quantifier complexity, and so is formalizable in th language of ACA_0. We conclude that Lemma 5.3.18, with bounded satisfaction relation, is provable in ACA_0 from Lemma 5.2.12. This bounded satisfaction relation incorporates the constants from M.

section 5.4. All within ACA_0, from Lemma 5.3.18. The quantifiers in E formulas of L(E) are required to be bounded in the structure M. Hence the E formulas of L(E) are covered by the bounded satisfaction relation for M. Since only E formulas of L(E) are considered, Lemma 5.4.17 is provable in ACA_0 from Lemma 5.3.18.

section 5.5. All within ACA', from Lemma 5.4.17. Lemma 5.5.1 involves arbitrary formulas of L(E), and so it needs

ACA' to formulate, using partial satisfaction relations for M. The induction hypothesis as stated in the proof of Lemma 5.5.1 is Σ^1_1 (or Π^1_1), and therefore the induction, as it stands, is not formalizable in ACA'. However, this can be fixed. We fix n, the number of quantifiers, and form the satisfaction relation for n quantifier formulas, for M, in ACA'. We then prove the displayed equivalence by all $0 \leq n' \leq n$ by induction on n'. This modification reduces the induction to an arithmetical induction, well within ACA'. Note that we can use Lemma 5.5.1 to construct the full satisfaction relation for M from the bounded satisfaction relation for M, within ACA_0. Also, the construction of the sets X_k can easily be formalized in ACA'. In the proof of Lemma 5.5.4, second order quantification in formulas of the language L*(E) are removed. This removal allows us to construct the satisfaction relation for M* from the satisfaction relation for M, within ACA_0. This allows us to argue freely within ACA_0 throughout the rest of section 5.5. We conclude that Lemma 5.5.8, with satisfaction relation, is provable in ACA' from Lemma 5.4.17.

section 5.6. The formalization in ACA_0 is straightforward through the development of internal arithmetic in Lemma 5.6.12, via the internal structure M(I). The substructure M|rng(h) is defined arithmetically, with an arithmetic isomorphism from M(I) onto M|rng(h). The satisfaction relation for M|rng(h) is constructed from the satisfaction relation for M(I) via the isomorphism, within ACA_0. Hence the statement and proof that M|rng(h) satisfies PA(L) + TR(Π^0_1,L) lie within ACA_0. It immediately follows, in ACA_0, that M(I) satisfies PA(L) + TR(Π^0_1,L). It is clear that the use of h and M|rng(h) is an unnecessary convenience that causes no difficulties within ACA_0. The conversion to linearly ordered set theory is by explicit definition, and so Lemma 5.6.20, with satisfaction relation, is provable in ACA_0 from Lemma 5.5.8.

section 5.7. The development through Lemma 5.7.28 is internal to M#, and so cause no difficulties within ACA_0. In the subsequent construction of M^, we use equivalence classes under a definable equivalence relation as points. Instead of using the actual equivalence classes, we can instead use the equivalence relation as the equality relation. The sets Y_k become families of relations that respect the equality relation. The construction is by explicit definition, and so we obtain a version of the M^ of Lemma 5.7.30 using this equality relation, with a satisfaction relation. We can then factor out by the equality relation, using a set of representatives of the equivalence classes. Specifically, taking the numerically least element of each equivalence class as the representative of that equivalence class. All of this can easily be done in ACA_0. Hence Lemma 5.7.30, with satisfaction relation, is provable in ACA_0 from Lemma 5.6.20.

section 5.8. All within ACA_0 from Lemma 5.7.30. This is an inner model construction that is totally definable. Hence Lemma 5.8.37, with satisfaction relation, is provable in ACA_0 from Lemma 5.7.30.

section 5.9. Using the satisfaction relation for M^+, we see that M^+ satisfies ZFC + V = L + SMAH + Π^0_1(L), within ACA_0. Again using the satisfaction relation for M^+, we have 1-Con(SMAH), within ACA_0.

From these considerations, we see that ACA' + Proposition C proves 1-Con(SMAH). Since B $\rightarrow$ A $\rightarrow$ C in RCA_0, we have that ACA' + Proposition A, and ACA' + Proposition B, also prove 1-Con(SMAH). QED

We conjecture that RCA_0 proves that Propositions A,B,C each imply 1-Con(SMAH).

DEFINITION 5.9.8. The system EFA = exponential function arithmetic is in the language 0,<,S,+,-,•,↑,log, and consists of the axioms for successor, defining equations for <,+,-,•,↑,log and induction for all Δ_0 formulas in 0,<,S,+,-,•,↑,log.

EFA is essentially the same as the system $I\Sigma_0$(exp). See [HP93].

Also recall the following result from Chapter 4.

THEOREM 4.4.11. Propositions A,B,C are provable in ACA' + 1-Con(MAH).

Thus we have

THEOREM 5.9.11. ACA' proves the equivalence of each of Propositions A,B,C and 1-Con(MAH), 1-Con(SMAH).

Proof: We have only to remark that EFA proves 1-Con(MAH) → 1-Con(SMAH). This is from Lemma 4.1.7. QED

THEOREM 5.9.12. None of Propositions A,B,C are provable in any set of consequences of SMAH that is consistent with ACA'. The preceding claim is provable in RCA_0. For finite sets of consequences, the first claim is provable in EFA.

Proof: Suppose Proposition C is provable in T, where

SMAH proves T.
T + ACA' is consistent.
T proves Proposition C.

Let T* be finitely axiomatized, where

SMAH proves T*.
T* + ACA' is consistent.
T* proves Proposition C.

By Theorem 5.9.11, T* proves 1-Con(SMAH). In particular, T* proves Con(T* + ACA'), using that T* + ACA' is finite, and SMAH proves T* + ACA'. By Gödel's second incompleteness theorem, T* + ACA' is inconsistent. This is a contradiction. The argument is obviously formalizable in RCA_0. If T is already finite, then there is no need for RCA_0, and we can use EFA = $I\Sigma_0(\exp)$ instead. QED