CONCEPT CALCULUS: MUCH BETTER THAN

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ABSTRACT. This is the initial publication on Concept Calculus, which establishes mutual interpretability between formal systems based on informal commonsense concepts and formal systems for mathematics through abstract set theory. Here we work with axioms for "better than" and "much better than", and the Zermelo and Zermelo Frankel axioms for set theory.

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1. INTRODUCTION.

We have discovered an unexpectedly close connection between the logic of mathematical concepts and the logic of informal concepts from common sense thinking. Our results indicate that they are, in a certain precise sense, equivalent.

This connection is new and there is the promise of establishing similar connections involving a very wide range of informal concepts.

We call this development the Concept Calculus. In this paper, we focus on just one context for Concept Calculus. We use two particular informal concepts from common sense thinking. These are the informal binary relations

BETTER THAN.
MUCH BETTER THAN.

As discussed briefly in section 10, these relations can be looked at mereologically, using the part/whole and the infinitesimal part/whole relation.

Sections 2,3 contain background information about interpretability between theories, which should be informative for readers not familiar with this fundamentally important concept credited to Alfred Tarski. We are now preparing a book on this topic (see [FVxx]).

In section 4, we present our basic axioms involving "better than", "much better than", and identity. These axioms are of a simple character, and range from obvious to intriguingly plausible.

We anticipate that Concept Calculus is extremely flexible, so that axioms can be chosen to accommodate many diverse points of view - while still maintaining the mutual interpretability with systems such as Z and ZF that we establish here.

For instance, the axioms investigated here preclude there being a best object. There are important viewpoints where a best object is an essential component. We anticipate a formulation accommodating a best object that stands in relation to the system here as does class theory to set theory.

In section 4, you will find three groups of axioms.

BASIC.

DIVERSE EXACTNESS.
STRONG DIVERSE EXACTNESS.
VERY STRONG DIVERSE EXACTNESS.
SUPER STRONG DIVERSE EXACTNESS.

UNLIMITED IMPROVEMENT.
STRONG UNLIMITED IMPROVEMENT.

We put primary emphasis on the system MBT (much better than) = Basic + Diverse Exactness + Strong Unlimited Improvement. We prove that MBT is mutually interpretable with ZF (and hence ZFC, as ZF and ZFC are mutually interpretable).

However, there are other meritorious combinations that we show are mutually interpretable with ZF.

In fact, we show that if we choose one axiom from each of the three groups (thus Basic must be included), then we get a system mutually interpretable with ZF - with exactly one exception: Basic + Diverse Exactness + Unlimited Improvement is interpretable in ZF/P (ZF without the power set), and may be much weaker still.

Zermelo set theory (Z) is a particularly important relatively strong fragment of ZF of substantial foundational significance. In particular, ZC (Z with the axiom of choice) forms a very smooth and workable foundation for mathematics that is nearly as comprehensive, in practice, as ZFC. Of course, known exceptions to this are particularly interesting and noteworthy. See [Fr09] for a discussion.

We show that Basic plus any of the last three forms of Diverse Exactness form a system that is mutually interpretable with Z.

We close with section 10, where we give a very brief discussion of some further developments.

A Corollary of the results here is a proof of the equivalence of the consistency of MBT and the consistency of ZF(C), within a weak fragment of arithmetic such as EFA = exponential function arithmetic. In particular, this provides a proof of the consistency of mathematics (as formalized by ZFC), assuming the consistency of any of MBT. The same holds for the variants discussed above that are mutually interpretable with ZF.

We have also obtained a number of results in Concept Calculus involving a variety of other informal concepts, and a variety of formal systems including ZF and beyond. We are planning a comprehensive book on Concept Calculus.

2. INTERPRETATION POWER.

The notion of interpretation plays a crucial role in Concept Calculus.

Interpretability between formal systems was first precisely defined by Alfred Tarski. We work in the usual framework of first order predicate calculus with equality.

DEFINITION. An interpretation of S in T consists of
i. A one place relation defined in T which is meant to carve out the domain of objects that S is referring to, from the point of view of T.
ii. A definition of the constants, relations, and functions in the language of S by formulas in the language of T, whose free variables are restricted to the domain of objects that S is referring to (in the sense of i).
iii. It is required that every axiom of S, when translated into the language of T by means of i,ii, becomes a theorem of T.

It is now standard to allow quite a lot of flexibility in i-iii. Specifically

a. Parameters are allowed in all definitions.
b. The domain objects can be tuples.
c. The equality relation in S need not be interpreted as equality - but, instead, as an equivalence relation. The interpretations of the domain, constants, relations must respect this equivalence relation. Functions are interpreted as "functional" relations that respect this equivalence relation.

A detailed discussion interpretations between theories will appear in the forthcoming book [FVxx].

We caution the reader that interpretations may not preserve truth. They only preserve provability.

We give two illustrative examples.

S consists of the axioms for linear order, together with “there is a least element”.

i. $\neg(x < x)$.
ii. $(x < y \wedge y < z) \rightarrow x < z$.
iii. $x < y \vee y < x \vee x = y$.
iv. $(\exists x)(\forall y)(x < y \vee x = y)$.

T consists of the axioms for linear order, together with “there is a greatest element”.

i. $\neg(x < x)$.
ii. $(x < y \wedge y < z) \rightarrow x < z$.
iii. $x < y \vee y < x \vee x = y$.
iv. $(\exists x)(\forall y)(y < x \vee x = y)$.

S,T are theories in first order predicate calculus with equality, in the same language: just <.

CLAIM: S is interpretable in T and vice versa. They are mutually interpretable.

Obvious interpretation of S in T: In T, take the objects of S to be everything (according to T). Define x < y of S to be y < x in T. Define x = y of S to be x = y in T.

Interpretation of the axioms of S formally yields

i'. $\neg(x < x)$.
ii'. $(y < x \wedge z < y) \rightarrow z < x$.
iii'. $y < x \vee x < y \vee x = y$.
iv'. $(\exists x)(\forall y)(y < x \vee x = y)$.

These are obviously theorems of T.

The obvious interpretation of T in S is the same! The interpretation of the axioms of T formally yields

i''. $\neg(x < x)$.
ii''. $(y < x \wedge z < y) \rightarrow z < x$.
iii''. $y < x \vee x < y \vee x = y$.
iv''. $(\exists x)(\forall y)(x < y \vee x = y)$.

These are obviously theorems of S.

We now discuss a much more sophisticated example.

Let PA = Peano Arithmeitc be the theory in 0,S,+,• with successor axioms, defining equations for +,•, and the scheme of induction for all formulas in this language.

Now consider "finite set theory". By this, we mean ZF with the axiom of infinity replaced by its negation; i.e., ZF\I + ¬I.

THEOREM (well known). PA, ZF\I + ¬I are mutually interpretable.

The theorem is probably due to Tarski.

To interpret PA in ZF\I + ¬I, nonnegative integers are interpreted as the finite von Neumann ordinals in ZF\I + ¬I. 0,S,+,•,= are interpreted in the normal way on the finite von Neumann ordinals in ZF\I + ¬I.

To interpret ZF\I + ¬I in PA, sets are coded by the natural numbers in PA. A common method writes $n = 2^{m_1} + ... + 2^{m_k}$, and has n coding the set of sets coded by the m's. This uses all of the natural numbers in PA, with = interpreted as =.

In many examples of mutual interpretability, the considerably stronger relation of synonymy holds. The strongest notion of synonymy normally considered is that of having a common definitional extension. There are some important weaker notions.

Notions of synonymy and other topics concerning interpretability, are treated systematically and extensively in the forthcoming book [FVxx].

Synonymy and its natural variants exhibit many delicate phenomena. It is obvious that S,T above are synonymous. It is proved in [KW07] that PA and ZF\I + ¬I are synonymous, if we formulate the axiom of foundation in ZF as a scheme.

However, it is proved in [ESV08] that PA and ZF\I + ¬I are not synonymous, if foundation is formulated in the more usual way as a single sentence.

3. BASIC FACTS ABOUT INTERPRETATION POWER.

Every theory is interpretable in any inconsistent theory. Thus, the most powerful level of interpretation power is inconsistency.

Fundamental fact: there is no maximal interpretation power - short of inconsistency.

THEOREM 3.1. (In ordinary predicate calculus with equality). Let S be a consistent recursively axiomatized theory. There exists a consistent finitely axiomatized extension T of S which is not interpretable in S.

This is proved using Gödel's second incompleteness theorem. Consider T = EFA + Con(S), where EFA is exponential function arithmetic. If T is interpretable in S then EFA proves Con(S) → Con(EFA + Con(S)). By Gödel's second incompleteness theorem, EFA + Con(S) is inconsistent, which is a contradiction.

COMPARABILITY(?). Let S,T be recursively axiomatized theories. Then S is interpretable in T or T is interpretable in S?

There are plenty of natural and interesting examples of incomparability for finitely axiomatized theories that are rather weak.

Let T_1 be the theory of strict linear orderings where every element has an immediate predecessor and immediate successor.

Let T_2 be the theory of strict linear orderings where between any two elements there is a third, and there are no least or greatest elements.

We use (Q,<), where Q is the set of all rational numbers, which forms a model of T_2.

LEMMA 3.2. There is no model (D,R,≡) of T_1 definable in (Q,<).

Proof: Let (D,R,≡) be a model of T_1 that is definable in (Q,<). Let P be the set of parameters used to define (D,R,≡). Here D is the domain, R is the interpreted linear order relation, and ≡ is the interpreted equality relation. Let k be such that every element of D is a tuple of length at most k.

Let x ∈ D. Consider [x],[x}+1,[x]+2,..., the equivalence classes under ≡ that are the successive immediate successors of [x] in (D,R,≡). Let $\alpha_0,\alpha_1,...$ be such that each $\alpha_i \in$ [x]+i. Since the lengths of the α's are bounded by k, we see that by Ramsey's theorem, we can find i < j < k such that (α_i,α_j), (α_i,α_k), (α_j,α_k) are all of the same order type over P. Hence there is an automorphism of (Q,<) which is the identity on P, which sends (α_i,α_j) to (α_i,α_k). Hence there is an automorphism of (D,R,≡) which sends (α_i,α_j) to (α_i,α_k). But (D,R,≡) satisfies

the distance from α_i to α_j is j-i.
the distance from α_i to α_k is k-i.

This is a contradiction. QED

We now use (Z,<), where Z is the set of all integers, which forms a model of T_1. It is well known that T_2 has elimination of quantifiers, if we add the +1 and -1 functions.

Let p,r ≥ 1. The p,r types of the $x \in Z^r$ are the set of all true formulas $\varphi(x)$ in (Z,<), where φ has at most p quantifiers, and no parameters are allowed.

LEMMA 3.3. Let (E,S,~) be a model of T_2, definable in (Z,<). There is a nondegenerate interval I in (E,S,~), r ≥ 1, a partial r tuple u from Z, and a p,r type σ, such that the following holds. For all nondegenerate subintervals J of I and integers t, J contains an extension of u, of p,r type σ, where the new coordinates are all > t.

Proof: Let E,S,~ be as given. If we partition E into finitely many pieces, one of the pieces must be somewhere dense in (E,S,~). Therefore we can assume without loss of generality that σ is a p,r type, I1 is a nondegenerate interval of (E,S,~), and the $x \in D^r$ of p,r type σ are dense in I1.

If possible, fix an integer t_1 and a coordinate position 1 ≤ r_i ≤ r such that the $x \in D^r$ of p,r type σ whose r_1-th coordinate is t_1 are dense in some nondegenerate subinterval I_2 of I_1.

If possible, choose an integer t_2 and another coordinate position 1 ≤ r_2 ≤ r so that the $x \in D^r$ of p,r type σ whose r_1-th coordinate is t_1 and whose r_2-th coordinate is t_2 are dense in some nondegenerate subinterval $I_3 \subseteq I_2$ of (E,S,~).

Continue in this way as long as possible. This results in a partial r tuple u and a nondegenerate subinterval I, such that the extensions of u of p,r type σ are dense in I.

Let J be a nondegenerate subinterval of I and t be an integer. Suppose J does not contain an extension of u, of p,r type σ, where the new coordinates are all > t. Then the union over the remaining coordinate positions i ≤ r of the extensions of u of p,r type σ, whose i-th coordinate is ≤ t, are dense in J. Hence one of these sets is dense in some nondegenerate subinterval of J. Therefore we have $t' \leq t$ and a remaining coordinate position $i \leq r$ such that the extensions of u of p,r type σ, whose i-th coordinate is t', are dense in some nondegenerate subinterval of J. This contradicts that we could not continue the process. Hence J does contain an extension of u, of p,r type σ, where the new coordinates are all > t. QED

LEMMA 3.4. There is no model (E,S,~) of T_2, definable in (Z,<).

Proof: Let (E,S,~) be a model of T_2 definable in (Z,<), using q quantifiers, with no parameters. Let p >> q.

