Kernel Structure Theory

by
Harvey M. Friedman
Ohio State University
August 29, 2010

ABSTRACT

One of the major directions of research not touched on in the BRT book is

to find an explicitly Π^0_1 sentence which can only be proved using large cardinals, and which arguably represents clear and compelling information in the finite mathematical realm.

We have been recently engaged in this search, and have announced a long series of successively simpler and more convincing examples. See [Fr09-10].

We present a recent version of this work in progress.

1. Digraphs, Kernels, Downward, Order Invariance.
2. Kernel Closure Theorem.
3. Kernel Closure Templates.
4. Kernel Tower Theorems.
5. Kernel Tower Templates.
6. Towards Unification with BRT.

1. Digraphs, Kernels, Downward, Order Invariance.

A digraph is a pair G = (V,R), where R ⊆ V × V. The vertices of G are the elements of V, and the edges of G are the elements of R. We say that G is a digraph on V.

We say that E is independent in G if and only if E is a subset of V, where there is no edge from any element of E to any element of E.

We say that E is a kernel in G if and only if E is independent in G, and for all x in V\E, there is an edge from x to some element of E.

A kernel of A in G is a kernel of the induced subdigraph G|A.

Let Q be the set of all rational numbers. We say that a digraph on Q^k is downward if and only if for all edges (x,y), max(x) > max(y).

We say that $x,y \in Q^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

We say that $A \subseteq Q^k$ is order invariant if and only if for all order equivalent $x,y \in Q^k$, $x \in A \leftrightarrow y \in A$.

A digraph on Q^k is order invariant if and only if its edge set is order invariant as a subset of Q^{2k}.

2. Kernel Closure Theorem.

Let $A \subseteq Q^k$. The upper shift of A is the set of all $y \in Q^k$ obtained by starting with some $x \in A$, and adding 1 to all nonnegative coordinates of x.

PROPOSITION 2.1. Kernel Closure Theorem. In every downward order invariant digraph on Q^k, the kernel of some $(E \cup \{0\})^k$ contains its upper shift.

THEOREM 2.2. Proposition 2.1 is provable in SRP^+ but not from any consequence of SRP that is consistent with RCA_0. Proposition 2.1 is provably equivalent, over WKL_0, to Con(SRP). These results hold if we fix k to be any particular sufficiently large integer in Proposition 2.1.

3. Kernel Closure Templates.

Note how the Kernel Closure Theorem is based on the functions $ush:Q^k \to Q^k$, where ush(x) is the result of adding 1 to all nonnegative coordinates of x.

What else can be used in Proposition 2.1 in addition to ush?

We can of course view the one dimensional upper shift $ush:Q \to Q$ as fundamental here, where $ush:Q^k \to Q^k$ is obtained by ush acting on the coordinates.

What else can be used in Proposition 2.1 in addition to the one dimensional ush?

Let $f:Q^k \to Q^k$ and $A \subseteq Q^k$. We say that A is f closed if and only if $f[A] \subseteq A$.

Let $g:Q \to Q$ and $A \subseteq Q^k$. We say that A is g closed if and only if every $(g(x_1),...,g(x_k))$, $(x_1,...,x_k) \in A$, lies in A.

Note that the upper shift is a function that can be presented in a rather concrete way. In particular, the upper shift is
rational semilinear - a well known and well studied category of objects.

Specifically, we say that $A \subseteq Q^k$ is rational semilinear if and only if it is in the Boolean algebra generated by the open rational half planes

$$q_1x_1 + \ldots + q_kx_k + q_{k+1} > 0$$

where $q_1,...,q_{k+1}$ are rationals. This is the same as A being a finite union of finite intersections of rational half planes

$$q_1x_1 + \ldots + q_kx_k + q_{k+1} > 0$$
$$q_1x_1 + \ldots + q_kx_k + q_{k+1} \geq 0.$$

We say that a function is rational semilinear if and only if its graph is rational semilinear.

TEMPLATE A. Let $f:Q \to Q$ be rational semilinear. In every downward order invariant digraph on Q^k, some $(E \cup \{0\})^k$ has an f closed kernel.

The instances of Template A can be taken to quantify over k, or have k fixed. Our results are not sensitive to this choice.

