# BOOLEAN RELATION THEORY AND MORE

**by**
**Harvey M. Friedman**
**Ohio State University**
**MIT LOGIC SEMINAR**
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**friedman@math.ohio-state.edu**
**www.math.ohio-state.edu/~friedman**
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We will start with the "MORE".

We present the Unprovable Upper Shift Fixed Point Theorem. It asserts the existence of relations on Q - and in fact, arithmetical relations on Q.

It is provable from large cardinals but not in ZFC. It is provably equivalent to consistency of a large cardinal axiom system, and so is, at least indirectly, $\Pi^0_1$.

It has simple finite forms that are explicitly $\Pi^0_2$ and $\Pi^0_1$.

It has a natural Template that supports a general theory.

# SOME EASY DEFINITIONS

We use Q for the rationals.

We say that $x,y \in Q^k$ are order equivalent iff for all $1 \leq i,j \leq k$, $x_i < x_j \Leftrightarrow y_i < y_j$.

We say that $A \subseteq Q^k$ is order invariant iff for all order equivalent $x,y \in Q^k$, $x \in A \Leftrightarrow y \in A$.

We say that $R \subseteq Q^k \times Q^k$ is order invariant iff R is order invariant as a subset of $Q^{2k}$.

We say that $R \subseteq Q^k \times Q^k$ is strictly dominating iff for all $x,y \in Q^k$, $R(x,y) \Rightarrow \max(x) < \max(y)$.

Write $SDOI(Q^k,Q^k)$ for the set of all strictly dominating order invariant $R \subseteq Q^k \times Q^k$.

We define R[A] for $A \subseteq Q^k$, as $\{y: (\exists x \in Q^k)(R(x,y))\}$.

For $x \in Q^k$, define the upper shift us(x) to be the result of adding 1 to every nonnegative coordinate of x.

For $A \subseteq Q^k$, let $us(A) = \{us(x): x \in A\}$.

Write cube(A,0) for the least $B^k$ such that $A \subseteq B^k \wedge 0 \in B$.

**UNPROVABLE UPPER SHIFT FIXED POINT THEOREM.** For all $R \in SDOI(Q^k,Q^k)$, some $A = cube(A,0)\backslash R[A]$ contains us(A).

Relevant large cardinals required most memorably stated in terms of the k-SRP = order k stationary Ramsey property.

λ is k-SRP iff λ is a limit ordinal where every partition of the unordered k-tuples from λ into two pieces has a stationary homogenous set.

For $k \geq 2$, the least k-SRP is strongly inaccessible, even weakly compact, even totally indescribable, etc. But below $\kappa \rightarrow \omega$.

Stationary Ramsey property hierarchy.
Subtle cardinal hierarchy.
Almost ineffable hierarchy.
Ineffable hierarchy.

All provably intertwined. See

H. Friedman, Subtle Cardinals and Linear Orderings, Annals of Pure and Applied Logic 107 (2001), 1-34.

SRP+ = ZFC + (∀k)(there exists a k-SRP ordinal). SRP = ZFC + $\{$there exists a k-SRP ordinal$\}_k$.

USFP provable in SRP+ but not in any consistent fragment of SRP.
USFP provably equivalent, over $WKL_0$, to Con(SRP).

We prove USFP in $WKL_0$ + Con(SRP).

LEMMA 1. There is a countable model M of ZFC + V = L with ordinals $\alpha_0 < \alpha_1 < \ldots$ which are strong indiscernibles in the following sense. (M may have nonstandard integers).

i. Any tuples $(\alpha_{i1},\ldots\alpha_{in},\beta_1,\ldots,\beta_m)$ and $(\alpha_{j1},\ldots,\alpha_{jn},\beta_1,\ldots,\beta_m)$ of the same order type, where $\max(\beta_1,\ldots,\beta_m) < \min(\alpha_{i1},\ldots,\alpha_{in},\alpha_{j1},\ldots,\alpha_{jn})$, look the same in M.

ii. Any ordinal M definable from ordinals $< \alpha_i$ is $< \alpha_{i+1}$.