Let I,r,u be as given by Lemma 3.3. Let J < K be nondegenerate subintervals of I in (E,S,~). By Lemma 3.3, let $x \in J \cap D^r$ extend u, with p,r type σ, where all new coordinates of x are >> pmax(u). By Lemma 3.3, let $y \in K \cap D^r$ extend u, with p,r type σ, where all new coordinates of y are >> pmax(x). Then x S y. By Lemma 3.3, let z be an r tuple of p,r type σ, extending u, from the interval (x,y) in (E,S,~), such that all new coordinates of z are >> pmax(x,y). By the quantifier elimination in (Z,<), we see that (x,z) and (y,z) satisfy the same q quantifier formulas without parameters, in (Z,<). Hence $x \, S \, z \leftrightarrow y \, S \, z$. But x S z S y. This is a contradiction. QED

THEOREM 3.5. T_1 is not interpretable in T_2. T_2 is not interpretable in T_1.

Proof: By Lemmas 3.2 and 3.4. QED

THEOREM 3.6. Let S be a consistent recursively axiomatized theory. There exist consistent finitely axiomatized theories T_1,T_2, both in a single binary relation symbol, such that
i) S is provable in T_1,T_2;
ii) T_1 not interpretable in T_2;
iii) T_2 is not interpretable in T_1.

For the proof of a sharper result, see Theorem 2.7 in [Fr07].

BUT, are there examples of incomparability between natural theories that are metamathematically strong? E.g., where PA is interpretable?

STARTLING OBSERVATION. ***Any two natural theories S,T, known to interpret PA, are known (with small numbers of exceptions) to have: S is interpretable in T or T is interpretable in S. The exceptions are believed to also have comparability.***

As a consequence, there has emerged a rather large linearly ordered table of “interpretation powers” represented by natural formal systems. Several natural systems may occupy the same position.

We call this growing table, the ***Interpretation Hierarchy***. See [Fr07], section 7.

4. BETTER THAN, MUCH BETTER THAN.

We use the informal notions: better than (>), and much better than (>>). These are binary relations. Passing from > to >> is an example of what we call *concept amplification*. Equality is taken for granted.

We need to consider properties of things. The properties that we consider are to be given by first order formulas. Their extensions are called "ranges of things".

When informally presenting axioms, we prefer to use "range of things" rather than "set of things", as we do not want to commit to set theory here.

We say that x is (much) better than a given range of things if and only if x is (much) better than every element of that range of things.

Note that by transitivity of better than (see Basic below), if x is better than a given range of things, then it is also better than everything that something in that range of things is better than.

We say that x is exactly better than a given range of things if and only if x is better than every element of that range of things, and everything that something in that range of things is better than, and nothing else.

Thus among the x that are better than a given range of things, the x that are exactly better are better than the fewest possible things.

We now introduce some important axiom groups.

BASIC (B). Nothing is better than itself. If x is better than y and y is better than z, then x is better than z. If x is much better than y, then x is better than y. If x is much better than y and y is better than z, then x is much better than z. If x is better than y and y is much better than z, then x is much better than z. There is something that is much better than any given x,y. If x is much better than y then x is much better than something better than y.

The following basic principle asserts that "we can find any given bounded level of goodness in a variety of ways".

DIVERSE EXACTNESS (DE). Let x be better than a given range of things. There is something that is exactly better than the given range of things, that x is not better than.

In DE, ranges of things are given by formulas in L(>,>>,=), with side parameters allowed.

Our next basic principle asserts that "if x is much better than y, and bears a relation to y, then there is no limit to how much x can be improved while still maintaining that relation to y".

UNLIMITED IMPROVEMENT (UI). Assume that x is much better than and related to y by a given binary relation. Then arbitrarily good x are related to y by the given binary relation.

In UI, binary relations are given by formulas in L(>,=) with no side parameters.

In order to obtain mutual interpretability with strong set theories, we strengthen Unlimited Improvement in the following natural way.

STRONG UNLIMITED IMPROVEMENT (SUI). Let x be much better than something, as well as everything x is related to by a given binary relation. Then arbitrarily good x are related to the same things that x is related to, by the given binary relation.

In SUI, binary relations are given by formulas in L(>,=) with no side parameters.

We place the greatest emphasis on the system MBT (much better than) = B + DE + SUI. We prove that MBT is mutually interpretable with ZF.

We isolate the following three strengthenings of Diverse Exactness.

STRONG DIVERSE EXACTNESS (SDE). Let x be much better than something better than a given range of things. Then x is better than some, but not all, things exactly better than the given range of things.

VERY STRONG DIVERSE EXACTNESS (VSDE). Let x be much better than something better than a given range of things. Then x is much better than some, but not all, things exactly better than the given range of things.

SUPER STRONG DIVERSE EXACTNESS (SSDE). Let x be much better than something, and a given range of things. Then x is better than some, but not all, things exactly better than the given range of things.

In these three, ranges of things are given by formulas in L(>,>>,=), with side parameters allowed.

We derive VSDE and UI from MBT. We also show that MBT does not derive SSDE.

We show that B + SDE is mutually interpretable with Z. Here Z is Zermelo set theory, a fragment of ZF of considerable strength. This result holds if we use VSDE or even SSDE.

We show that B + SDE + UI is mutually interpretable with ZF. We prove this result even if we use VSDE + SSDE instead of SDE.

Finally, consider B + SSDE + SUI. We show that this system is also mutually interpretable with ZF.

To avoid any ambiguities, we now present these axioms formally.

Let φ be a formula in L(>,>>,=), where y is not free in φ. We define

$y > \varphi$ iff $(\forall x)(\varphi \rightarrow y > x)$.

$y \gg \varphi$ iff $(\forall x)(\varphi \rightarrow y \gg x)$.

$y >_{ex} \varphi$ iff $(\forall z)(y > z \leftrightarrow (\exists x)(\varphi \wedge x = z \vee x > z))$.

Note that in the above, we think of $\varphi = \varphi(x)$. Thus the variable x has a special status. The other variables, y,z, are allowed to be any distinct variables other than x, that do not appear in φ.

BASIC (B). $\neg x > x$. $x > y \wedge y > z \rightarrow x > z$. $x \gg y \Rightarrow x > y$. $x \gg y \wedge y > z \rightarrow x \gg z$. $x > y \wedge y \gg z \rightarrow x \gg z$. $(\exists z)(z \gg x \wedge z \gg y)$. $x \gg y \rightarrow (\exists z)(x \gg z \wedge z > y)$.

DIVERSE EXACTNESS (DE). $y > \varphi \rightarrow (\exists z)(\neg y > z \wedge z >_{ex} \varphi)$, where φ is a formula in L(>,>>,=) in which y,z are not free.

UNLIMITED IMPROVEMENT (UI). $x \gg y \wedge \varphi \rightarrow (\forall z)(\exists w)(\varphi[x/w])$, where φ is a formula of L(>,=) with at most the free variables x,y.

STRONG UNLIMITED IMPROVEMENT (SUI). $(\exists y)(x \gg y) \wedge (\forall y)(\varphi \rightarrow x \gg y) \rightarrow (\forall z)(\exists w)(w > z \wedge (\forall y)(\varphi \leftrightarrow \varphi[x/w])$, where φ is a formula of L(>,=) with at most the free variables x,y.

STRONG DIVERSE EXACTNESS (SDE). $y \gg z > \varphi \rightarrow (\exists z)(y > z >_{ex} \varphi) \wedge (\exists z)(\neg y > z \wedge z >_{ex} \varphi)$, where φ is a formula in L(>,>>,=) in which y,z are not free.

VERY STRONG DIVERSE EXACTNESS (VSDE). $y \gg z > \varphi \rightarrow (\exists z)(y \gg z >_{ex} \varphi) \wedge (\exists z)(\neg y \gg z \wedge z >_{ex} \varphi)$, where φ is a formula in L(>,>>,=) in which y,z are not free.

SUPER STRONG DIVERSE EXACTNESS (SSDE). $y \gg \varphi \wedge (\exists x)(y \gg x) \rightarrow (\exists z)(y > z >_{ex} \varphi) \wedge (\exists z)(\neg y > z \wedge z >_{ex} \varphi)$, where φ is a formula in L(>,>>,=) in which y,z are not free.

MBT = B + DE + SUI.

We take Z to be pairing, extensionality, union, power set, separation, and infinity. It is customary to take the axiom of infinity to be in either of two forms.

The first is the same weak form that is most commonly used in ZF. It asserts the existence of a set containing the element ∅, and closed under the operation that sends x to x ∪ {x}. It is well known that this form of Z does not suffice to prove the existence of V(ω).

The second form is the stronger form based on the operation that sends x,y to x ∪ {y}. This form of Z does suffice to prove the existence of V(ω).

What happens to Russell's Paradox in this context? In sets, we start with

there is a set whose elements are exactly the sets with a given property

and obtain a contradiction that Frege missed and Russell saw. The corresponding principle here is

there is something which is better than, exactly, the things with a given property and those things they are better than.

This immediately leads to a contradiction. Even the much weaker

there is something which is better than the things with a given property

gives an immediate contradiction, because there cannot be anything which is better than all things - by irreflexivity. I.e., nothing can be better than itself.

Thus Russell's Paradox now becomes entirely transparent and never would have trapped anyone: it disappears as a Paradox. Clearly there is no residual feeling of mystery here as there is in the context of sets and properties.

5. SOME IMPLICATIONS.

We establish a number of implications, some of which are needed for section 9.

THEOREM 5.1. The following are provable in B. SSDE → SDE → DE. VSDE → SDE → DE.

Proof: Assume SSDE. Let x >> y > S. Then x >> S, and apply SSDE.

Assume SDE. Let $x > S$. Let $y >> x > S$. By SDE, let $z >_{ex} S$, $\neg z < y$. Then $\neg z < x$.

The last claim is immediate. QED

THEROEM 5.2. MBT proves UI.

Proof: Let $x >> y$ and $R(x,y)$. By B, Let $z >_{ex} \{y\}$, $\neg z < x$. Let $w >_{ex} \{x,z\}$. Note that if $x = z$ then $x >_{ex} \{y\}$, which contradicts the last axiom of B. Hence $x \neq z$.

We claim that $P(w)$ holds, where $P(w)$ asserts that

*) there exists unique b,c,d such that $w >_{ex} \{b,c\} \wedge b \neq c \wedge d < b \wedge c >_{ex} \{d\}$.

For existence, set $b = x$, $c = z$, $d = y$.

For uniqueness, let b,c,d witness *). Then $\{b,c\} = \{x,z\}$. Suppose $b = z \wedge c = x$. Let $d < z \wedge x >_{ex} \{d\}$. By $z >_{ex} \{y\}$, we have $d < y < x$, contradicting $x >_{ex} \{d\}$.

Now define $S(u,v)$ if and only if $P(u) \wedge (\exists b,c,d)$(b,c,d witnesses *) $\wedge R(b,v) \wedge v = d)$.

Obviously $(\forall v)(S(w,v) \leftrightarrow v = y)$, and so w is related to exactly y by S. Since $w >> y$, the hypothesis of SUI holds. Hence there are arbitrarily good w' related to exactly y by S.

Suppose w' is related to exactly y by S. Then $P(w')$. Let b,c,d witness *). Then $R(b,d)$, $d = y$. Also $c >_{ex} \{y\}$, $w' >_{ex} \{b,c\}$.

It suffices to show that the b's that arise from the w' that are related to exactly y by S, are arbitrarily good. Let $u > y$ be given. We can take $w' > u$. But then $u \leq b \vee u \leq c$. Hence $u \leq b$ as required. QED

THEOREM 5.3. MBT proves SSDE.

Proof: The diversity follows from DE. Let $x >> E$, where x is much better than something.

We first dispense with the case where E is empty. We assume that $(\forall w < x)(\neg w >_{ex} E)$. Define $R(u,v)$ if and only if $(\exists w < u)(w >_{ex} E)$. Then x is related to nothing by R. By SUI, there are arbitrarily good x' that are related to nothing by R. Let x' > y $>_{ex}$ E, where x' is related to nothing by R. Then x' is related to everything by R, and so we have a contradiction.

We now assume that E is nonempty.

Let y $>_{ex}$ E, ¬y < x. We assume that (∀u < x)(¬u $>_{ex}$ E), and derive a contradiction. Let z $>_{ex}$ {x,y}.

We claim that P(z) holds, where P(z) asserts that

*) there exists unique b,c such that z $>_{ex}$ {b,c} ∧ b ≠ c ∧ (∀w < c)(w < b) ∧ (∀w < b)(w,c are not equivalent))

where equivalent means "being better than the same things".

For existence, set b = x, c = y.

For uniqueness, let b,c witness *). Then {b,c} = {x,y}. Suppose b = y ∧ c = x. Then (∀u < x)(u < y). But (∀w < y)(w < x). Hence x >ex E. This is a contradiction.

Now define R(u,v) if and only if P(u) ∧ (∃b,c)(b,c witnesses *) ∧ c > v).

Note that z is related to exactly E by R, z >> E, and z is much better than something. Hence we can apply SUI. There are arbitrarily good z' which are related to exactly E by S.

Suppose z' is related to exactly E by R. Since E is nonempty, P(z') holds. Let b,c witness *). We have b ≠ c, z' $>_{ex}$ {b,c}, (∀w < c)(w < b), (∀w < b)(w,c are not equivalent). Clearly z' is related, by R, to exactly the v with c > v. Hence c >ex E. By *), (∀w < b)(¬w $>_{ex}$ E).

Now let w* > w $>_{ex}$ E. Then ¬w < b. However, we can assume that w* < z'. Hence w* ≤ b ∨ w* ≤ c, w < b ∨ w < c. Hence w < c. But then w < b, which is a contradiction. QED

6. INTERPRETATION OF MBT IN ZF.

We will use the following axioms for ZF here. The primitives are ∈,=.

