This is an MS Word formatted version of my FOM posting #258 of November 27, 2005, at

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Quite a number of typographical errors in the FOM posting have been corrected.

Π^0_1 INCOMPLETENESS

by

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"Beautiful" is a word used by mathematicians with a semi rigorous meaning.

We give "arguably beautiful" explicitly Pi01 sentences independent of ZFC. See Proposition A from section 1 and Proposition B from section 2, and variants.

Proposition A is simplest and has a graph theoretic flavor, with algebraic overtones. Proposition B has a fixed point flavor.

1. Π^0_1 INDEPENDENT STATEMENTS USING ANTICHAINS.

We use [1,n] for the discrete interval {1,...,n}.

Let $A \subseteq [1,n]^k$. We write $A' = [1,n]^k\backslash A$. This treats $[1,n]^k$ as the ambient space.

Let $R \subseteq [1,n]^{2k}$. We define

$$RA = R[A] = \{y \in [1,n]^k: (\exists x \in A)(R(x,y))\}.$$

We say that R is strictly dominating if and only if for all $x,y \in [1,n]^k$, if R(x,y) then max(x) < max(y).

We start with the basic 'complementation theorem for RA'.

THEOREM 1.1. For all k,n ≥ 1 and strictly dominating $R \subseteq [1,n]^{2k}$, there exists $A \subseteq [1,n]^k$ such that RA = A'. Furthermore, $A \subseteq [1,n]^k$ is unique.

For $A \subseteq [1,n]^k$ and t ≥ 1, we write $A\backslash t = \{x \in A$: t is not a coordinate of x} = "A with t omitted".

Here is a modification of Theorem 1.1 which we call the 'complementation theorem for R[A\(8k)!]'.

THEOREM 1.2. For all k,n ≥ 1 and strictly dominating $R \subseteq [1,n]^{2k}$, there exists $A \subseteq [1,n]^k$ such that R[A\(8k)!] = A'. Furthermore, $A \subseteq [1,n]^k$ is unique.

We now incorporate the antichain concept.

Let $R \subseteq [1,n]^{2k}$. We say that A is an antichain for R if and only if $A \subseteq [1,n]^k$ and A,RA are disjoint.

Of course, we have the following "maximal antichain" theorem.

THEOREM 1.3. For all k,n ≥ 1, every $R \subseteq [1,n]^{2k}$ has a maximal antichain.

Note that Theorem 1.3 is virtually contentless, since we are in a finite context where every nonempty class has a maximal element.

Note that Theorem 1.1 provides a much stronger kind of antichain, which we call a 'complete' antichain. We refer to the following as a 'complete antichain' theorem.

THEOREM 1.4. For all k,n ≥ 1, every strictly dominating $R \subseteq [1,n]^{2k}$ has an antichain A such that RA contains A'. Furthermore, A is unique.

However, the analog of Theorem 1.4 for R[A\(8k)!] is false.

THEOREM 1.5. The following is false. For all k,n ≥ 1, every strictly dominating $R \subseteq [1,n]^{2k}$ has an antichain A such that R[A\(8k)!] contains A'.

We will weaken the conclusion in a simple way.

Our development depends heavily on a very strong regularity condition on R.

We say that $R \subseteq [1,n]^k$ is order invariant if and only if for all x,y in $[1,n]^k$ of the same order type, R(x) iff R(y). The number of such R is bounded by an exponential expression in k that does not depend on n.

The imposition of order invariance is still not sufficient:

THEOREM 1.6. The following is false. For all k,n ≥ 1, every strictly dominating order invariant $R \subseteq [1,n]^{2k}$ has an antichain A such that R[A\(8k)!] contains A’.

However, we can assert that R[A\(8k)!] contains a "significant part" of A’ as follows.

The powers of t are the numbers t^i, i ≥ 0.

THEOREM 1.7. For all k,n ≥ 1, every strictly dominating order invariant $R \subseteq [1,n]^{2k}$ has an antichain A such that R[A\(8k)!] contains all k tuples of powers of (8k)!+1, lying in A’.