Proof: We apply compactness. The finite approximations involve constant symbols $\alpha_0 < \ldots < \alpha_k$ and the first k properties in M. We can, provably in SRP, set M = L($\lambda$), where $\lambda$ is the least k+1-SRP (and even a stationary set of $\alpha$'s). Hence using just $WKL_0$ + Con(SRP), we see that each finite approximation is consistent. Now apply compactness to obtain the countable model M satisfying ZFC + V = L with the appropriate $\alpha$'s. QED

Fix M according to Lemma 1. Fix $\alpha_0 < \alpha_1 < \ldots$ . Let M* consist of the elements of M that are M definable from the $\alpha$'s. Then M* is an elementary submodel of M containing the $\alpha$'s, and the $\alpha$'s are strong indiscernibles in M*. Evidently, the $\alpha$'s are limit ordinals in M*. Furthermore, every element of M* is M* definable from the $\alpha$'s.

LEMMA 1. There is a countable model M of ZFC + V = L with ordinals $\alpha_0 < \alpha_1 < \ldots$ which are strong indiscernibles in the following sense. (M may have nonstandard integers).
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M* = submodel of points M definable from the α's.

We now define $j:M^* \to M^*$. Let $x \in M^*$. Let x be the unique x such that $\varphi(x,\alpha_{i1},\ldots\alpha_{ik})$ in M*. By indiscernibility, let j(x) be the unique y such that $\varphi(y,\alpha_{i1+1},\ldots\alpha_{ik+1})$ in M*.

LEMMA 2. j is well defined. j is an elementary embedding from M* into M* with critical point $\alpha_0$. $j(\alpha_i) = \alpha_{i+1}$. j is one-one increasing from $[\alpha_i,\alpha_{i+1})$ into $[\alpha_{i+1},\alpha_{i+2})$. The α's are unbounded in M*. M* satisfies ZFC + V = L.

Now fix $R \in SDOI(Q^k,Q^k)$. Then R lifts canonically to the ordinals ord(M*) of M*. By internal transfinite recursion in M*, define $B \subseteq \mathrm{ord}(M^*)^k$ uniquely according to

$$x \in B \Leftrightarrow (\forall y \in B)(\neg R(y,x)).$$

$$B = \mathrm{ord}(M^*)^k \setminus R[B].$$

LEMMA 3. B is M* definable. B contains j<B>. B = cube(B,0)\R[B].

NOTE: We use < > here for forward images of functions.

LEMMA 4. $(\mathrm{ord}(M^*),\alpha_0,<,j)$ is isomorphically embeddable into (Q, 0,<,us), where us is the upper shift from Q into Q.

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Proof: We define the embedding $h:\mathrm{ord}(M^*) \to Q$ as follows. Let $\beta_1,\beta_2,\ldots$ be an enumeration of ord(M*)\rng(j) without repetition, where $\beta_1 = \alpha_0$. Define $E_i$ = the closure of $(-\infty,\alpha_0) \cup E_{i-1} \cup \{\beta_1,\ldots,\beta_i\}$ under j. Then each $E_i$ has only finitely many elements from any $[\alpha_0,\alpha_p]$. For each successive $n \geq 1$, define $h_n$ to be an embedding from $(E_n,\alpha_0,<,j)$ into (Q,0,<,us), where the $h_n$'s form a chain under inclusion. QED

Fix h as given by Lemma 4. Define h<B> = {h(x): x ∈ B}, where h acts coordinatewise. Recall B = $\mathrm{ord}(M^*)^k$\R[B].

LEMMA 5. h<B> = cube(h<B>,0)\R[h<B>]. h<B> contains us(h<B>).

Proof: By Lemmas 3,4. QED

We have now proved:

**UNPROVABLE UPPER SHIFT FIXED POINT THEOREM.** For all R ∈ $\mathrm{SDOI}(Q^k,Q^k)$, some A = cube(A,0)\R[A] contains us(A).

For the reversal, first construct R ∈ $\mathrm{SDOI}(Q^{3k},Q^{3k})$ carefully so that the equation A = $\mathrm{On}^{3k}$\R[A] is "complete for transfinite recursion" when performed on the actual ordinals.

**UNPROVABLE UPPER SHIFT FIXED POINT THEOREM.** For all R ∈ SDOI($Q^k$,$Q^k$), some A = cube(A,0)\R[A] contains us(A).

Now we actually have to work with the equation

A = cube(A,0)\R[A], A ⊆ $Q^{3k}$.