EXTENSIONALITY. (∀x)(x ∈ y ↔ x ∈ z) → y = z.

PAIRING. $(\exists x)(y \in x \wedge z \in x)$.
UNION. $(\exists x)(\forall y)(\forall z)(y \in z \wedge z \in w \rightarrow y \in x)$.
SEPARATION. $(\exists x)(\forall y)(y \in x \leftrightarrow y \in z \wedge \varphi)$, where x is not free in φ.
POWER SET. $(\exists x)(\forall y)((\forall z)(z \in y \rightarrow z \in w) \rightarrow y \in x)$.
INFINITY. $(\exists x)(\emptyset \in x \wedge (\forall y,z)(y \in x \wedge z \in x \rightarrow y \cup \{z\} \in x))$.
FOUNDATION. $x \neq \emptyset \rightarrow (\exists y)(y \in x \wedge \neg(\exists z)(z \in y \wedge z \in x))$.
COLLECTION. $(\forall y)(y \in x \rightarrow (\exists z)(\varphi)) \rightarrow (\exists w)(\forall y)(y \in x \rightarrow (\exists z)(z \in w \wedge \varphi))$, where w is not free in φ.

Infinity is more usually formulated in the weaker form where the above $y \cup \{z\}$ is replaced by $y \cup \{y\}$. Replacement is more commonly used than Collection, where the above $(\exists z)(\varphi)$ in the antecedent is replaced by $(\exists! z)(\varphi)$. It is well known that when both these changes are made, the resulting system is logically equivalent to the above.

We will give an interpretation of MBT in ZF. In fact, the construction interprets MBT + VSDE + SSDE in ZF. Hence by Theorems 5.1 - 5.3, we have interpreted every axiom considered here, simultaneously in ZF.

In light of the axiom in Basic asserting that for any two things, there is something much better, it is natural to strengthen the last axiom in Basic to the following:

#) $x >> y \wedge x >> z \rightarrow (\exists w)(x >> w \wedge w > y \wedge w > z)$.

DEFINITION. Basic' is Basic where the last axiom is replaced by #). We take MBT_0' - MBT_3' to be the same as MBT_0 - MBT_3, respectively, except that we use Basic' instead of Basic.

In this section, we give an interpretation of MBT in ZF. In fact, the same interpretation interprets MBT + VSDE in ZF.

We work within ZF. We first form a transfinite hierarchy of structures $(D_\alpha,>_\alpha)$.

For any binary relation >, we define $x < y \leftrightarrow y > x$, $x \leq y \leftrightarrow x < y \vee x = y$, $x \geq y \leftrightarrow x > y \vee x = y$.

DEFINITION. Let (E,>) be transitive and irreflexive, and $K \subseteq E$. We say that x is an exact upper bound of K over (E,>) if and only if for all $y \in E$, $x > y \leftrightarrow (\exists z \in K)(z \geq y)$. We say that K is (E,>) transitive if and only if $K \subseteq E$ and for all $x,y,z \in K$, $x > y \wedge y > z \rightarrow x > z$.

DEFINITION. We define pairs $(D_\alpha,>_\alpha)$, for all ordinals α. Define $(D_0,>_0) = (\emptyset,\emptyset)$. Suppose $(D_\alpha,>_\alpha)$ has been defined, and is transitive and irreflexive. Define $(D_{\alpha+1},>_{\alpha+1})$ to extend $(D_\alpha,>_\alpha)$ by adding an exact upper bound of every $(D_\alpha,>_\alpha)$ transitive set - even if this set already has an exact upper bound over $(D_\alpha,>_\alpha)$ lying in D_α. For limit ordinals λ, define $D_\lambda = \cup_{\alpha<\lambda}D_\alpha$, $>_\lambda = \cup_{\alpha<\lambda}>_\alpha$.

It will be convenient to have a form of this definition that uses a purely set theoretic construction. This is accomplished by giving an explicit definition of the exact upper bounds.

DEFINITION'. We define pairs $(D_\alpha,>_\alpha)$, for ordinals α, where $>_\alpha \subseteq D_\alpha \times D_\alpha$. Define $(D_\alpha,>_\alpha) = (\emptyset,\emptyset)$. Suppose $(D_\alpha,>_\alpha)$ has been defined, where $>_\alpha$ is transitive and irreflexive. We set $D_{\alpha+1} = D_\alpha \cup \{(\alpha,A)\text{: A is } (D_\alpha,>_\alpha) \text{ transitive}\}$. We define $x >_{\alpha+1} y \leftrightarrow x >_\alpha y \vee (\exists A)(x = (\alpha,A) \wedge y \in A)$. For limit ordinals λ, set $D_\lambda = \cup_{\beta<\lambda}D_\beta$, $>_\lambda = \cup_{\beta<\lambda}>_\beta$.

LEMMA 6.1. Let $\alpha,\beta < \lambda$.
i. $\alpha \leq \beta \rightarrow D_\alpha \subseteq D_\beta \wedge >_\alpha \subseteq >_\beta$.
ii. $x \in D_{\alpha+1}\backslash D_\alpha \rightarrow x = (\alpha,\{y\text{: } x >_{\alpha+1} y\})$.
iii. $x \in D_\alpha \rightarrow (\exists\beta < \alpha)(x = (\beta,\{y\text{: } x >_{\beta+1} y\}))$.
iv. $x >_\alpha y \rightarrow (\exists\beta < \alpha)(y \in D_\beta)$.
v. $x,y \in D_\alpha \rightarrow (x >_\beta y \rightarrow x >_\alpha y)$.

Proof: Let $\alpha,\beta < \lambda$. Claim i is proved by fixing α and applying transfinite induction to $\beta \geq \alpha$. For claim ii, let $x \in D_{\alpha+1}\backslash D_\alpha$. Let $x = (\alpha,A)$, where A is $(D_\alpha,>_\alpha)$ transitive. Hence $A \subseteq D_\alpha$. We claim that $A = \{y\text{: } x >_{\alpha+1} y\}$. To see this, suppose $x >_{\alpha+1} y$. Now $x >_\alpha y$ is impossible since $x \notin D_\alpha$. Hence $y \in A$. Conversely, suppose $y \in A$. Then $x >_{\alpha+1} y$.

For iii, let $x \in D_\alpha$. Let α' be least such that $x \in D_{\alpha'}$. Then $\alpha' \leq \alpha$ and $x \in D_{\alpha'}\backslash D_{\alpha'-1}$. By ii, $x = (\alpha'-1,\{y\text{: } x >_{\alpha'} y\})$. Set $\beta = \alpha'-1$.

We prove iv by transfinite induction on α. For $\alpha = 0$, the statement is vacuously true since $D_0 = \emptyset$. Suppose

$$(\forall x,y)(x >_\alpha y \rightarrow (\exists\beta < \alpha)(y \in D_\beta)).$$

Let $x >_{\alpha+1} y$. If $x >_\alpha y$ then $(\exists\beta < \alpha)(y \in D_\beta)$, and so $(\exists\beta < \alpha+1)(y \in D_\beta)$. Otherwise, let $x = (\alpha,A)$, $y \in A$, where A is $(D_\alpha,>_\alpha)$ transitive. Then $y \in D\alpha$, and so $(\exists\beta < \alpha+1)(y \in D_\beta)$.

Finally, suppose λ is a limit ordinal, and for all $\alpha < \lambda$,

$$(\forall x,y)(x >_\alpha y \rightarrow (\exists\beta < \alpha)(y \in D_\beta)).$$

Let $x >_\lambda y$. Then $y \in D_\lambda$, and so $(\exists\beta < \lambda)(y \in D_\beta)$.

We prove v by fixing α, and applying transfinite induction to $\beta \geq \alpha$. For $\beta = \alpha$, the statement is trivial. Let $\beta \geq \alpha$, and suppose

$$(\forall x,y)(x,y \in D_\alpha \rightarrow (x >_\beta y \rightarrow x >_\alpha y)).$$

Let $x,y \in D_\alpha$, $x >_{\beta+1} y$. If $x >_\beta y$ then $x >_\alpha y$. Otherwise, let $x = (\beta,A)$, $y \in A$, where A is $(D_\beta,>_\beta)$ transitive. By iii, $\beta < \alpha$, which is impossible.

Finally, suppose λ is a limit ordinal, and for all $\alpha < \lambda$,

$$(\forall x,y)(x,y \in D_\alpha \rightarrow (x >_\lambda y \rightarrow x >_\alpha y)).$$

Let $x >_\beta y$, $\beta < \lambda$. Then $x >_\alpha y$. QED

LEMMA 6.2. For all α, $(D_\alpha,>_\alpha)$ is irreflexive and transitive.

Proof: We prove this by transfinite induction on α. Clearly $(D_0,>_0)$ is irreflexive and transitive. Suppose $(D_\alpha,>_\alpha)$ is irreflexive and transitive. Suppose $x >_{\alpha+1} x$. By Lemma 5.1, $x \in D_\alpha$, and so $x >_\alpha x$. This violates the irreflexivity of $>_\alpha$.

To see that $(D_{\alpha+1},>_{\alpha+1})$ is transitive. Let $x >_{\alpha+1} y$, $y >_{\alpha+1} z$, where $x,y,z \in D_{\alpha+1}$. Then $y \in D_\alpha$, $z \in D_\alpha$. If $x \in D_\alpha$ then $x >_\alpha z$, $x >_{\alpha+1} z$. Suppose $x \in D_{\alpha+1}\backslash D_\alpha$. Let $x = (\alpha,A)$, where A is $(D_\alpha,>_\alpha)$ transitive. Then $y \in A$, $y >_\alpha z$. Hence $z \in A$, and so $x >_{\alpha+1} z$. Hence $(D_{\alpha+1},>_{\alpha+1})$ is irreflexive and transitive.

Suppose for all $\beta < \lambda$, $(D_\beta,>_\beta)$ is irreflexive and transitive. Then obviously $(D_\lambda,>_\lambda)$ is irreflexive. Let $x >_\lambda y \wedge y >_\lambda z$, where $x,y,z \in D_\lambda$. Let $x,y,z \in D_\beta$, $\beta < \lambda$. Then $x >_\beta y \wedge y >_\beta z$. Therefore $x >_\beta z$. Hence $(D_\lambda,>_\lambda)$ is transitive. QED

DEFINITION. Fix S to be a nonempty set of limit ordinals, with no greatest element, whose union is the limit ordinal λ. We define M[S] to be the following structure $(D_\lambda,>_\lambda,>>_S)$.

$D_\lambda, >_\lambda$ have already been defined. We define $x >>_S y$ if and only if

$$x,y \in D_\lambda \wedge (\exists \alpha,\beta \in S)(\alpha < \beta \wedge y \in D_\alpha \wedge (\forall w \in D_\beta)(x >_\lambda w)).$$

REMARK. We could have defined $x >>_S y$ if and only if

$$x,y \in D_\lambda \wedge (\exists \alpha \in S)(\alpha < \beta \wedge y \in D_\alpha \wedge (\forall w \in D_\alpha)(x >_\lambda w)).$$

but then it can be shown that under this definition, Strong Diverse Exactness would fail in M[S].

LEMMA 6.3. M[S] satisfies Basic'.

Proof: Since $(D_\lambda, >_\lambda)$ is irreflexive and transitive, we have the first two statements in Basic. The third is obvious.

Suppose $x >>_S y$, $y >_\lambda z$. Let $\alpha < \beta$ be from S, $y \in D_\alpha$, $(\forall w \in D_\beta)(x >_\lambda w)$. Then $z \in D_\alpha$, and so $x >>_S z$.

Suppose $x >_\lambda y$, $y >>_S z$. Let $\alpha < \beta$ be from S, $z \in D_\alpha$, $(\forall w \in D_\beta)(y >_\lambda w)$. Then $(\forall w \in D_\beta)(x >_\lambda w)$, and so $x >>_S z$.

Let $y,z \in D_\lambda$. Let $y,z \in D_\alpha$. Let $\beta < \gamma$ be from S, where $\alpha < \beta$. Then $(\gamma, D_\gamma) >>_S y,z$.

Let $x >>_S y,z$. Let $\alpha_1 < \beta_1$ be from S, where $y \in D_{\alpha\_1} \wedge (\forall w \in D_{\beta\_1})(x >> w)$. Let $\alpha_2 < \beta_2$ be from S, where $z \in D_{\alpha\_2} \wedge (\forall w \in D_{\beta\_2})(x >> w)$. Let $\alpha = \max(\alpha_1,\alpha_2)$, $\beta = \max(\beta_1,\beta_2)$. Then $y,z \in D_\alpha \wedge (\forall w \in D_\beta)(x >> w)$. Since α is a limit ordinal, let $y,z \in D_\gamma$, $\gamma < \alpha$. Then $(\gamma, D_\gamma) >_\lambda y,z$. Also $x >>_S (\gamma, D_\gamma)$. QED

For SDE, we need a further condition on S. Recall that Diverse Exactness follows easily from Diverse Exactness in the presence of Basic.

LEMMA 6.4. Suppose S is of order type ω. Then M[S] satisfies VSDE + SSDE.

