Π^0_1, Π^0_0 Incompleteness: finite graph theory

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February 25, 2006

In this abstract, a digraph is a directed graph with no loops and no multiple edges. Thus all digraphs will be simple. The results will be the same if we allow loops.

A dag is a directed graph with no cycles.

Let G be a digraph. We write V(G) for the set of all vertices in G, and E(G) for the set of all edges in G.

Let A $\subseteq$ V(G). We write GA for the set of all destinations of edges in G whose origins lie in A. I.e., GA = {y: $(\exists x)((x,y) \in E(G))$}.

We begin by quoting a well known theorem about directed acyclic graphs, or so called dags. We call it the complementation theorem, but we have been told that it is rather ordinary fundamental fare in dag theory.

COMPLEMENTATION THEOREM (finite dags). Let G be a finite dag. There is a unique set A $\subseteq$ V(G) such that GA = V(G)\A.

We can look at the Complementation Theorem in terms of a large independent set. We say that A $\subseteq$ V(G) is independent in G if and only if there is no edge connecting any two elements of A.

COMPLEMENTATION THEOREM (finite dags). Every finite dag has a unique independent set A such that V(G)\A $\subseteq$ GA.

A digraph on a set E is a digraph G where V(G) = E.

We will focus on digraphs whose vertex set is of the form $[1,n]^k$. Here $k,n \geq 1$ and $[1,n] = \{1,2,...,n\}$.

An upgraph on $[1,n]^k$ is a digraph on $[1,n]^k$ such that for all $(x,y) \in E(G)$, $\max(x) < \max(y)$.

The following is an immediate consequence of the Complementation Theorem (finite dags) since upgraphs are obviously dags.

COMPLEMENTATION THEOREM (upgraphs). For all $k,n,r \geq 1$, every upgraph on $[1,n]^k$ has a unique independent set A such that $V(G)\backslash A$ □ GA.

Our development relies on what we call order invariant digraphs on $[1,n]^k$. These are the digraphs G on $[1,n]^k$ where only the relative order of coordinates of pairs of vertices determine if they are connected by an edge.

More formally, let u,v □ $\{1,2,3,...\}^p$. We say that u,v are order equivalent if and only if for all 1 □ i,j □ p,

$$u_i < u_j \ \square \ v_i < v_j.$$

Let G be a digraph on $[1,n]^k$. We say that G is order invariant if and only if the following holds. For all x,y,z,w □ $[1,n]^k$, if (x,y) and (z,w) are order equivalent (as 2k tuples), then

$$(x,y) \ \square \ E(G) \ \square \ (z,w) \ \square \ E(G).$$

Note that an order invariant digraph on $[1,n]^k$ is completely determined, among digraphs on $[1,n]^k$, by the subdigraph induced by $[1,2k]^k$ - regardless of how large n is. Thus the number of order invariant digraphs on $[1,n]^k$ is bounded by $(2k)^k$.

Let $x_1,...,x_s,y_1,...,y_s$ be vertices in the digraph G. We say that $x_1,...,x_k$ and $y_1,...,y_s$ are G equivalent if and only if for all 1 □ i,j □ k,

$$(x_i,x_j) \ \square \ E(G) \ \square \ (y_i,y_j) \ \square \ E(G).$$

We can add an additional clause to this definition, without affecting any of the results. Specifically,

$$x_i = x_j \ \square \ y_i = y_j.$$

We write x! when x is a tuple of nonnegative integers. Here $x! = (x_1!,...,x_k!)$, where x has length k.

For A □ $[1,n]^k$ and $m \geq 1$, we write $A\backslash m$ for the set of all vectors in A in which m does not appear as a coordinate.

PROPOSITION A. For all $k,n,r \geq 1$, every order invariant upgraph G on $[1,n]^k$ has an independent set A such that every

$x!,y_1,...,y_r$ from V(G)\A is G equivalent to some $x!,z_1,...,z_r$ from GA\((8kr)!-1).

PROPOSITION B. For all $n,r \geq 1$, every order invariant upgraph G on $[1,n]^8$ has an independent set A such that every $x!,y_1,...,y_r$ from V(G)\A is G equivalent to some $x!,z_1,...,z_r$ from GA\((8r)!-1).

Note that Proposition B is obtained from Proposition A by setting k = 8. We can get away with 8r here instead of 64r.

PROPOSITION C. For all $k,n,r \geq 1$, every order invariant upgraph G on $[1,n]^k$ has an independent set A such that for all strictly increasing $x,y \in [8kr,n]^k$, every $x!,z_1,...,z_r$ from V(G)\A is G equivalent to some $y!,w_1,...,w_r$ from GA\((8kr)!-1).

PROPOSITION D. For all $n,r \geq 1$, every order invariant upgraph G on $[1,n]^8$ has an independent set A such that for all strictly increasing $x,y \in [8r,n]^8$, every $x!,z_1,...,z_r$ from V(G)\A is G equivalent to some $y!,w_1,...,w_r$ from GA\((8r)!-1).

PROPOSITION E. For all $r \geq 1$, every order invariant upgraph G on $[1,(8r)!!]^8$ has an independent set A such that for all strictly increasing $x,y \in \{(8r)!,8r+1)!,...,(8r)!!\}^8$, every $x,z_1,...,z_r$ from V(G)\A is G equivalent to some $y,w_1,...,w_r$ from GA\((8r)!-1).

PROPOSITION F. Every order invariant upgraph G on $[1,8!!!!]^8$ has an independent set A such that for all strictly increasing $x,y \in \{8!!!,(8!!+1)!,...,8!!!!\}^8$, every $x,z_1,...,z_{7!!}$ from V(G)\A is G equivalent to some $y,w_1,...,w_{7!!}$ from GA\(8!!!-1).

Note that if we use GA instead of GA\((8r)!-1) in Propositions A,B, then the resulting Proposition follows immediately from the Compelementation Theorem (finite upgraphs).

Propositions A-E can be proved with large cardinals but not without. Note that Propositions A-E are explicitly Π^0_1. Note that Proposition F is explicitly Π^0_0.

Here is more detailed information.

Let MAH = ZFC + $\{$there exists a strongly n-Mahlo cardinal$\}_n$.

Let MAH+ = ZFC + "for all n there exists a strongly n-Mahlo cardinal".

THEOREM 1. MAH+ proves Propositions A-E. However, Propositions A,C are not provable in any consistent fragment of MAH that derives Z = Zermelo set theory. In particular, Propositions A-E are not provable in ZFC, provided ZFC is consistent. These facts are provable in RCA_0.

THEOREM 2. MAH+ proves Propositions B,D,E. However, Propositions B,D,E are not provable in ZFC + "there exists a strongly Mahlo cardinal".

THEOREM 3. EFA + Con(MAH) proves Propositions A-E.

THEOREM 4. It is provable in ACA that Propositions A,C are equivalent to Con(MAH). It is provable in EFA that Proposition C is equivalent to Con(MAH).

THEOREM 5. Proposition F can be proved in MAH+ with full abbreviation power using at most 10^6 symbols. Every proof of Proposition F in ZFC with full abbreviation power uses at least 10^{1000} symbols.

The numbers in Proposition F are expected to come down substantially. Note that we have chosen to use factorial notation for ease of reading, which does make the numbers bigger than they would be otherwise.