Obviously we can apply R once or more times to both sides of the equations in Theorems 1.1 and 1.2, and to both sides of the containments in Theorems 1.3 and 1.7. This is because, e.g.,

RRRRA = RRR[A’]
RRRR[A\(8k)!] = RRR[A’]
RRRRA contains RRR[A’]

follow immediately from

RA = A’
R[A\(8k)!] = A’
RA contains A’

respectively. However,

RRRR[A\(8k)!] contains all k tuples of powers of (8k)!+1, lying in RRR[A’]

does NOT follow from

R[A\(8k)!] contains all k tuples of powers of (8k)!+1, lying in A’.

The use of p R's has an obvious graph theoretic interpretation: paths in R (viewed as a graph) of length p.

So what happens?

PROPOSITION A. For all $k,n \geq 1$, every strictly dominating order invariant $R \subseteq [1,n]^{2k}$ has an antichain A such that RRRR[A\(8k)!] contains all k tuples of powers of (8k)!+1, lying in RRR[A'].

Proposition A is obviously an explicitly Π^0_1 sentence. It is independent of ZFC. Here is much more precise information.

Let MAH = ZFC + $\{$there exists a strongly n-Mahlo cardinal$\}_n$.

Let MAH+ = ZFC + "for all n there exists a strongly n-Mahlo cardinal".

THEOREM 1.8. MAH+ proves Proposition A. However, Proposition A is not provable in any consistent fragment of MAH that derives Z = Zermelo set theory. In particular, Proposition A is not provable in ZFC, provided ZFC is consistent. These facts are provable in RCA_0.

THEOREM 1.9. It is provable in ACA that Proposition A is equivalent to Con(MAH).

There are some variants of Proposition A which have the same status (as given by Theorems 1.8 and 1.9).

a. We can use more R's in the two expressions, as long as the first uses exactly one more than the second, and the first uses at least 4 R's. When the number of R's get substantial relative to k, we have to raise (8k)!.

b. We can use t instead of (8k)!, and hypothesize that t is at least (8k)!. The use of (8k)! is merely safe and convenient, and not anything near optimal.

c. In A\(8k)!, we are eliminating all vectors in which (8k)! appear. We can instead only eliminate a single vector. In particular, we can just eliminate the k tuple ((8k)!,...,(8k)!). Thus we can replace A\(8k)! with $A\backslash\{(8k)!\}^k$. Of course, this weakens the statement and is also not so pretty. But the idea behind A\(8k)! is that we are perturbing A just a little bit. With $A\backslash\{8k)!\}^k$, we are perturbing A by even less, so there is something to be said for this modification.

d. We can combine some or even all of a - d.

Note that Theorems 1.3 - 1.7 and Proposition A have an obvious graph theoretic flavor, with lattice points as the vertex set.

2. Π^0_1 INDEPENDENT STATEMENTS WITHOUT USING ANTICHAINS.

Here we avoid using antichains, and instead emphasize the fixed point aspects.

We start with the basic unique fixed point theorem for the operator R[A′].

THEOREM 2.1. For all $k,n \geq 1$ and strictly dominating $R \subseteq [1,n]^{2k}$, there exists $A \subseteq [1,n]^k$ such that R[A′] = A. Furthermore, $A \subseteq [1,n]^k$ is unique.

The "omission" operator R[A′\(8k)!] also has a unique fixed point.

THEOREM 2.2. For all $k,n \geq 1$ and strictly dominating $R \subseteq [1,n]^{2k}$, there exists $A \subseteq [1,n]^k$ such that R[A′\(8k)!] = A. Furthermore, $A \subseteq [1,n]^k$ is unique.

However, we can't get a COMMON fixed point

R[A′\(8k)!] = R[A′] = A.