Proof: Let S be as given. We begin with VSDE. Let $y >>_S z > \varphi$. Let $B = \{x: \varphi(x)$ holds in M[S]$\}$. Let $\alpha < \beta$ be from S such that $z \in D_\alpha \wedge (\forall w \in D_\beta)(y >_\lambda w)$. Since α is a limit ordinal, let $z \in D_\gamma$, $\gamma < \alpha$. Then $(\gamma+1,B)$ is an exact upper bound for B. Evidently, $y >>_S (\gamma+1,B)$.

We turn to SSDE. Let $y >>_S \varphi$, and $y >>_S u$ for some u. Let B = {x: $\varphi(x)$ holds in M[S]}. First assume B = $\emptyset$. Let $\alpha < \beta$ be from S, where $u \in D_\alpha$, $(\forall w \in D_\beta)(y >_\lambda w)$. Then $y >>_S (1,\emptyset)$.

Now assume B $\neq \emptyset$. For each $x \in B$, let $\alpha_x < \beta_x$ be from S such that $x \in D_{\alpha\_x} \wedge (\forall w \in D_{\beta\_x})(y >_\lambda w)$. Then the β_x are bounded below λ, and so the β_x have a max, β. Also, the α_x are bounded below λ, and also have a max, α. Obviously $\alpha < \beta$ and $(\forall w \in D_\beta)(y >_\lambda w)$. Clearly (α,B) is an exact upper bound for B. Since $(\alpha,B) \in D_\beta$, clearly $y >_\lambda (\alpha,B)$.

Let $y \in D_\gamma$, $\gamma < \lambda$. Note that (γ,y) is also an exact upper bound for B. Clearly $\neg y >_\lambda (\beta,z)$. QED

Obviously we have given an interpretation of MBT_0' within ZF. Actually, we only need $V(\omega^2)$ for this construction, taking S = $\{\omega,\omega\times2,\omega\times3,...\}$, $\lambda = \omega^2$. Thus M[S] does not provide an interpretation of MBT_0' in Z = Zermelo set theory. We will modify M[S] so that it does provide an interpretation of MBT_0' in Z, in section 6.

We now come to Unlimited Improvement and Strong Unlimited Improvement. Here we need further conditions on S.

LEMMA 6.5. For all $k \geq 1$ there exists $r \geq 1$ such that the following holds. Suppose S has order type ω, where for all α from S, $V(\alpha)$ is an elementary submodel of $V(\lambda)$ for formulas with at most r quantifiers. Then M[S] is a model of MBT, where in Strong Unlimited Improvement, the formula φ has at most k quantifiers.

Proof: By Lemmas 6.3 and 6.4, it suffices to verify that SUI holds in M[S].
Let k,r,S be as above. Unless stated otherwise, φ is always evaluated in $(D_\lambda,>_\lambda)$. Let $\varphi(x,y)$ define the relation for SUI, for "x is related to y".

Assume that

7) $(\forall y)(\varphi(x,y) \rightarrow x >>S\ y) \wedge x >>S\ u$

For each y with $\varphi(x,y)$, choose $\alpha_y < \beta_y$ from S such that $x \in D_{\alpha\_y} \wedge (\forall w \in D_{\beta\_y})(x >_\lambda w)$. Since S has order type ω, let α be the min of the α_y, and β be the min of the β_y. If there are no y's, take α,β to be the first two elements of S. Then

8) $(\forall y)(\varphi(x,y) \rightarrow y \in D_\alpha) \wedge (\forall w \in D_\beta)(x >_\lambda w)$

Let B = {y: φ(x,y)}. Then B $\in$ V($\alpha+\omega$) $\subseteq$ V(β). In L(λ), we have

9) {y: φ(x,y)} = B $\wedge$ ($\forall$w $\in$ D_β)(x $>_\lambda$ w)

10) ($\exists$x)({y: φ(x,y)} = B $\wedge$ ($\forall$w $\in$ D_β)(x $>_\lambda$ w))

We claim that the ordinals $\beta < \lambda$ for which 10) is true in L(λ) is unbounded. If they are bounded, then their sup is definable over L(λ) from B. So by elementary substructure considerations, and the fact that B $\in$ V(β), the sup is < β, which is a contradiction.

Since the ordinals for which 10) is true is unbounded in λ, the proof is complete. QED

THEOREM 5.6. The system consisting of Basic, the four versions of Diverse Exactness, and the two versions of Unlimited Improvement, is interpretable in ZF. We can even use Basic'.

Proof: By Lemmas 6.3 - 6.5, and Theorems 5.1 - 5.3.

7. INTERPRETATION OF B + VSDE + SSDE IN Z.

Z is the system in $\in$,= with the following axioms.

EXTENSIONALITY. ($\forall$x)(x $\in$ y $\leftrightarrow$ x $\in$ z) $\rightarrow$ y = z.
PAIRING. ($\exists$x)(y $\in$ x $\wedge$ z $\in$ x).
UNION. ($\exists$x)($\forall$y)($\forall$z)(y $\in$ z $\wedge$ z $\in$ w $\rightarrow$ y $\in$ x).
SEPARATION. ($\exists$x)($\forall$y)(y $\in$ x $\leftrightarrow$ y $\in$ z $\wedge$ φ), where x is not free in φ.
POWER SET. ($\exists$x)($\forall$y)(($\forall$z)(z $\in$ y $\rightarrow$ z $\in$ w) $\rightarrow$ y $\in$ x).
INFINITY. ($\exists$x)($\emptyset$ $\in$ x $\wedge$ ($\forall$y,z)(y $\in$ x $\wedge$ z $\in$ x $\rightarrow$ y $\cup$ {z} $\in$ x)).

The above are the first six axioms (schemes) for ZF.

We now give an interpretation of B + VSDE + SSDE in Z. As remarked in section 6, the construction there requires all of the V($\omega\times$n), n < ω. Even V($\omega\times$2) is not available as a set in Z, and V($\omega\times$3) is not available as a class in Z.

LEMMA 7.1. There are definable operators *,**, with no parameters, such that the following is provable in Z. Let (E,>) be an irreflexive transitive relation, where E $\subseteq$

$V(\omega+n)$, $n < \omega$. Then $(E^*,>^{**})$ is an irreflexive transitive relation such that
i. $E^* \subseteq V(\omega+n+6)$.
ii. $E \subseteq E^*$, $> \subseteq >^{**}$.
iii. $x \in E \rightarrow (x >^{**} y \leftrightarrow x > y)$.
iv. Any finite subset of E^* has a strict upper bound in $(E^*,>^{**})$.

Proof: Let $(E,>) \in V(\omega+n)$, $n < \omega$, be irreflexive and transitive. We let E^* be E together with all ordered pairs $(E,(x_1,\ldots,x_k))$, where $k \geq 1$ and each $x_i \in E$. For $\alpha,\beta \in E^*$, we define $\alpha >^{**} \beta$ as follows.

case 1. $\alpha,\beta \in E$. Then $\alpha >^{**} \beta \leftrightarrow R(\beta,\alpha)$.

case 2. $\alpha \notin E$, $\beta \in E$. Let $\alpha = (E,(x_1,\ldots,x_k))$. Then $\alpha >^{**} \beta \leftrightarrow (\exists i)(\beta = x_i))$.

case 3. $\alpha,\beta \notin E$. Let $\alpha = (E,(x_1,\ldots,x_k))$, $\beta = (E,(y_1,\ldots,y_n))$. Then $\alpha >^{**} \beta \leftrightarrow (x_1,\ldots,x_k)$ is a (not necessarily consecutive) subsequence of $(y_1,\ldots,y_n)$.

case 4. $\alpha \in E$, $\beta \notin E$. Then $\neg\alpha >^{**} \beta$.

We represent a finite sequence $(x_1,\ldots,x_k)$ from E by a function with domain k. Thus the elements of the finite sequence are of the form $\{\{i\},\{i,x\}\}$, where $x \in A$, and so lie in $V(\omega+n+2)$. Hence the finite sequences from E all lie in $V(\omega+n+3)$. Therefore each $(E,(x_1,\ldots,x_n)) = \{\{E\},\{E,(x_1,\ldots,x_n)\}\}$ lies in $V(\omega+n+5)$. Hence $E^* \in V(\omega+n+6)$.
QED

QED

We are now ready to define pairs $(E_n,>_n)$, for all integers $n < \omega$. Define $E_0 = \emptyset$, $>_0 = \emptyset$. Suppose $(E_n,>_n)$ has been defined. We now define $(E_{n+1},>_{n+1})$.

We define $E_{n+1} = E_n^* \cup \{(n,0,A): A \text{ is } (E_n^*,>_n^{**}) \text{ transitive}\}$.
We define $x >_{n+1} y \leftrightarrow x >_n^{**} y \vee (\exists A)(x = (n,0,A) \wedge y \in A)$.

We use $(n,0,A)$ instead of (n,A) because $(n,0,A)$ is an ordered triple, and so cannot duplicate ordered pairs in the construction.

We can now create $E_\omega = \bigcup_{n<\omega} E_n$, $>_\omega = \bigcup_{n<\omega} >_n$, as proper classes in Z.

We define >> by $x >> y \leftrightarrow (\exists n)(y \in E_n^* \wedge (\forall w \in E_{n+1})(x >_\omega w))$.

We claim that Z proves that $(E_\omega,>_\omega,>>)$ forms a model of B + VSDE + SSDE. More precisely, for each axiom of B + VSDE + SSDE, Z proves that the axiom holds in $(E_\omega,>_\omega,>>)$.

The verification of Basic' is straightforward, except for the last assertion.

LEMMA 7.2. $(E_\omega,>_\omega,>>)$ satisfies Basic'.

Proof: This follows the proof of Lemma 6.3, except for the last axiom of Basic'. Let $x >> y,z$. Let $y \in E_n^*$, $(\forall w \in E_{n+1})(x >_\omega w)$. Let $z \in E_m^*$, $(\forall w \in E_{m+1})(x >_\omega w)$. Let $r = \max(n,m)$. Then $y,z \in E_r^*$, $(\forall w \in E_{r+1})(x >_\omega w)$. Let $u \in E_r^*$, $u >_r^{**} y,z$. Then $u >_\omega y,z$ and $x >> u$. QED

LEMMA 7.3. $(E_\omega,>_\omega,>>)$ satisfies SSDE.

Proof: Let $y >> \varphi$. Let $B = \{x: (\exists y)(x \geq_\omega y \wedge \varphi(x)$ holds in $(E_\omega,>_\omega,>>)$. For each $x \in B$, let $f(x)$ be least such that $x \in E_{f(x)}^*$, $(\forall w \in E_{f(x)+1})(y >_\omega w)$. Let n be the maximum of the $f(x)$. Then $(\forall w \in E_{n+1})(y >_\omega w)$. Clearly $(n,0,B)$ is an exact upper bound for all x such that $\varphi(x)$ holds in $(E_\omega,>_\omega,>>)$, over $(E_\omega,>_\omega)$. Since $(n,0,B) \in E_{n+1}$, we have $y >_\omega (n,0,B)$.

Let $y \in E_t$. Clearly $(t,0,B)$ is another exact upper bound for B, and $\neg y >_\omega (t,0,B)$. QED

LEMMA 7.4. $(E_\omega,>_\omega,>>)$ satisfies VSDE.

Proof: Let $y >> z > \varphi$. Let $z \in E_{n+1}^*$, $(\forall w \in E_{n+2})(y >_\omega w)$. Let $B = \{x: (\exists y)(x \geq_\omega y \wedge \varphi(x)$ holds in $(E_\omega,>_\omega,>>)\}$. Then $B \subseteq E_n^*$, and so $(n,0,B)$ is an exact upper bound for all x such that $\varphi(x)$ holds in $(E_\omega,>_\omega,>>)$, over $(E_\omega,>_\omega)$. Since $(n,0,B) \in E_{n+1}$, we have $y >> (n,0,B)$.

THEOREM 7.5. B + VSDE + SSDE is interpretable in Z. We can even use B'.

Proof: By Lemmas 7.2, 7.3. QED

8. INTERPRETATION OF Z IN B + SDE.

In this section, we give an interpretation of Z in B + SDE. We use the same interpretation in section 9. We will work entirely in B + SDE.

As in section 6, we formulate Z as follows.

EXTENSIONALITY. $(\forall x)(x \in y \leftrightarrow x \in z) \rightarrow y = z$.
PAIRING. $(\exists x)(y \in x \wedge z \in x)$.
UNION. $(\exists x)(\forall y)(\forall z)(y \in z \wedge z \in w \rightarrow y \in x)$.
SEPARATION. $(\exists x)(\forall y)(y \in x \leftrightarrow y \in z \wedge \varphi)$, where x is not free in φ.
POWER SET. $(\exists x)(\forall y)((\forall z)(z \in y \rightarrow z \in w) \rightarrow w \in x)$.
INFINITY. $(\exists x)(\emptyset \in x \wedge (\forall y,z)(y \in x \wedge z \in x \rightarrow y \cup \{z\} \in x))$.