TEMPLATE B. Let $f:Q^k \to Q^k$ be rational semilinear. In every downward order invariant digraph on Q^k, some $(E \cup \{0\})^k$ has an f closed kernel.

Note that the Kernel Closure Theorem asserts Template A for the particular case of $ush:Q \to Q$.

We have been able to completely analyze Template A.

THEOREM 3.1. Every instance of Template A is provable or refutable in SRP^+. In fact, every instance of Template A is provable in WKL_0 + Con(SRP) or refutable in RCA_0. It is not the case that every instance of Template A is provable or refutable in SRP - assuming SRP does not prove its own inconsistency. According to SRP^+, SRP does not prove its own inconsistency.

In Template B, we can set f to be the extension of ush to Q^k, with k sufficiently large. By the last claim of Theorem 0.14D3.1, we have the following.

THEOREM 3.2. There is an instance of Template B that is not provable or refutable in SRP - assuming SRP does not prove its own inconsistency.

We can also template the domain set in Templates A,B - i.e, template the construction $(E \cup \{0\})^k$. In doing so, it is highly preferable to treat the domain set and the kernel set uniformly. The natural way to do this is to use $P \subseteq Q^{3k}$ in the following way.

Let $A \subseteq Q^k$. We say that A is P closed if and only if $P[A^2] \subseteq A$.

But before templating both the domain set and the kernel set, we can just template the kernel set - and also use $P \subseteq Q^{2k}$. Thus we are led to these two preliminary Templates which are modifications of Template B.

TEMPLATE C. Let $P \subseteq Q^{2k}$ be rational semilinear. In every downward order invariant digraph on Q^k, some $(E \cup \{0\})^k$ has a P closed kernel.

TEMPLATE D. Let $P \subseteq Q^{3k}$ be rational semilinear. In every downward order invariant digraph on Q^k, some $(E \cup \{0\})^k$ has a P closed kernel.

Now we template both the domain set and the kernel set.

TEMPLATE E. Let $P,J \subseteq Q^{3k}$ be rational semilinear. In every downward order invariant digraph on Q^k, some nonempty P closed set has a J closed kernel.

THEOREM 3.4. There are instances of Templates A - E that are not provable or refutable in SRP - assuming SRP does not prove its own inconsistency.

Proof: For Templates C,D,E, we use the following P's and J's, where k is sufficiently large.

$P_k(x,y,z)$ if and only if $x,y,z \in Q^k$ and every coordinate of z is a coordinate of x, a coordinate of y, or 0.

$J_k(x,y,z)$ if and only if $x,y \in Q^k$ and z is obtained by starting with x or y, and then adding 1 to all nonnegative coordinates.

$J_k(x,y,z)$ if and only if $x,y,z \in Q^k$ and z is obtained by starting with x or y, and then adding 1 to all nonnegative coordinates.

QED

CONJECTURE. Every instance of Templates A - E is provable or refutable in SRP^+. In fact, every instance of Templates A - E is provable in SRP^+ or refutable in RCA_0.

4. Kernel Tower Theorems.

A sharp tower is a finite sequence $A_1,\ldots,A_n \subseteq Q^k$ such that for all $1 \leq i < n$, some $(E \cup \{0\})^k$ lies between A_i and A_{i+1}, in the sense of set inclusion.

An upper shift tower is a finite sequence $A_1,\ldots,A_n \subseteq Q^k$ such that for all $1 \leq i < n$, A_i and the upper shift of A_i is contained in A_{i+1}.

PROPOSITION 4.1. In all downward order invariant digraphs on Q^k, there is a sharp tower of finite sets of length r with an upper shift tower of kernels.

Note that Proposition 4.1 is explicitly Π^0_2. It can be seen that for Proposition 4.1, there are a priori upper bounds for the finite sets that put this in explicitly Π^0_1 form. However, we can also incorporate a bound directly.

PROPOSITION 4.2. In all downward order invariant digraphs on Q^k, there is a sharp tower of finite sets of length r, involving only numerators and denominators of magnitude at most (8kr)!!, with an upper shift tower of kernels.

Note that Proposition 4.2 is explicitly Π^0_1.