THEOREM 2.3. The following is false. For all $k,n \geq 1$ and strictly dominating $R \subseteq [1,n]^{2k}$, there exists $A \subseteq [1,n]^k$ such that R[A'\(8k)!] = R[A′] = A.

We will weaken this three term common fixed point equation so that it is attainable. The weakening involves replacing = both with $\subseteq$ and with "having the same elements of a certain kind".

Order invariance will not be sufficient:

THEOREM 2.4. The following is false. For all $k,n \geq 1$ and strictly dominating order invariant $R \subseteq [1,n]^{2k}$, there exists $A \subseteq [1,n]^k$ such that R[A′\(8k)!] = R[A′] = A.

The powers of t are the numbers t^i, $i \geq 0$.

THEOREM 2.5. For all $k,n \geq 1$ and strictly dominating order invariant $R \subseteq [1,n]^{2k}$, there exists $A \subseteq [1,n]^k$ such that the three sets $R[A'\backslash(8k)!] \subseteq R[A'] \subseteq A$ contain the same k tuples of powers of $(8k)!+1$.

Obviously we can apply R once or more times to both sides of the equations in Theorems 2.1 and 2.2. This is because, e.g.,

$$RRRR[A'] = RRRA$$
$$RRRR[A'\backslash(8k)!] = RRRA$$

respectively follow immediately from

$$R[A'] = A$$
$$R[A'\backslash(8k)!] = A$$

What happens if we apply R, once or more, to the expressions in the three term tower of Theorem 2.5?

Again

1) $RRRR[A'\backslash(8k)!] \subseteq RRRR[A'] \subseteq RRRA$

follows immediately from

2) $R[A'\backslash(8k)!] \subseteq R[A'] \subseteq A$.

However,

'the three sets in 1) above contain the same k tuples of powers of $(8k)!+1$'

does not follow from

'the three sets in 2) above contain the same k tuples of powers of $(8k)!+1$.'

PROPOSITION B. For all $k,n \geq 1$ and strictly dominating order invariant $R \subseteq [1,n]^{2k}$, there exists $A \subseteq [1,n]^k$ such that the three sets $RRRR[A'\backslash(8k)!] \subseteq RRRR[A'] \subseteq RRRA$ contain the same k tuples of powers of $(8k)!+1$.

Proposition B is obviously an explicitly Π^0_1 sentence. It is independent of ZFC. Here is much more precise information.

Let MAH = ZFC + $\{$there exists a strongly n-Mahlo cardinal$\}_n$.

Let MAH+ = ZFC + ‘for all n there exists a strongly n-Mahlo cardinal’.

THEOREM 2.6. MAH+ proves Proposition B. However, Proposition B is not provable in any consistent fragment of MAH that derives Z = Zermelo set theory. In particular, Proposition B is not provable in ZFC, provided ZFC is consistent. These facts are provable in RCA_0.

THEOREM 2.7. It is provable in ACA that Proposition B is equivalent to Con(MAH).

There are some variants of Proposition B which have the same status (as given by Theorems 2.6 and 2.7).

a. We can use more R's in the three expressions, as long as the first two use the same number of R's, with at least 4 R's, and the third uses one fewer. When the number of R's get substantial relative to k, we have to raise (8k)!.

b. We can use t instead of (8k)!, and hypothesize that t is at least (8k)!. The use of (8k)! is merely safe and convenient, and not anything near optimal.

c. We can eliminate a single vector instead of all vectors in which (8k)!. Thus we can replace A'\(8k)! with A’\{(8k)!}k. Of course, this weakens the statement and is also not so pretty. But the idea behind A’\(8k)! is that we are perturbing A' just a little bit. With A'\{8k)!}k, we are perturbing A' by even less, so there is something to be said for this modification.

d. We can combine some or even all of a - d.

3. CONTROLLING PROPOSITION A.

We wish to control the strength of Propositions A,B by weakening them in simple ways. This seems to be a large subject, with much to work out, and we merely scratch the surface of it here.