DEFINITION. We define
$x < y \leftrightarrow y > x$.
$x \leq y \leftrightarrow x < y \vee x = y$.
$x << y \leftrightarrow y >> x$.
$x \geq y \leftrightarrow x > y \vee x = y$.
$x \neq y \leftrightarrow \neg x = y$.
$x \text{ inc } y \leftrightarrow \neg x \leq y \wedge \neg y \leq x$.
$x \equiv y \leftrightarrow (\forall z)(z < x \leftrightarrow z < y)$.
$x \sim \emptyset \leftrightarrow (\forall y)(\neg x > y)$.
$y >_{ex} \{x_1,\ldots,x_n\}$ if $(\forall x)(x < y \leftrightarrow x \leq x_1 \vee \ldots \vee x \leq x_n)$.
$x <^* y \leftrightarrow x < y \wedge \neg(\exists z)(x < z < y)$.
$A(x,y) \leftrightarrow y \sim \emptyset \wedge \neg y \leq x$.
$A(x,y_1,\ldots,y_n) \leftrightarrow A(x,y_1) \wedge \ldots \wedge A(x,y_n) \wedge y_1,\ldots,y_n$ are distinct.

In the above, $n \geq 1$. The "ex" in $>_{ex}$ means "exact".

LEMMA 8.1. $(\exists y)(x_1,\ldots,x_n << y)$. $(\exists y)(y >_{ex} \{x_1,\ldots,x_n\})$. $(\forall x)(\exists y_1,\ldots,y_n)(A(x,y_1,\ldots,y_n))$.

Proof: The first claim is proved by induction on n using $(\forall x,y)(\exists z)(x,y << z)$.

For the second claim, let $x_1,\ldots,x_n$. By the first claim, let $y > x_1,\ldots,x_n$. By DE, $(\exists z)(z >_{ex} \varphi)$, where φ is $x = x_1 \vee \ldots \vee x = x_n$.

The third claim is proved by induction on n. Let x be given, and let $A(x,y_1,\ldots,y_{n-1})$, where this is considered vacuously true if n = 1. Let $x' > x,y_1,\ldots,y_{n-1}$. Obviously $x' > \varphi$, where φ is $x \neq x$. By DE,

$$(\exists y_n)(\neg y_n < x' \wedge y_n >_{ex} \varphi).$$
$$(\exists y_n)(\neg y_n \leq x,y_1,\ldots,y_{n-1} \wedge y_n \sim \emptyset).$$
$$(\exists y_n)(\neg y_n \leq x \wedge y_n \neq y_1,\ldots,y_{n-1} \wedge y_n \sim \emptyset)$$
$$A(x,y_1,\ldots,y_n).$$

QED

LEMMA 8.2. Let φ be a formula of $L(>,>>,=)$ in which u,v do not appear. Suppose $y >_{ex} \varphi$, where $(\forall u,v)(\varphi[x/u] \wedge \varphi[x/v] \rightarrow \neg u < v)$. Then $(\forall x)(x <^* y \leftrightarrow \varphi)$.

Proof: Let y,φ be as given. Then

$$(\forall z)(z < y \leftrightarrow (\exists x)(\varphi \wedge z \leq x)).$$

Suppose φ. Then $x < y$. Also $\varphi[x/w] \rightarrow \neg x < w < y$. Hence $x <^* y$. On the other hand, suppose $x <^* y$. Then $x < y$, and so let $\varphi[x/u] \wedge x \leq u$. Clearly $u < y$, and so $x \leq u < y$. By $x <^* y$, we have $x = u$. Hence φ. QED

DEFINITION. $z >_{ex*} \varphi \leftrightarrow (\forall x)(x <^* z \leftrightarrow \varphi)$.

LEMMA 8.3. Suppose $w >> y > \varphi \wedge (\forall u,v)(\varphi[x/u] \wedge \varphi[x/v] \rightarrow \neg u < v)$. There exists $z < w$ such that $z >_{ex*} \varphi$.

Proof: Let w,y,φ be as given. By SDE, let $w > z >_{ex} \varphi$. By Lemma 8.2, $z >_{ex*} \varphi$. QED

LEMMA 8.4. Suppose $A(z,w) \wedge u,v \leq z \wedge x >_{ex} \{u,w\} \wedge y >_{ex} \{v,w\}$. Then $\neg x < y$. Furthermore, if $x = y$ then $u = v$. Also, $(\forall x')(x' <^* x \leftrightarrow x' = u \vee x' = w)$.

Proof: Let z,w,u,v,x,y be as given. Assume $x < y$. Then $x \leq v \vee x \leq w$, $x \leq v$, $x < y$, which is a contradiction.

Suppose $x = y$. Then $u < x$, $u < y$, $u \leq v \vee u \leq w$, $u \leq v \vee u = w$, $u \leq v \vee w \leq z$, $u \leq v$. Similarly, $v < y$, $v < x$, $v \leq u \vee v \leq w$, $v \leq u \vee v = w$, $v \leq u \vee w \leq z$, $v \leq u$. Hence $u = v$.

Suppose $x' <^* x$. Then $x' \leq u \vee x' \leq w$, $x' \leq su \vee x' = w$. If $x' < u$ then $x' < u < x$, violating $x' <^* x$. Hence $x' = u \vee x' = w$.

To see that $u <^* x$, note that $u < x$, and let $u < b < x$. Then $b \leq u \vee b \leq w$, $b \leq w$, which is a contradiction.

To see that $w <^* x$, note that $w < x$, and let $w < b < x$. Then $b \leq u \vee b \leq w$, $b \leq u$, $w < u$, $w < z$, which is a contradiction. QED

LEMMA 8.5. Let φ be a formula of $L(>,>>,=)$ in which y,z,u,v,c do not appear. Assume $A(y,z)$, $v >> u >> u' > y,z$. Then $y > \varphi \rightarrow (\exists c < v)(\forall x)(\varphi \leftrightarrow x < y \wedge (\exists x')(x,z <^* x' <^* c))$.

Proof: Let φ,y,z,u,v be as given. Let $v >> u >> u' > y,z$. Assume $y > \varphi$.

By SDE, if φ then there exists $x' >_{ex} \{x,z\}$ with $x' < u$. Furthermore, by Lemma 8.4, these various $x' < u$ are incomparable under $<$ (i.e., no x' is $<$ any other x'). Hence by Lemma 8.3, there exists c such that these various $x' < u$ are the $x' <^* c$. By Lemma 8.3, we can choose $c < v$. Then $x,z <^* x' <^* c$.

Now suppose $x < y \wedge x,z <^* x' <^* c$. Then $x' < u$. We want to show that φ. Let x^* be such that $x' >_{ex} \{x^*,z\}$, $\varphi[x/x^*]$. By Lemma 8.4, since $x <^* x'$, we have $x = x^* \vee x = z$, $x = x^* \vee z < y$, $x = x^*$, φ. QED

DEFINITION. (scheme). The 'sets' are constructs of the form $\{x: \varphi\}$, where φ is a formula in $L(>,>>,=)$, with the property that there exists y such that $(\forall x)(\varphi \rightarrow x < y)$. I.e., we require that 'sets' be bounded above. (Here y is not in φ, but φ may have any other variables). A 'set' is $<$ y if and only if every element is $< y$.

Fortunately the above schematic definition is merely an important expositional convenience because of the following definition and Lemma.

DEFINITION. We say that a 'set' B is coded by c using y,z if and only if $A(y,z) \wedge B = \{x < y: (\exists x')(x,z <^* x' <^* c)\}$. We say that a 'set' B is fully coded by c,y,z if and only if it is coded by c using y,z.

LEMMA 8.6. Let $A(y,z)$, $v >> u >> u' >> y,z$. Every 'set' $< y$ is coded by some $c < v$ using y,z. Every 'set' has a full code.

Proof: The first claim is by Lemma 8.5. The second claim follows from the first claim using Lemma 8.1 and Basic. QED

From Lemma 8.6, we see that we can quantify over 'sets', by quantifying over full codes.

DEFINITION. Let $n \geq 2$. We say that $x_1,...,x_n$ is coded by c using $y,a_1,...,a_n$ if and only if $x_1,...,x_n < y \wedge A(y,a_1,...,a_n) \wedge (\exists u_1,...,u_n)(u_1 >_{ex} \{x_1,a_1\} \wedge ... \wedge u_n >_{ex} \{x_n,a_n\} \wedge c >_{ex} \{u_1,...,u_n\})$.

LEMMA 8.7. Let $n \geq 2$, $A(y,a_1,...,a_n)$, $v >> u >> u' >> y,a_1,...,a_n$. Every $x_1,...,x_n < y$ is coded by some $c < v$ using $y,a_1,...,a_n$.

Proof: Let $n,y,a_1,...,a_n,v,u$ be as given. Let $x_1,...,x_n < y$. By SDE, , let $u_1,...,u_n < u$ be such that each $u_i >_{ex} \{x_i,a_i\}$. By SDE, $(\exists c < v)(c >_{ex} \{u_1,...,u_n\})$. QED

LEMMA 8.8. Let $n \geq 2$. Suppose $x_1,...,x_n$ is coded by c using $y,a_1,...,a_n$, and $x_1',...,x_n'$ is coded by c using $y,a_1,...,a_n$. Then each $x_i = x_i'$.

Proof: Let $n,y,a_1,...,a_n,x_1,...,x_n,x_1',...,x_n',c$ be as given. Then $A(y,a_1,...,a_n) \wedge x_1,...,x_n \leq y \wedge x_1',...,x_n' < y$. Let $u_i >_{ex} \{x_i,a_i\}$, $u_i' >_{ex} \{x_i',a_i\}$, $c >_{ex} \{u_1,...,u_n\}$, $c >_{ex} \{u_1',...,u_n'\}$.

If $a_i < u_j$ then $a_i \leq x_j \vee a_i \leq a_j$, $a_i \leq w \vee a_i = a_j$, $a_i = a_j$, $i = j$. Similarly $a_i < u_j' \rightarrow i = j$.

We have $u_i < x$, $u_i \leq u_1' \vee ... \vee u_i \leq u_n'$. Now $a_i < u_i$. Hence $u_i \leq u_j'$ implies $a_i < u_j'$, $a_i \leq x_j' \vee a_i \leq a_j$, $a_i \leq w \vee a_i \leq a_j$, $a_i \leq a_j$, $a_i = a_j$, $i = j$. Hence $u_i \leq u_i'$.

We have $u_i' < x$, $u_i' \leq u_1 \vee ... \vee u_i' \leq u_n$. Now $a_i < u_i'$. Hence $u_i' \leq u_j$ implies $a_i < u_j$, $a_i \leq x_j \vee a_i \leq a_j$, $a_i \leq w \vee a_i \leq a_j$, $a_i \leq a_j$, $a_i = a_j$, $i = j$. Hence $u_i \leq u_i'$.

We have shown that $u_i = u_i'$. We have $x_i < u_i$, $x_i < u_i'$, $x_i \leq x_i' \vee x_i \leq a_i$, $x_i \leq x_i' \vee x_i = a_i$, $x_i \leq x_i' \vee a_i \leq w$, $x_i \leq x_i'$.

We have $x_i' < u_i'$, $x_i' < u_i$, $x_i' \leq x_i \vee x_i' \leq a_i$, $x_i' \leq x_i \vee x_i' = a_i$, $x_i' \leq x_i \vee a_i \leq w$, $x_i' \leq x_i$. Hence $x_i = x_i'$. QED

DEFINITION. (scheme). Let $n \geq 2$. The n-ary 'relations' are constructs of the form $\{<x_1,...,x_n>: \varphi\}$, where φ is a formula in $L(>,>>,=)$, with the property that there exists y such that $(\forall x_1,...,x_n)(\varphi \rightarrow x < y)$. (Here y is not in φ, but

φ may have any other variables). An n-ary 'relation' is $< y$ if and only if it holds only of arguments $< y$.

Fortunately, the above schematic definition is merely an important expositional convenience because of the following definition and Lemma.

DEFINITION. Let $n \geq 2$. We say that an n-ary 'relation' R is coded by c using $y,w,a_1,\ldots,a_{n+1}$ if and only if $R = \{<x_1,\ldots,x_n>: x_1,\ldots,x_n < y \wedge (\exists c')(c'$ codes $x_1,\ldots,x_n$ using $y,a_1,\ldots,a_n \wedge c'$ lies in the 'set' coded by c using $w,a_{n+1})\}$. We say that an n-ary 'relation' R is fully coded by $c,y,w,a_1,\ldots,a_{n+1}$ if and only if it is coded by c using $y,w,a_1,\ldots,a_{n+1}$.

Note that full codes for 'sets' have 3 components. Note that for $n \geq 2$, full codes for n-ary 'relations' have n+4 components.

LEMMA 8.9. Let $n \geq 2$, $w >> z >> y,a_1,\ldots,a_n \wedge v >> u >> u' >> w,a_{n+1} \wedge A(y,a_1,\ldots,a_n) \wedge A(w,a_{n+1})$. Every n-ary 'relation' $R < y$ is coded by some $c < v$ using $y,w,a_1,\ldots,a_{n+1}$. Every n-ary 'relation' has a full code.

Proof: Let $n,v,u,w,z,y,a_1,\ldots,a_{n+1}$ be as given. Let R be an n-ary 'relation' with $R < y$.

