NOTE: This talk was prepared at the requrest of the organizers of the Gödel Centenary, in case Professor Georg Kreisel was unable to deliver his talk. Professor Kreisel gave his talk as scheduled, and this talk was not devliered. Excerpts were presented at our regularly scheduled later in the meeting.

REMARKS ON GÖDEL PHENOMENA AND THE FIELD OF REALS

by
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Gödel Centenary
April 27, 2006

I wish to make some remarks on the Gödel phenomena generally, and on the Gödel phenomena within the field of real numbers.

A lot of the well known impact of the Gödel phenomena is in the form of painful messages telling us that certain major mathematical programs cannot be completed as intended. This aspect of Gödel – the delivery of bad news –is not welcomed, and defensive measures are now in place:

1. <u>In Decision Procedures</u>. “We only really wanted a decision procedure in less generality, closer to what we have worked with successfully so far. Can you do this for various restricted decision procedures?”

2. <u>In Decision Procedures</u>. “We only really wanted a decision procedure in less generality, closer to what we have worked with successfully so far. Here are restricted decision procedures covering a significant portion of what we are interested in.”

3. <u>In Incompleteness</u>. “This problem you have shown is independent is too set theoretic, and pathology is the cause of the indepen-dence. When you remove the pathology by imposing regularity conditions, it is no longer independent.”

4. <u>In Incompleteness</u>. “The problem you have shown is independent has no pathology, but was not previously worked on by mathematicians. Can you do this for something we are working on?”

Of these, number 3 is most difficult to answer, and in fact is the one where I have real sympathy. So I will focus on 1,2,4.

I regard these objections as totally natural and expected.

When the Wright Brothers first got a plane off the ground for long enough to qualify as “flight”, obvious natural and expected reactions are”

Can it be sustained to really go somewhere?
If it can go somewhere, can it go there in a reasonable amount of time?
If it can go there in a reasonable amount of time, can it go there safely?
If it can go there safely, can it go there economically?

The answer to these and many other crucial questions, is YES. In fact, a bigger, more resounding YES then could have ever been imagined at that time.

But to establish yes answers, there had to be massively greater amounts of effort by massively more people, involving massive amounts new science and engineering, than were involved in the original breakthrough.

And so it is with much of Gödel. To reap anything like the full consequences of his great insights, it is going to take far greater efforts over many years than we have seen. Consider Diophantine equations. A decision procedure for Diophantine equations over Z or Q has been one of the holy grails of mathematics. We know that this is impossible for Z and suspect it is impossible for Q.

Already this bad news represents a rather substantial body of work by many people over many years, far more than what it took for Gödel to show this for some class of almost Diophantine equations over Z.

The number of variables needed presently for this is 9. For 9, the degrees needed are also gigantic.

An absolutely fantastic improvement would be, say, that the Diophantine equations over Z with 5 variables of degree at most 10 is recursively unsolvable.

Want to get very dramatic? Cubics in three variables.

Such things, assuming they are true, will take massively more effort than has been devoted thus far.

Specifically, nobody thinks that the present undecidability proof for Diophantine equations over Z is even remotely the “right” proof. Yet there has not been a serious change in this proof since 1970, when it appeared.

Yet, of the so called “leading logicians” in the world, how many have made a sustain-ed effort to find a better proof? How many of the lead-ing recursion theorists, set theorists, proof theorists, foundationalists, and, yes, model theorists? Almost none.

We now turn to the incompleteness phenomena. The fact that the plane flies at all comes from the original Gödel first incompleteness theorem. That you can fly somewhere comes from the Gödel second incompleteness theorem and Gödel/Cohen work. Upon reflection after many years, we now realize that we want very considerable flexibility in where we can fly.

In fact, there will be a virtually unending set of stronger and stronger requirements as to where we want to go with incompleteness.