So we are working with a universe V = fld(A) ⊆ Q. We can arrange that the (q,q,...,q) ∉ R[$Q^{3k}$], so that 0 ∈ V, and (0,...,0) ∈ A. Since A contains us(A), we see that each (n,...,n) ∈ A, n ≥ 0. So N ⊆ V.

The relation A is an encoding of the constructible hierarchy on the universe (V,<), and internal well foundedness holds. The upper shift gives us a system

(V,A,<,us)

where us:V → V, us<A> ⊆ A. In particular, us has critical point 0, and us is an elementary embedding with respect to bounded formulas. It is now clear that there must be negative rationals in V, and they are fixed by us.

**UNPROVABLE UPPER SHIFT FIXED POINT THEOREM.** For all $R \in \mathrm{SDOI}(Q^k,Q^k)$, some $A = \mathrm{cube}(A,0)\backslash R[A]$ contains us(A).

Now (0,1,...,k),(1,...,k+1) look the same with respect to bounded formulas in $(V,A,<)$, with k negative parameters. This is enough to show that

i. The L(k-1) of $(V,A,<)$ satisfies ZFC + V = L.
ii. In L(k-1), 0 looks like a (k-3)-subtle cardinal.

This completes the sketch of the reversal. I.e., the construction of models of the subtle cardinal hierarchy, which is intertwined with the stationary Ramsey property hierarchy.

**FINITE FORMS**

SEQUENTIAL UNPROVABLE UPPER SHIFT FIXED POINT THEOREM. For all $R \in \mathrm{SDOI}(Q^k,Q^k)$, there exist finite $A_1,A_2,... \subseteq Q^k$ such that each $A_{i+1} = \mathrm{cube}(A_{i+1},0)\backslash R[A_{i+2}]$ contains $A_i \cup \mathrm{us}(A_i)$.

FINITE SEQUENTIAL UNPROVABLE UPPER SHIFT FIXED POINT THEOREM. For all $R \in \mathrm{SDOI}(Q^k,Q^k)$, there exist finite $A_1,...,A_k \subseteq Q^k$ such that each $A_{i+1} = \mathrm{cube}(A_{i+1},0)\backslash R[A_{i+2}]$ contains $A_i \cup \mathrm{us}(A_i)$.

ESTIMATED SEQUENTIAL UNPROVABLE UPPER SHIFT FIXED POINT THEOREM. The magnitudes of the denominators/numerators bounded by (8k)!.

**TEMPLATE**

The Unprovable Upper Shift Fixed Point Theorem is, quite naturally, an instance of a large family of propositions. Note that the upper shift is the obvious lifting of the one dimensional upper shift from Q into Q to higher dimensions.

We let PPL(Q) be the family of partial $f:Q \to Q$ given by

$a_1x + b_1$ if $x \in I_1$
...
$a_nx + b_n$ if $x \in I_n$

where $n \geq 1$, the a's,b's are rationals, and the I's are pairwise disjoint nonempty intervals with rational endpoints (or $\pm\infty$).

A PPL(Q) system consists of a finite list of elements of PPL(Q).

Let M be a PPL(Q) system. We say that V is M closed iff V is f closed for all components f of M.

TEMPLATE. Let M be a PPL(Q) system. Is it the case that for all $R \in SDOI(Q^k,Q^k)$, some $A = cube(A,0)\backslash R[A]$ contains M[A]?

**TEMPLATE. Let M be a PPL(Q) system. Is it the case that for all R ∈ SDOI($Q^k$,$Q^k$), some A = cube(A,0)\R[A] contains M[A]?**

THEOREM. The Template is false for the single function f:Q → Q, where for all x ∈ Q, f(x) = x+1. (The shift). The Template is true for the single function f:[0,∞) into [0,∞), where for all x ≥ 0, f(x) = x+1. (The nonnegative shift). These assertions are provable in $RCA_0$.

CONJECTURE. Every instance of the Template is refutable in $RCA_0$ or provable in SRP+.

WHAT ABOUT MUCH BIGGER LARGE CARDINALS?

We are still researching this, but do not yet have something that is comparable to the upper shift fixed point theorem.