Let B be the 'set' $\{c' < w: (\exists x_1,\ldots,x_n)(R(x_1,\ldots,x_n) \wedge c'$ codes $x_1,\ldots,x_n$ using $y,a_1,\ldots,a_n\}$. By Lemma 8.6, let $c < v$ code the 'set' B using w,a_{n+1}. We claim that R is coded by c using $y,w,a_1,\ldots,a_{n+1}$. To see this, let $R(x_1,\ldots,x_n)$. Then $x_1,\ldots,x_n < y$. By Lemma 8.7, let $c' < w$ code $x_1,\ldots,x_n$ using $y,a_1,\ldots,a_n$. Then c' lies in the 'set' coded by c using w,a_{n+1}, since $c' \in B$. Conversely, suppose $x_1,\ldots,x_n < y \wedge c'$ codes $x_1,\ldots,x_n$ using $y,a_1,\ldots,a_n \wedge c'$ lies in the 'set' coded by c using w,a_{n+1}. Then $(\exists x_1,\ldots,x_n)(R(x_1,\ldots,x_n) \wedge c'$ codes $x_1,\ldots,x_n$ using $y,a_1,\ldots,a_n)$. By Lemma 8.8, $R(x_1,\ldots,x_n)$.

The second claim follows from the first claim using Lemma 8.1 and Basic. QED

From Lemma 8.9, we see that for each $n \geq 2$, we can quantify over n-ary 'relations', by quantifying over full codes.

DEFINITION. We say that R is a pre linear ordering if and only if R is a 2-ary 'relation' such that for all $x,y,z \in fld(R)$,
i. x R x.
ii. $x R y \wedge y R z \rightarrow x R z$.
iii. $x R y \vee y R x$.

DEFINITION. Let R be a pre linear ordering. We write $x <_R y \leftrightarrow x R y \wedge \neg y R x$, $x =_R y \leftrightarrow x R y \wedge y R x$, $x \neq_R y \leftrightarrow \neg x =_R y$.

DEFINITION. A pre well ordering is a pre linear ordering where every nonempty 'subset' of its field has a least element.

DEFINITION. Let R be a pre linear ordering and $x \in fld(R)$. We write R|<x for the restriction of R to the $y \in fld(R)$ with $y <_R x$. Note that R|<x is a pre linear ordering.

DEFINITION. Let R,S be pre linear orderings. A comparison relation between R and S is an isomorphism 'relation' from R onto S, or from R onto some S|<x, or from S onto some R|<x.

LEMMA 8.10. Let R,S be pre well orderings. There is a unique comparison 'relation' between R and S. In case it is onto R|<x then x is unique up to $=_R$.

Proof: Let R,S be pre well orderings. Let x be an upper bound on the field of R and the field of S. Let y >> y' >> x. Now argue in the usual way to develop the unique comparison relation. QED

DEFINITION. We write $R \leq_{PWO} S \leftrightarrow$ R,S are pre well orderings $\wedge$ the comparison relation between R,S is from R. $R <_{PWO} S \leftrightarrow R \leq_{PWO} S \wedge \neg S \leq_{PWO} R$. $R =_{PWO} S \leftrightarrow R \leq_{PWO} S \wedge S \leq_{PWO} R$.

LEMMA 8.11. $\leq_{PWO}$ is reflexive, transitive, and connected. $<_{PWO}$ is irreflexive and transitive. $R \leq_{PWO} S \leftrightarrow R <_{PWO} S \vee R =_{PWO} S$. $R =_{PWO} S \leftrightarrow$ R,S are pre well orderings $\wedge$ the comparison relation between R,S is from R onto S. $R <_{PWO} S \leftrightarrow$ R,S are pre well orderings $\wedge$ the comparison relation between R,S is from R onto some S|<x. Let R,S be pre well orderings. Then $R <_{PWO} S \vee S <_{PWO} R \vee R =_{PWO} S$, where the disjunct is unique.

Proof: $\leq_{PWO}$ is obviously reflexive. It is transitive by using composition of binary 'relations'. It is connected by Lemma 8.11.

If R $<_{PWO}$ R then we have two different comparison relations between R,R, which is impossible. Hence $<_{PWO}$ is irreflexive. Suppose R $<_{PWO}$ S ∧ S $<_{PWO}$ T. Then R $<_{PWO}$ T by composition of 'relations'.

Suppose R $\leq_{PWO}$ S. If ¬R $<_{PWO}$ S then S $\leq_{PWO}$ R, and so R $\leq_{PWO}$ S. Conversely, obviously R $<_{PWO}$ S and R $=_{PWO}$ S separately imply R $\leq_{PWO}$ S.

Suppose R $=_{PWO}$ S. Then the comparison relation between R,S is from R, and the comparison relation between S,R is from S. If either comparison relation is not onto, then by composition, we get a comparison relation from R onto S that is not onto, or a comparison relation from S onto R that is not onto. This contradicts R $=_{PWO}$ S and the unique of comparison relations. Hence both comparison relations are onto. Conversely, if the comparison relation between R,S is from R onto S then R $\leq_{PWO}$ S ∧ S $\leq_{PWO}$ R.

Suppose R $<_{PWO}$ S. If the comparison relation between R,S is onto S then S $\leq_{PWO}$ R, which is a contradiction. If the comparison relation between R,S is from S, then S $\leq_{PWO}$ R, which is a contradiction.

Suppose the comparison relation between R,S is from R onto some S|<x. Then R $\leq_{PWO}$ S. If S $\leq_{PWO}$ R, then the comparison relation between R,S is either from R onto S or from S onto some R|<x. This violates uniqueness of the comparison relation.

Let R,S be pre well orderings. Then R $\leq_{PWO}$ S ∨ S $\leq_{PWO}$ R. Hence R $<_{PWO}$ S ∨ S $<_{PWO}$ R ∨ R $=_{PWO}$ S. If more than one disjunct holds, then we violate the irreflexivity of $<_{PWO}$. QED

LEMMA 8.12. Let R be a nonempty 6-ary 'relation' of full codes for pre well orderings. There is a pre well ordering S with a full code in R such that every pre well ordering T with a full code in R has S $\leq_{PWO}$ T.

Proof: Let S be a pre well ordering with a code in R. Look at the 'set' B of all x ∈ fld(S) such that the comparison relation between some S' with full code in R, and S, maps S' onto S|<x. If B is empty, then S is as required. Suppose

B is nonempty. Let y be an S least element of B. Let T have full code in R, where the comparison relation between T,S maps T onto S|<y. Then T is as required. QED

DEFINITION. A finite well ordering is a pre well ordering whose equality relation is identity, where every nonempty 'subset' of the field has a greatest element.

DEFINITION. A finite 'set' is the field of some finite well ordering R.

DEFINITION. We write $x \leq_{fin} y$ if and only if x,y are finite 'sets' such that there is a one-one function from x into y. $x <_{fin} y \leftrightarrow x \leq_{fin} y \wedge \neg y \leq_{fin} x$. $x =_{fin} y \leftrightarrow x \leq_{fin} y \wedge y \leq_{fin} x$.

LEMMA 8.13. Every one-one function from a finite 'set' into itself is onto. Let x,y be finite 'sets'. Then $x <_{fin} y \leftrightarrow$ there is a one-one function from x into y that is not onto.

Proof: Let R be a finite well ordering. We first prove that for all x ∈ A, every f:{y: y R x} → {y: y R x} is onto. Let x be an R least counterexample. Let f:{y: y R x} → {y: y R x} be one-one and not onto. Then f is not empty. Let u be the R greatest element of dom(f). We can obviously adjust f so that is one-one and not onto, and f(u) = u. Now delete f(u) = u, to obtain g:{y: y R u} → {y: y R u} which is one-one and not onto. This is a contradiction.

Suppose $x <_{fin} y$. Let f:x → y be one-one. If f is onto then by taking the inverse function, $y \leq_{fin} x$, which is impossible.

Suppose f:x → y is one-one but not onto. Then $x \leq_{fin} y$. If $y \leq_{fin} x$ then by composition, we obtain a one-one function from x into x which is not onto. This contradicts the first claim. QED

LEMMA 8.14. Let R,S be finite well orderings. Then $fld(R) \leq_{fin} fld(S) \leftrightarrow R \leq_{PWO} S$, $fld(R) <_{fin} fld(S) \leftrightarrow R <_{PWO} S$, $fld(R) =_{fin} fld(S) \leftrightarrow R =_{PWO} S$. If A,B are finite 'sets' and A is a proper 'subset' of B, then $A <_{fin} B$.

Proof: Let R,S be finite well orderings. Suppose $R <_{PWO} S$. Let f be the comparison map from R onto S|<z. Obviously $fld(R) \leq_{fin} fld(S)$. If $fld(S) \leq_{fin} fld(R)$ then by composition, we obtain a one-one function from fld(R) into fld(R) which

is not onto. This contradicts Lemma 8.13. Hence fld(R) $<_{fin}$ fld(S).

Suppose fld(R) $<_{fin}$ fld(S). There is a one-one function from fld(R) into fld(S) which is not onto. There cannot be a one-one function from fld(S) into fld(R), since by composition we would violate the first claim of Lemma 8.14. Hence ¬S $\leq_{PWO}$ R, and so R $<_{PWO}$ S.

Suppose fld(R) $\leq_{fin}$ fld(S). Then fld(R) $<_{fin}$ fld(S) ∨ fld(S) $\leq_{fin}$ fld(R). In the first case, R $<_{PWO}$ S. In the second case, we have one-one f:fld(R) → fld(S), g:fld(S) → fld(R). The compositions are one-one, and therefore onto. Hence f,g are onto, and so R $=_{PWO}$ S.

Suppose (x,R) ≤ (y,S). Then evidently x $\leq_{fin}$ y.

The fourth claim follows immediately from the first claim.

Now let A,B be finite 'sets', where A is a proper subset of B. Obviously A $\leq_{fin}$ B. If B $\leq_{fin}$ A then we obtain a one-one function from B into B that is not onto, violating Lemma 8.13. QED

LEMMA 8.15. Let R be a nonempty 3-ary 'relation' of full codes for finite sets. There is a finite 'set' x with a full code in R such that every finite 'set' y with a full code in R has x $\leq_{fin}$ y.

Proof: Let R be as given. Let u be a finite 'set' with a full code in R. Let S be a finite well ordering. Let B be the 'set' of all v in u such that some 'set' x with a full code in R has x $\leq_{fin}$ {w: w S v}. If B is empty then u is as required. Otherwise, let v be the R least element of B. Let x have a full code in R, where x $\leq_{fin}$ {w: w S v}. Then x is as required. QED

LEMMA 8.16. Let x >> y. Let A be a finite 'subset' of {z: z << x}. There exists z << x such that z is not in A. There exists a finite 'subset' B of {z: z << x} such that A $<_{fin}$ B.

Proof: Let x,y,A be as given. Let R be a finite well ordering with field A. We claim that

$$(\forall y \in A)(\exists z \ll x)\neg(\exists w)(w \; R \; y \wedge z \leq w).$$

This is proved by R induction on y. For the basis case, let y be R least. Choose y < z << x. Now let y ∈ A and y' be the immediate R successor of y. Let z << x, ¬(∃w)(w R y ∧ z ≤ w). If ¬z ≤ y' then keep the same z. If z ≤ y' then use z* such that y' < z* << x.

Now set y to be R greatest. Using w = y, we see that the corresponding z << x cannot lie in A.

For the second claim, choose B to be A together with any z << x that lies outside A, and apply Lemma 8.14. QED

LEMMA 8.17. There exists a pre well ordering R such that
i. R has no greatest element.
ii. For all x ∈ fld(R) and nonempty 'set' B ⊆ {y: y R x} has an R greatest element.

Proof: Let x >> y. We work with the finite 'subsets' of {z: z << x}. Let A(x,z), v >> u >> x,z. By Lemma 8.6, every 'set' < x is coded by some c < v using x,z. In particular, each finite 'subset' of {z: z << x} has a code c < v using x,z. Let R = {<c,c'>: c,c' < v, and there exists finite 'subsets' B,C of {z: z << x} which are coded by c,c' using x,z, respectively, where B $\leq_{fin}$ C}. QED

We are now ready to interpret Z. The interpreted sets will be certain systems (A,E,R,x), defined as follows.

DEFINITION. We say that (A,E,R,x) is a system if and only if A is a 'set', E,R are binary 'relations' on A, and x ∈ A, such that the following holds.
i. E is an equivalence relation with field A.
ii. E(a,b) ∧ E(c,d) ∧ R(a,c) → R(b,d).
iii. Suppose (∀x ∈ A)(R(x,a) ↔ R(x,b)). Then E(a,b).
iv. Let B ⊆ A be a nonempty coded 'set'. (∃y ∈ B)(∀z ∈ B)(¬R(z,y)).
v. Let B ⊆ A be a coded 'set'. Let x ∈ B, and B be closed under R,E predecessors. Then A ⊆ B.

Here B is closed under R,E predecessors means (∀y,z)(z ∈ B ∧ R(y,z) → y ∈ B) ∧ (∀y,z)(z ∈ B ∧ E(y,z) → y ∈ B).

Also, (A,E,R,x) is not a quadruple, but rather is a notational convenience that groups together the four items into a system.

DEFINITION. For systems (A,E,R,x), and y ∈ A, define y# be the least 'subset' of A containing y, and closed under E,R predecessors. Define
(A,E,R,x)|y to be (y#,E|y#,R|y#,y). If we write (A,E,R,x)|y then (A,E,R,x) is required to be a system and y is required to lie in A.