THEOREM 4.3. Propositions 4.1, 4.2 are provable in SRP^+ but not from any consequence of SRP that is consistent with EFA. Propositions 4.1, 4.2 are provably equivalent, over EFA, to Con(SRP). These results hold if we fix k to be any particular sufficiently large integer in Propositions 4.1, 4.2.

Recall the definition of EFA in section 0.5.

5. Kernel Tower Templates.

Let $P \subseteq Q^{3k}$. A P closed tower is a finite sequence $A_1,...,A_n \subseteq Q^k$ such that for all $1 \leq i < n$, $A_i \cup P[A_i^2) \subseteq A_{i+1}$.

TEMPLATE F. Let $P,J \subseteq Q^{3k}$ be rational semilinear. In every downward order invariant graph on Q^k, some P closed tower of nonempty finite sets of length r has a J closed tower of kernels.

TEMPLATE G. Let $P,J \subseteq Q^{3k}$ be rational semilinear with coefficients from {0,1}. In every downward order invariant graph on Q^k, some P closed tower of nonempty sets of length r, involving only numerators and denominators of magnitude at most (8kr)!!, has a J closed tower of kernels.

Note that Propositions 0.14D4.1 and 0.14D4.2 are instances of Templates F,G, respectively.

THEOREM 0.14D5.1. There are instances of Templates A - E that are not provable or refutable in SRP - unless SRP proves its own inconsistency.

CONJECTURE. Every instance of Templates F,G is provable or refutable in SRP^+. In fact, every instance of Templates F,G is provable in SRP^+ or refutable in RCA_0.

6. Towards Unification with BRT.

We present another kind of Templating that suggests a unification of the BRT approach and Kernel Structure Theory.

In this approach, kernels are not given any special status. They arise in connection with Boolean combinations of sets and their forward images.

Of course, Kernel Structure Theory is more concrete than BRT, generating equivalents of Con(SRP), and when we give the tower forms, they are in fact explicitly Π^0_1.

It should be pointed out that there is a general methodology whereby we can give a general conversion of the infinitary sentences arising from KST to provably equivalent Π^0_1 sentences.

We use three operators on the subsets A of Q^k.

1. Two specific unary expansion operators, A# and A*. These take any A ⊆ Q^k and create A ⊆ A#,A* ⊆ Q^k.
2. One specific upper image operator R<A. This takes any R ⊆
Q^{2k} and A contained in Q^k, and creates R<A contained in Q^k.

Here we initially set

A# = (fld(A) ∪ {0})^k.
A* = A ∪ the upper shift of A.
R<A = {y: (∃x ∈ A)(max(x) < max(y) ∧ R(x,y)}.

Note how elementary these three operators are, in the sense that they just involve order on Q, 0 in Q, and the +1 function on Q.

As in BRT, we consider the 2^{16} statements in

TEMPLATE #*R<. For all order invariant R contained in Q^{2k}, there exists A ⊆ Q^k such that a given Boolean equation in A,R<A,A#,A* holds.

We know that the following instance of Template #*R< is provablycequivalent, in WKL_0, to Con(SRP):

PROPOSITION #*R<. For all order invariant R ⊆ Q^{2k}, there exists A ⊆ Q^k such that A* ⊆ A = A#\R<A.

CONJECTURE #*R<. Every instance of Template #*R< is provable or refutable in SRP+. In fact, every instance of Template R<#* is provable in WKL_0 + Con(SRP) or refutable in RCA_0.

This is a manageable Conjecture. From here, we can attempt to template #, template *, and template R<. The plan, after establishing the Conjecture, is to first template *.

I have not had the time to come up with the constant dimension k that drives the unremoveable connection with large cardinals. Initial investigations suggest k = 4 or 5.

We close with a discussion of finite forms. The following is provably equivalent to Con(SRP) over EFA.

TINITE PROPOSITION #*R<. For all order invariant R ⊆ Q^{2k}, there exists finite A_1,...,A_r contained in Q^k such that for all 1 ≤ i < j < p ≤ r, A_i* ⊆ A_j = A_j#\R<A_p. Furthermore, we can bound the magnitudes of the numerators and denominators appearing in $A_1,...,A_r$ by a double exponential in k,r.

The above can be subjected to various template investigations as well. We will not go into this here.