PROPOSITION A1. For all k,n ≥ 1, every strictly dominating order invariant $R \subseteq [1,n]^{2k}$ has an antichain A such that RRRR[A\(8k)!] contains all k tuples from the first 3 powers of (8k)!+1, lying in RRR[A’].

PROPOSITION A2. For all k,n ≥ 1, every strictly dominating order invariant R □ $[1,n]^{2k}$ has an antichain A such that RRRR[A\(8k)!] contains all k tuples from the first k powers of (8k)!+1, lying in RRR[A’].

PROPOSITION A3. For all k,n ≥ 1, every strictly dominating order invariant R □ $[1,n]^{2k}$ has an antichain A such that RRRR[A\(8k)!] contains all k tuples from the first (8k)! powers of (8k)!+1, lying in RRR[A’].

PROPOSITION A4. For all k,n,p ≥ 1, every strictly dominating order invariant R □ [1,n]2k has an antichain A such that RRRR[A\(8k)!] contains all k tuples from 1,(8k)!+1,..., (8k)!+p, lying in RRR[A'].

THEOREM 3.1. Propositions A1,A2,A4 are provable in ACA but not in PA. It is provable in PRA that Propositions A1,A2,A4 are equivalent to Con(PA).

THEOREM 3.2. For fixed p, Proposition A4 is provable in PA. The set of Propositions obtained by fixing p in Proposition A4 is equivalent, over PRA, to {Con(PA_n): n ≥ 1}. For fixed p, Proposition A4 requires approximately p quantifier induction (exact calculation is being postponed).

THEOREM 3.3. Proposition A3 is provable in Z but not in WZC. It is provable in PRA that Proposition A3 is equivalent to Con(WZC).

Here Z = Zermelo set theory, and WZC = Zermelo set theory with only bounded separation, with the axiom of choice. The equivalence of Con(WZC) and the consistently of Russell's type theory with infinity is provable in PRA.

4. CONTROLLING PROPOSITION B.

We wish to control the strength of Proposition B by weakening it in simple ways. This seems to be a large subject, with much to work out, and we merely scratch the surface of it here.

PROPOSITION B1. For all k,n ≥ 1 and strictly dominating order invariant R □ [1,n]2k, there exist A □ $[1,n]^k$ such that the
three sets RRRR[A’\(8k)!] □ RRRR[A’] □ RRRA contain the same k tuples from the first three powers of (8k)!+1.

PROPOSITION B2. For all k,n ≥ 1 and strictly dominating order invariant R □ $[1,n]^{2k}$, there exist A □ $[1,n]^k$ such that the three sets RRRR[A'\(8k)!] □ RRRR[A'] □ RRRA contain the same k tuples from the first k powers of (8k)!+1.

PROPOSITION B3. For all k,n ≥ 1 and strictly dominating order invariant R □ $[1,n]^{2k}$, there exist A □ $[1,n]^k$ such that the
three sets RRRR[A'\(8k)!] □ RRRR[A'] □ RRRA contain the same k tuples from the first (8k)! powers of (8k)!+1.

PROPOSITION B4. For all k,n ≥ 1 and strictly dominating order invariant R □ $[1,n]^{2k}$, there exist A □ $[1,n]^k$ such that the
three sets RRRR[A'\(8k)!] □ RRRR[A'] □ RRRA contain the same k tuples from 1,(8k)!+1,...,(8k)!+p.

THEOREM 4.1. Propositions B1,B2,B4 are provable in ACA but not in PA. It is provable in PRA that Propositions B1,B2,B4 are equivalent to Con(PA).

THEOREM 4.2. For fixed p, Proposition B4 is provable in PA. The set of Propositions obtained by fixing p in Proposition B4 is equivalent, over PRA, to {Con(PA_n): n ≥ 1}. For fixed p, Proposition B4 requires approximately p quantifier induction (exact calculation is being postponed).

THEOREM 4.3. Proposition B3 is provable in Z but not in WZC. It is provable in PRA that Proposition B3 is equivalent to Con(WZC).