DEFINITION. Let (A,E,R,x) be a system. We say that c is a code for (A,E,R,x) using $y,w,a_1,...,a_7$ if and only if this holds where (A,E,R,x) is treated as the 6-ary 'relation' {<a,b,c,d,e,f>: a ∈ A ∧ E(b,c) ∧ R(d,e) ∧ f = x}. We say that $c,y,w,a_1,...,a_7$ is a full code for (A,E,R,x) if and only if c is a code for (A,E,R,x) using $y,w,a_1,...,a_7$.

LEMMA 8.18. Let (A,E,R,x) be a system. Then the R maximal elements of A are exactly the points E equivalent to x. Let y ∈ A. Then (A,E,R,x)|y is a system.

Proof: Let A,E,R,x,y be as given. It is obvious that z# is closed under E,R predecessors. Let B = {z: x ∉ z#\[x]}, where [x] is the E equivalence class of x. Then obviously [x] ⊆ B. Let x ∉ z#\[x], E(w,z). Then x ∉ w#\[x] since z# = w#. Let R(w,z). Then clearly w# ⊆ z#, by considering w# ∩ z#. Hence w#\[x] ⊆ z#\[x], and so x ∉ w#\[x]. Hence B contains [x] and is closed under E,R predecessors. So B = A.

Thus we have established that for all z in A, x ∉ z#\[x]. Therefore x is R maximal. Now let C = {z ∈ A: ¬E(z,x) ∧ z is not R maximal}. Then C contains [x] and is closed under E,R predecessors. Hence C = A. Therefore every R maximal point is E equivalent to x.

For the second claim, let y ∈ A. To see that (y#,E|y#,R|y#,y) is a system, it suffices to show that

i. every C ⊆ y# that contains [y] and is closed under E|y#,R|y# predecessors is C. This is immediate from the definition of y#.

ii. every nonempty C ⊆ y# has an R|y# minimal element. Any R minimal element is an R|y# minimal element. QED

DEFINITION. We say that S is an isomorphism relation from (A,E,R,x) onto (A',E',R',x') if and only if
i. (A,E,R,x) and (A',E',R',x') are systems.
ii. S is a binary 'relation'.

iii. $S \subseteq A \times A'$.
iv. $E(a,b) \wedge E'(c,d) \rightarrow (S(a,c) \leftrightarrow S(b,d))$.
v. $S(a,b) \wedge S(a,c) \rightarrow E'(b,c)$.
vi. $S(a,b) \wedge S(c,b) \rightarrow E(a,c)$.
vii. $(\forall a \in A)(\exists b \in A')(S(a,b))$.
viii. $(\forall b \in A')(\exists a \in A)(S(a,b))$.
ix. $S(a,b) \wedge S(c,d) \rightarrow (R(a,c) \leftrightarrow R'(b,d))$.
We write $(A,E,R,x) \approx (A',E',R',x')$ if and only if there is an isomorphism relation from (A,E,R,x) onto (A',E',R',x').

LEMMA 8.19. $\approx$ is an equivalence relation on systems. If S is an isomorphism from (A,E,R,x) onto (A',E',R',x') then S^{-1} is an isomorphism from (A',E',R',x') onto (A,E,R,x).

Proof: Left to the reader. QED

LEMMA 8.20. Let S be an isomorphism relation from (A,E,R,x) onto (A',E',R',x'). Let $a \in A$, $b \in A'$. Then $S(a,b) \leftrightarrow (\forall c)(R(c,a) \leftrightarrow (\exists d)(S(c,d) \wedge R'(d,b)))$. $S(a,b) \leftrightarrow (\forall d)(R'(d,b) \leftrightarrow (\exists c)(S(c,d) \wedge R(c,a)))$.

Proof: Let S,A,E,R,x,A',E',R',x' be as given. Let $a \in A$, $b \in A'$. The forward direction of the equivalence follows immediately from the definition. Now we assume

1) $(\forall c)(R(c,a) \leftrightarrow (\exists d)(S(c,d) \wedge R'(d,b)))$.

We claim

2) $(\forall b \in A')(\forall d)(R'(d,b) \leftrightarrow (\exists c)(S(c,d) \wedge R(c,a)))$.

To see this, suppose $R'(d,b)$. Let $S(c,d)$. By 1), $R(c,a)$. Suppose $S(c,d) \wedge R(c,a)$. By 1), let $S(c,d') \wedge R'(d',b)$. Then $E(d,d')$, $R'(d,b)$. This establishes 2).

Let $S(a,b')$. By 2),

3) $(\forall d)(R'(d,b') \leftrightarrow (\exists c)(S(c,d) \wedge R(c,a)))$.
4) $(\forall d)(R'(d,b) \leftrightarrow (\exists c)(S(c,d) \wedge R(c,a)))$.

Hence $E'(b,b')$, $S(a,b)$.

For the second claim, by Lemma 8.19, S^{-1} is an isomorphism from (A',E',R',x') onto (A,E,R,x). Apply the first claim to S^{-1}. QED

LEMMA 8.21. Let S be an isomorphism relation from (A,E,R,x) onto (A',E',R',x'). Then S(x,x').

Proof: Let S,A,E,R,x,A',E',R',x' be as given. By Lemma 8.18, x is R maximal. Let S(x,y). We claim that y is R' maximal. Suppose R'(y,z). Let S(w,z). Then R(x,w), violating the R maximality x. By Lemma 8.18, E'(y,x'). Hence S(x,x'). QED

LEMMA 8.22. Let S be an isomorphism relation from (A,E,R,x)|y onto (A',E',R',x')|y'. Let S' be an isomorphism relation from (A,E,R,x)|z onto (A',E',R',x')|z'. Then for all a ∈ dom(S) ∩ dom(S'), (∀b)(S(a,b) ↔ S'(a,b)).

Proof: Let S,A,E,R,x,y,A',E',R',x',y',S',z,z' be as given. Let a be R minimal such that a ∈ dom(S) ∩ dom(S'), (∃b)(S(a,b) ↔ ¬S'(a,b)). Fix b with S(a,b) ↔ ¬S'(a,b). By Lemma 8.20,

1) S(a,b) ↔ (∀c)(R(c,a) ↔ (∃d)(S(c,d) ∧ R'(d,b))).
2) S'(a,b) ↔ (∀c)(R(c,a) ↔ (∃d)(S(c,d) ∧ R'(d,b))).

It suffices to prove that the right sides of 1),2) are equivalent. This is clear because if R(c,a), then c ∈ dom(S) ∩ dom(S'), and (∀d)(S(c,d) ↔ S'(c,d)). QED

LEMMA 8.23. Let S be an isomorphism relation from (A,E,R,x) onto (A',E',R',x')|y', and S' be an isomorphism relation from (A,E,R,x) onto (A',E',R',x')|z'. Then E(y',z') and S = S'.

Proof: Let S,S',A,E,R,x,A',E',R',x',y',z' be as given. Apply Lemma 8.22 to (A,E,R,x)|x and (A,E,R,x)|x, obtaining S = S'. By Lemma 8.21, E(y',z'), S = S'. QED

LEMMA 8.24. Let (A,E,R,x) be a system. Then y ∈ A ∧ ¬E(y,x) ↔ (∃z)(R(z,x) ∧ y ∈ z#).

Proof: Let A,E,R,x be as given. For the forward direction, it suffices to show that B = {y ∈ A: E(y,x) ∨ (∃z)(R(z,x) ∧ y ∈ z#)} contains [x] and is closed under E,R predecessors. Closure under E predecessors is obvious. Now let R(u,y), y ∈ B. If E(y,x) then R(u,x) ∧ u ∈ x#, and so u ∈ B. Suppose R(z,x) ∧ y ∈ z#. Then u ∈ z#, and so u ∈ B.

Conversely, let R(z,x) ∧ y ∈ z#. Suppose E(y,x). Then R(z,y), y ∈ z#, z ∈ z#, contradicting Lemma 8.18. QED

LEMMA 8.25. Suppose $(\forall a)(R(a,x) \rightarrow (\exists a')(R'(a',x') \wedge (A,E,R,x)|a \approx (A',E',R',x')|a'))$, $(\forall a')(R'(a',x') \rightarrow (\exists a)(R(a,x) \wedge (A',E',R',x')|a' \approx (A,E,R,x)|a))$. Then $(A,E,R,x) \approx (A',E',R',x')$.

Proof: Let A,E,R,x,A',E',R',x' be as given. By Lemma 8.23, the a' in the first hypothesis is unique up to E'. For each a such that R(a,x), let T_a be the unique isomorphism from (A,E,R,x)|a onto (A',E',R',x')|a', where a' is uniquely determined up to E'. The various T_a cohere, according to Lemma 8.22. Let T be the union of the various T_a. Let T* be T extended by {{<y,z>: E(x,y) ∧ E'(x',z)}.

By the coherence, T respects E,E', and is univalent up to E'. By the second hypothesis, T is onto the a' with R'(a',x').

Note that dom(T) is the union of the a#, R(a,x), which, by Lemma 8.24, is A\[x]. Similarly, rng(T) = A'\[x']. Hence T* also respects E,E', is univalent up to E' and is onto A'.

Suppose u,v ∈ A\[x]. Let T(u,u'), T(v,v'). We claim that

$$R(u,v) \leftrightarrow R'(u',v').$$

To see this, let u ∈ a#, R(a,x), v ∈ b#, R(b,x). Suppose R(u,v). Then u ∈ b#, and so u,v ∈ dom(T_b). By coherence, $T_b(u,u')$, $T_b(v,v')$. Hence R'(u',v'). The converse is analogous.

We also claim that

$$R(u,x) \leftrightarrow R'(u',x')$$
$$R(x,v) \leftrightarrow R'(x',v').$$

Suppose R(u,x). Let u* be such that R'(u*,x') and (A,E,R,x)|u ≈ (A',E',R',x')|u*. Then $T_u(u,u^*)$, and so T(u,u*). By coherence, E'(u',u*). Hence R(u',x').

Note that x ∉ a#, x' ∉ b#, making the second equivalence vacuously true. This is because if x ∈ a# ∨ x' ∈ b#, then x is a maximal element of a# ∨ x' is a maximal element of b#, violating Lemma 8.18. From this we see that T is an isomorphism relation from (A,E,R,x) onto (A',E',R',x'). QED

DEFINITION. We write (A,E,R,x) ∈' (A',E',R',x') if and only if (∃y)(R'(y,x') ∧ (A,E,R,x) ≈ (A',E',R',x')|y).

DEFINITION. The interpretation of L(∈,=) is as follows. The sets are the systems (A,E,R,x). The interpretation of ∈ is ∈'. The interpretation of = is ≈.

We now show that the interpretations of the axioms of Z hold. Recall that we are working in B + SDE. So we are going to show that the interpretations of the axioms of Z are provable in B + SDE.

LEMMA 8.26. The interpretation of Extensionality holds.

Proof: Straightforward, using Lemma 8.25. QED

LEMMA 8.27. Let P be a property of full codes of systems whose fields are < y. (We do not require that P is bounded). There exists a system (A,E,R,x) such that every (A',E',R',x') with a full code with property P has (A',E',R',x') ∈' (A,E,R,x), and every (A,E,R,x)|u, R(u,x), is isomorphic to a system with a full code with property P.

Proof: Let P,y be as given. Let $w >> z >> y,a_1,...,a_6 \wedge v >> u >> w,a_7 \wedge A(y,a_1,...,a_6) \wedge A(w,a_7)$. By Lemma 8.9, every 6-ary 'relation' < y is coded by some c < v using $y,w,a_1,...,a_7$. In particular, each 6-ary 'relation' representing a system whose field is < y is coded by some c < v using $y,w,a_1,...,a_7$.

Let A be the 'set' of all codes c < v using $y,w,a_1,...,a_7$, of systems (A',E',R',x')|y', where (A',E',R',x') has a full code with property P. We build a system with field A ∪ {y}.

Let E(c,c') if and only if c,c' ∈ A ∪ {y} ∧ (c = c' = y ∨ the systems coded by c,c' using $y,w,a_1,...,a_7$ are isomorphic).

Let R(c,c') if and only if c,c' ∈ A ∪ {y} ∧ ((c' = y ∧ $P(c,y,w,a_1,...,a_7)$) ∨ (c,c' ∈ A ∧ the systems coded by c,c' using $y,w,a_1,...,a_7$ bear the relation ∈')).

The system (A,E,R,y) is as required. QED

LEMMA 8.28. The interpretation of Separation holds.

Proof: Let (∃a)(∀b)(b ∈ a ↔ b ∈ c ∧ φ) be an instance of Separation, where a is not free in φ. Of course, parameters are allowed in φ. Let c be interpreted as the system (A,E,R,x). Let P be the property of full codes of systems (A',E',R',x') ∈' (A,E,R,x) that obey the interpretation of φ. Since the interpretation of φ involves only ≈ and ∈', we see that if two systems (A',E',R',x'),(A'',E'',R'',x'') ∈' (A,E,R,x) are isomorphic, then (A',E',R',x') has a full code with P if and only if (A'',E'',R'',x'') has a full code with P.

Let P* be P restricted to full codes of systems (A,E,R,x)|z with R(z,x). Let y > A. By Lemma 8.27, let (A*,E*,R*,x*) be a system such that every (A,E,R,x)|z with R(z,x) has (A,E,R,x)|z ∈' (A*,E*,R*,x*), and every (A*,E*,R*,x*)|u, R*(u,x*), is isomorphic to some (A,E,R,x)|z, R(z,x). Then (A*,E*,R*,x*) witnesses the interpretation of this instance of Separation. QED

LEMMA 8.29. The interpretation of Pairing holds.