I have merely scratched the surface of non set theoretic destinations for incomplete-ness, for 40 years. Almost alone – I started in the late 60’s: in 1977 I was not alone (Paris/Harrington for PA).

The amount of effort devoted to unusual destinations for the incompleteness phenomena is trivial. Well, ***I*** might be exhausted from working on this, but what does that amount to compared to, say, the airline industry after the Wright Brothers? Zero.

Most of my efforts have been towards finding that single mathematically dramatic Π^0_1 sentence whose proof requires far more than ZFC. Recently, I have shifted to searching for mathematically dramatic finite sets of Π^0_1 sentences all of which can be settled only by going well beyond the usual axioms of ZFC.

In the detailed work, perfection remains elusive. So far, the Π^0_1 (and other very concrete) statements going beyond ZFC still have a bit of undesirable detail. There is continually less and less undesirable detail. The sets of Π^0_1 sentences clearly have substantially less undesirable detail.
I strongly believe in this:

Every interesting substantial mathematical theorem can be recast as one among a natural finite set of statements, all of which can be decided using well studied extensions of ZFC, but not within ZFC itself.

Recasting of mathematical theorems as elements of natural finite sets of sstatements represents an inevitable general expansion of mathematical activity. This applies to any standard mathematical context. This program has been carried out, to some very limited extent, by BRT - details will be presented Saturday.

Now concerning the issue of: who cares if it is independent if it wasn't worked on before you showed it independent?

In my own feeble efforts on Gödel phenomena, sometimes it was worked on before. Witness Borel determinacy (Martin), Borel selection (Debs/Saint Raymond), Kruskal's tree theorem (J.B. Kruskal), and the graph minor theorem (Robertson/Seymour).

Mathematics as a professional activity with serious numbers of actors, is quite new. Let's say 100 years old - although that is a stretch.

Assuming the human race thrives, what is this compared to, say, 1000 more years? Probably a bunch of minor trivialities in comparison.

Now 1000 years is absolutely nothing. A more reasonable number is 1M years. And what does our present mathematics look like compared to that in 1M years time?

There is not even the slightest expectation that what we call mathematics now would be even remotely indicative of what we call mathematics in 1M years time. The same can be said for our present understanding of the Gödel phenomena.

Of course, 1M years time is also absolutely nothing. This Sun has several billion good years left. Mathematics in 1B years time?? I’m speechless.

I now come to the field of real numbers. The well known decision procedure of Tarski is often quoted as a deeply appreciated safe refuge from the Gödel phenomena.

However, a very interesting and modern close look reveals the Gödel phenomena in force.

It is known that the theory of the reals is nondeterministic exponential time hard, and exponential space easy. I have not heard that the gap has been eliminated.

This lower bound is proved in a Gödelian way, drenched with Turing machines and interpretations and arithmetizations.

Furthermore, there is another aspect that is also very Gödelian: lengths of proofs. I think that the least length or size of the proof/ refutation of any sentence in the field of reals has a double exponential upper and lower bound.

We can go further. Given a sentence in the field of reals, what can we say about the least length/size of a proof/refutation in ZFC? This has an exponential lower bound.

In fact, the situation even supports the kind of Gödelian project underway for the usual systems of f.o.m.

For instance, we can ask for a short sentence in the field of reals all of whose proofs in ZFC with abbreviation power are ridiculously long. We can even more ambitiously ask that the sentence be of clear mathematical interest.

The computational complexity of the field of reals is just barely high enough to support such results.

However, the “tameness” of the field of reals is most commonly applied not to sentences in primitive notation, but rather to sentences with mathematically convenient abbreviations.

For example, we can add quantifiers over all polynomials in n variables of degree $\leq d$, for every fixed n,d. Or we can add quantifiers over all semialgebraic functions in n dimensions, made up of $\leq r$ algebraic components of degree $\leq d$, with n,r,d fixed.

Presumably the resulting complexity will be far higher, involving a substantial increase in the height of the exponential stack.