It looks like I forgot to mention Theorem 4.3 in posting #255.

5. FIT, CLASSIFICATIONS.

We can take a general view of Proposition B and place it in a theory called FIT = finite inclusion theory.

Let us begin by recalling

THEOREM 2.5. For all k,n ≥ 1 and strictly dominating order invariant R □ $[1,n]^{2k}$, there exists A □ $[1,n]_k$ such that the three sets R[A'\(8k)!] □ R[A'] □ A contain the same k tuples of powers of (8k)!+1.

We begin our classification theory by considering all three expressions appearing in Theorem 2.5.

A
R[A′]
R[A′\(8k)!]

Theorem 2.5 uses these two inclusion statements.

s ⊆ t
s ⊆* t

where s,t are among the above five expressions. Here s ⊆* t means that every k tuple of powers of (8k)!+1 lying in s, also lies in t.

Thus we can rewrite Theorem 2.5 as follows.

THEOREM 2.5′. For all k,n ≥ 1 and strictly dominating order invariant R ⊆ [1,n]2k, there exists A ⊆ $[1,n]^k$ such that
1. R[A′] ⊆ A.
2. A ⊆* R[A′\(8k)!].

Note how the conclusion of Theorem 2.5 reduces to only two inclusions. This is because

R[A′\(8k)!] ⊆ R[A′] ⊆ A
R[A′\(8k)!] =* R[A′] =* A

follow trivially from 1,2 in Theorem 2.5′.

Let K1 be the set of all inclusion statements of the above form. K1 has 18 elements, six of which are trivial because their left and right sides are identical. So there are really 12 elements of K1 that need to be considered.

We now wish to classify (i.e., determine the truth or falsity of) all statements of the following form.

TEMPLATE K1. For all k,n ≥ 1 and strictly dominating order invariant R ⊆ [1,n]2k, there exists A ⊆ $[1,n]^k$ such that a given subset of K1 holds conjunctively.

There are 2^{12} subsets of K1 that need to be considered, which is 4096.

We have been able to carry out this classification; i.e., determine the truth values of all instances of Template K1.

Now what exactly does this mean?

For one thing, we can actually exhibit this table, since 4096 is not too big.

But we have a methodology for determining these truth values without exhibiting the entire table. Instead, we use a tree like methodology which does exhibit all of the maximal subsets of K1 for which Template K1 holds.

The fundamental finding that can be stated without exhibiting anything is:

THEOREM 5.1. Every instance of Template K1 is provable or refutable in EFA (exponential function arithmetic).

In the course of this analysis, we did discover a variant of Theorem 2.5 that does not seem to follow formally from Theorem 2.5, or formally imply Theorem 2.5:

THEOREM 2.5*. For all k,n ≥ 1 and strictly dominating order invariant R □ $[1,n]^{2k}$, there exists A □ $[1,n]^k$ such that the three sets A □ R[A’\(8k)!] □ R[A’] contain the same k tuples of powers of (8k)!+1.

We also discovered an interesting counterexample.

THEOREM 5.2. The following is false. For all k,n ≥ 1 and strictly dominating order invariant R □ $[1,n]^{2k}$, there exists A □ $[1,n]^k$ such that R[A’\(8k)!] □ A □ R[A'] and R[A'] □* R[A’\(8k)!].

Now recall

PROPOSITION B. For all k,n ≥ 1 and strictly dominating order invariant R □ $[1,n]^{2k}$, there exist A □ $[1,n]^k$ such that the three sets RRRR[A’\(8k)!] □ RRRR[A’] □ RRRA contain the same k tuples of powers of (8k)!+1.

We analogously considering all three expressions appearing in Proposition B.

RRRA
RRRR[A’]

RRRR[A’\(8k)!]

Proposition A uses these two inclusion statements.

$$s \subseteq t$$
$$s \subseteq^* t$$

where s,t are among the above five expressions. Here s $\subseteq^*$ t means that every k tuple of powers of (8k)!+1 lying in s, also lies in t.