Proof: Let (A,E,R,x),(A',E',R',x') be systems. Let y > A,A'. Let P hold of the full codes of (A,E,R,x),(A',E',R',x'). Now apply Lemma 8.27. This creates a witness for Pairing applied to these two systems. QED

LEMMA 8.30. The interpretation of Union holds.

Proof: Let (A,E,R,x) be a system. Let P hold of the full codes of the (A,E,R,x)|y such that for some z, R(z,x) ∧ R(y,z). Let A < y, and apply Lemma 8.27. This creates a system which witnesses Union applied to (A,E,R,x). QED

LEMMA 8.31. Let (A,E,R,x),(A',E',R',x') be systems. Suppose that for all systems (A*,E*,R*,x*) ∈' (A,E,R,x), we have (A*,E*,R*,x*) ∈' (A',E',R',x'). Then (A,E,R,x) is isomorphic to some system whose domain is a 'subset' of A'.

Proof: Let A,E,R,x,A',E',R',x' be as given. Let B be the 'set' consisting of the y such that R'(y,x') and (A',E',R',x')|y is isomorphic to some (A*,E*,R*,x*) ∈' (A,E,R,x). Let C be the 'set' consisting of the b that lie in the field of some (A',E',R',x')|y, y ∈ B. Let u > A,E,R,x,A',E',R',x'. Then (C ∪ {x'},E**,R**,x') is the desired system, where E** is E restricted to C ∪ {x'}, and R** is the binary 'relation' given by

$R^{**}(c,d) \leftrightarrow (R'(c,d) \wedge c,d \in C) \vee (c \in B \wedge d = x')$.

By Lemma 8.25, $(A,E,R,x) \approx (C \cup \{x'\},E^{**},R^{**},x')$. QED

LEMMA 8.32. The interpretation of Power Set holds.

Proof: Let (A,E,R,x) be a system. By Lemma 8.30, it suffices to form a system (A',E',R',x') such that for all systems α whose domain is a 'subset' of A, we have α ∈' (A',E',R',x'). This is clear by Lemma 8.31. QED

The following will be used in section 9.

LEMMA 8.33. The interpretation of Foundation holds. In fact, the interpretation of the schematic form of Foundation holds.

Proof: Let $(\exists x)(\varphi)$, where φ is a formula in ∈,=. Let the variable x be interpreted as the system (A,E,R,x). Let z be R minimal such that the interpretation of φ holds at (A,E,R,x)|z. Note that the interpretation of φ involves only ≈ and ∈'. Hence (A,E,R,x)|z witnesses foundation for $(\exists x)(\varphi)$. QED

The following is not needed, but is of interest.

LEMMA 8.34. The interpretation of $(\forall x)(\exists y)(x \subseteq y \wedge y$ is transitive) holds.

Proof: Let (A,E,R,x) be given. Then (A,E,R',x) is as required, where $R'(a,b) \leftrightarrow R(a,b) \vee (a \in A \wedge b = x)$. QED

LEMMA 8.35. The interpretation of Infinity holds.

Proof: By Lemma 8.17, let R be a pre well ordering with no greatest element, where every nonempty 'subset' of any {y: R(y,x)} has an R greatest element. Because of Lemmas 8.6 and 8.9, we are in the standard situation with the second form of the Peano axioms for the natural numbers. Therefore we can build arithmetic and definitions by induction. It is then straightforward to build a copy $(\text{fld}(R),S,=_R)$ of the set theorist's (V(ω),∈,=), where S is a binary 'relation' on fld(R) that respects $=_R$. We can put this in the form of a system $(\text{fld}(R),=_R,S,x)$, which will witness Infinity under our interpretation. QED

LEMMA 8.36. The interpretation of all axioms of Z hold. In addition, the interpretations of Foundation (scheme) and Transitive holds.

Proof: By Lemmas 8.26, 8.28, 8.29, 8.30, 8.32, 8.33, 8.34, 8.35. QED

THEOREM 8.37. Z is interpretable in B + SDE. In fact, Z + Foundation (scheme) + Transitive is interpretable in B + SDE.

Proof: By Lemma 8.36 and 8.34. QED

9. INTERPRETATION OF ZF IN MBT.

We now show that ZF is interpretable in B + SDE + UI.

By Lemma 8.36, we have only to prove the interpretation of Collection in B + SDE + UI - using the same interpretation of L(>,>>,=) in L(∈,=) as we used in section 8.

By Theorems 5.2 and 5.3, MBT proves B + SDE + UI. Hence in this section we will have established that the interpretation of section 8 is also an interpretation of ZF in MBT.

Before we prove the interpretation of Collection from ZF, we first show that B + SDE + UI proves a form of Collection in L(>,=).

LEMMA 9.1. Let $n \geq 2$. Suppose $x_1,\ldots,x_n$ is coded by c using $y,a_1,\ldots,a_n$. Then $(\forall u)((\exists v)(u <^* v <^* c) \leftrightarrow u = x_1 \vee \ldots \vee u = x_n \vee u = a_1 \vee \ldots \vee u = a_n)$.

Proof: Let $n,x_1,\ldots,x_n,c,y,a_1,\ldots,a_n$ be as given. Then $x_1,\ldots,x_n \leq y$, $A(y,a_1,\ldots,a_n)$. Also, let $u_1 >_{ex} \{x_1,a_1\} \wedge \ldots \wedge u_n >_{ex} \{x_n,a_n\} \wedge y >_{ex} \{u_1,\ldots,u_n\}$, where $A(y,a_1,\ldots,a_n)$.

We claim that the u's are incomparable under ≤. To see this, suppose $u_i \leq u_j$. Then $a_i < u_j$, $a_i \leq x_j \vee a_i \leq a_j$, $a_i \leq a_j$, $a_i = a_j$, which is a contradiction.

It follows that $v <^* c \leftrightarrow v = u_1 \vee \ldots \vee v = u_n$. Also for each i, $u <^* u_i \leftrightarrow u = x_i \vee u = a_i$, using $u_i >_{ex} \{x_i,a_i\}$ and $A(y,a_1,\ldots,a_n)$. QED

COLLECTION (>,=). $(\forall y_1,\ldots,y_n < x)(\exists z)(\varphi) \rightarrow (\exists w)(\forall y_1,\ldots,y_n < x)(\exists z < w)(\varphi)$, where $n \geq 1$ and φ is a formula of L(>,=) in which w is not free.

LEMMA 9.2. B + SDE + UI proves each instance of Collection (>,=).

Proof: Let n,φ be as given. We can assume that the free variables of φ are among $x_1,\ldots,x_m,y_1,\ldots,y_n,z$, which are distinct variables, and distinct from x,w.

Fix $x,x_1,\ldots,x_m$ such that the implication fails. Let $x' > x,x_1,\ldots,x_m$. Let $A(x',a_1,\ldots,a_{m+n+1})$. By Lemma 8.7, let v be such that

$$(\forall y_1,\ldots,y_n < x')(\exists c < v)$$
$$(c \text{ codes } x_1,\ldots,x_m,y_1,\ldots,y_n \text{ using } x',a_1,\ldots,a_{m+n}).$$

Let w >> v. Since the implication fails, let

$$y_1,\ldots,y_n < x.$$
$$(\exists z)(\varphi(x_1,\ldots,x_m,y_1,\ldots,y_n,z)).$$
$$\neg(\exists z < w)(\varphi(x_1,\ldots,x_m,y_1,\ldots,y_n,z)).$$

Now let c code $x_1,\ldots,x_m,y_1,\ldots,y_n$ using $x',a_1,\ldots,a_{m+n}$. By Lemma 9.1, we can define $x_1,\ldots,x_m,y_1,\ldots,y_n,a_1,\ldots,a_{m+n}$ from c, after repetitions are removed. However, we may not be able to define them in any particular order. So we say that we can define $\{x_1,\ldots,x_m,y_1,\ldots,y_n,a_1,\ldots,a_{m+n}\}$ from c.

Now let t be the number of m+n tuples $\alpha_1,\ldots,\alpha_{m+n}$ from $\{x_1,\ldots,x_m,y_1,\ldots,y_n,a_1,\ldots,a_{m+n}\}$ such that

$$\neg(\exists z < w)(\varphi(\alpha_1,\ldots,\alpha_{m+n},z).$$

I.e., the statement $\varphi(c,w)$ =

there are exactly t tuples $\alpha_1,\ldots,\alpha_{m+n}$ from $\{x_1,\ldots,x_m,y_1,\ldots,y_n,a_1,\ldots,a_{m+n}\}$ such that $\neg(\exists z < w)(\varphi(\alpha_1,\ldots,\alpha_{m+n},z)$

holds. Note that t is a standard integer, and not a parameter in the above. So we can apply Unlimited Improvement. Since w >> y, there are arbitrarily good w' > w such that $\varphi(c,w')$. For w' > w, the count must be at most t.

In fact, for sufficiently good w' > w, there is a choice of $\alpha_1,\ldots,\alpha_{m+n}$, namely $x_1,\ldots,x_m,y_1,\ldots,y_n$, such that

$$\neg(\exists z < w)(\varphi(\alpha_1,\ldots,\alpha_{m+n+1},z)$$

goes from true to false for w = w'. This is because $(\exists z)(\varphi(x_1,\ldots,x_m,y_1,\ldots,y_n,z))$. Therefore, for sufficiently good w' > w, the count is at least t+1. This is a contradiction. QED

LEMMA 9.3. The interpretation of Collection (∈,=) is provable in B + SDE + UI.

Proof: Let (A,R,E,x) be a system. Assume that

1) for all systems (A',E',R',x') ∈' (A,E,R,x), there exists a system (A*,E*,R*,x*) such that P((A',E',R',x'),(A*,E*,R*,x*)),

where P is expressed in terms of ∈',≈ on systems, with some systems used as parameters.

Let y > A. Let $w >> z >> y,a_1,\ldots,a_6 \wedge v >> u >> w,a_7 \wedge A(y,a_1,\ldots,a_6) \wedge A(w,a_7)$. By Lemma 7.9, every 6-ary 'relation' < y is coded by some c < v using $y,w,a_1,\ldots,a_7$. In particular, each 6-ary 'relation' representing a system whose field is < y is coded by some c < v using $y,w,a_1,\ldots,a_7$.

We have

2) for each c < v that codes a system (A,E,R,x)|z, R(z,x), using $y,w,a_1,\ldots,a_7$, there exists a full code for a system (A*,E*,R*,x*) such that P((A,E,R,x)|z,(A*,E*,R*,x*)).

By Collection (>,=), let w be such that

3) for each c < v that codes a system (A,E,R,x)|z, R(z,x), using $y,w,a_1,\ldots,a_7$, there exists a full code < w for a system (A*,E*,R*,x*) such that P((A,E,R,x)|z,(A*,E*,R*,x*)).

Using the bound w, and Lemma 7.27, we can construct a witness for the interpretation of this instance of Collection (∈,=) with the given system parameters. QED

THEOREM 9.4. ZF is interpretable in B + SDE + UI, and in MBT.

Proof: From Theorem 8.37, Lemmas 9.2, 9.3, and Theorems 5.1, 5.2. QED

10. SOME FURTHER RESULTS.

We have not determined most but not all of the implications between subsets of B, DE, SDE, VSDE, SSDE, UI, SUI. E.g., does MBT prove VSDE?

We have also considered some other variants. Here we will only give brief sketches of a few results. Full proofs will appear in [Frxx].

Suppose we eliminate the diversity in VSDE and SSDE.

VERY STRONG EXACTNESS (VSE). Let x be much better than something better than a given range of things. Then x is much better than something exactly better than the given range of things.

SUPER STRONG EXACTNESS (SSE). Let x be much better than something, and a given range of things. Then x is better than something exactly better than the given range of things.

We claim that $(\omega^{\omega+1},>,>>)$ is a model of B + VSE + SSE + UI + SUI, where we define $\alpha >> \beta \leftrightarrow \alpha > \beta + \omega^{\omega}$. This is evident except for UI and SUI. But here we can use an elimination of quantifiers for small ordinals. This provides an interpretation of B + VSE + SSE + UI + SUI within EFA = exponential function arithmetic, or, equivalently $I\Sigma_0(\exp)$. In fact, we obtain a consistency proof of B + VSE + SSE + UI + SUI within EFA.

To obtain a fragment that is mutually interpretable with PA = Peano Arithmetic, we need only use >,=. Drop the last axiom of B, and use

DIVERSE EXACTNESS (in >,=). $y > \varphi \rightarrow (\exists z)(z >_{ex} \varphi \wedge \neg z < y)$, where φ is a formula in L(>,=) in which y,z are not free.

If we strengthen MBT by replacing L(>,=) with L(>,>>,=) in SUI, then we obtain an inconsistent system. If we strengthen B + SDE + UI by replacing L(>,=) with L(>,>>,=) in UI, then we also obtain an inconsistent system.

There are substantial connections between these systems and mereology. For background on mereology, see [Ho09], [Si87], and [Va07].

Specifically, we can read x > y as "y is a proper part of x", and read x >> y as "y is an infinitesimal part of x".

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