Thus we can rewrite Proposition B as follows.

PROPOSITION B'. For all k,n ≥ 1 and strictly dominating order invariant R $\subseteq [1,n]^{2k}$, there exists A $\subseteq [1,n]^k$ such that
1. RRRR[A’] $\subseteq$ RRRA.
2. RRRA $\subseteq^*$ RRRR[A’\(8k)!].

Note how the conclusion of Theorem 2.4 reduces to only two inclusions. This is because

RRRR[A’\(8k)!] $\subseteq$ RRRR[A’] $\subseteq$ RRRA
RRRR[A’\(8k)!] =* RRRR[A’] =* RRRA

follow trivially from 1,2 in Proposition B’.

Let K2 be the set of all inclusion statements of the above form. K2 has 18 elements, six of which are trivial because their left and right sides are identical. So there are really 12 elements of K2 that need to be considered.

We now wish to classify (i.e., determine the truth or falsity of) all statements of the following form.

TEMPLATE K2. For all k,n ≥ 1 and strictly dominating order invariant R $\subseteq [1,n]^{2k}$, there exists A $\subseteq [1,n]^k$ such that a given subset of K2 holds conjunctively.

There are 2^{12} subsets of K2 that need to be considered, which is 4096.

THEOREM 5.3. Every instance of Template K2 can be proved or refuted in MAH+. In fact, every instance of Template K2 is either

i. Provable in EFA, or

ii. Refutable in EFA, or
iii. Provably equivalent, over ACA, to Con(MAH).

We have discovered a variant of Proposition B that does not seem to formally imply or be implied by Proposition B:

PROPOSITION B*. For all k,n ≥ 1 and strictly dominating order invariant R ⊆ $[1,n]^{2k}$, there exists A ⊆ $[1,n]^{k}$ such that the three sets RRRA ⊆ RRRR[A’\(8k)!] ⊆ RRRR[A’] contain the same k tuples of powers of (8k)!+1.

Proposition B* is also provably equivalent, over ACA, to Con(MAH).

It is more ambitious to work with the five expressions

A
A’
RA
R[A’]
R[A’\(8k)!].

And the two inclusion relations

s ⊆ t
s ⊆* t.

This comprises 50 inclusions. The idea is to classify Proposition B using, as conclusion, any of the 2^{50} subsets interpreted conjunctively. I.e., which are true and which are false, and show that each one can be decided in, say, EFA.

Getting rid of silly inclusions, we have 40 inclusions, and 2^{40} subsets. This is roughly 1 trillion statements.

Yet more ambitiously, we can generate a hierarchy of expressions as follows.

1. A,A’,A\(8k)!,A’\(8k)! are expressions.
2. If t is an expression, then so are Rt, (Rt)’, (Rt)\(8k)!, (Rt)’\(8k)!.

Atomic statements are again of the forms

3. s ⊆ t.
4. s ⊆* t.

We want to handle arbitrary finite sets of atomic statements, interpreted conjunctively.

At the level appropriate for Theorem 1.4, we have 4 expressions from 1 and another 16 from 2, for a total of 20 expressions. This yields 2(20)(19) = 760 nontrivial inclusions. The number of statements to be analyzed is 2^{760}.

At the level appropriate for Proposition B, we have 4 expressions from 1, 16 more from one application of 2, and 64 more from a second application of 2, 256 more from a third application of 2, and 1024 more from a fourth application of 2. The total number of expressions is 1300, yielding 2(1300)(1299) = 3,377,400 nontrivial inclusions. The number of statements to be analyzed is $2^{3,377,400}$.

It remains to be seen whether these classifications can actually be carried out, and whether EFA is enough for the 2^{760} classification, and whether Con(MAH) is enough for the $2^{3,377,400}$ classification. Expect massive amounts of silly trivialities. The problem is to tame what's left.