SUBTLE CARDINALS AND LINEAR ORDERINGS

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INTRODUCTION

The subtle, almost ineffable, and ineffable cardinals were introduced in an unpublished 1971 manuscript of R. Jensen and K. Kunen, and a number of basic facts were proved there. These concepts were extended to that of k-subtle, k-almost ineffable, and k-ineffable cardinals in [Ba75], where a highly developed theory is presented.

This important level of the large cardinal hierarchy was discussed in some detail in the survey article [KM78], section 20. However, discussion was omitted in the subsequent [Ka94]. This level is associated with the discrete/finite combinatorial independence results in [Fr98] and [Fr].

In section 1 we give a self contained treatment of the basic facts about this level of the large cardinal hierarchy, which were established in [Ba75]. In particular, we give a proof that the k-subtle, k-almost ineffable, and k-ineffable cardinals define three properly intertwined hierarchies with the same limit, lying strictly above “total indescribability” and strictly below “arrowing □.”

The innovation here is presented in section 2, where we take a distinctly minimalist approach. Here the subtle cardinal hierarchy is characterized by very elementary properties that do not mention closed unbounded or stationary sets. This development culminates in a characterization of the hierarchy by means of a striking universal second order property of linear orderings.

As is usual in set theory, we treat cardinals and ordinals as von Neumann ordinals. We use ω for the first limit ordinal, which is also N. For sets X and integers $k \geq 1$, we let $S_k(X)$ be the set of all k element subsets of X.

To orient the reader, we mention two results proved in this paper.

The first is an important result from [Ba75] which is perhaps the simplest way of defining the k-ineffable cardinals from the point of view of set theoretic Ramsey theory.

We say that λ is k-SRP if and only if λ is a limit ordinal, and for every $f:S_k(\lambda) \to \{0,1\}$, there exists a stationary $E \subseteq \lambda$ such that f is constant on $S_k(E)$. Here SRP stands for "stationary Ramsey property."

In [Ba75] it is proved that for all $k \geq 0$ and ordinals λ, λ is k-ineffable if and only if λ is a regular cardinal with (k+1)-SRP.

The second is a new result that is the most elementary way we know of defining the least k-subtle cardinal.

We say that a linear ordering (X,<) is k-critical if and only if it has no endpoints, and:

$$\text{for all regressive } f:X^k \to X, \text{ there exists } b_1 < \ldots < b_{k+1} \text{ such that } f(b_1,\ldots,b_k) = f(b_2,\ldots,b_{k+1}).$$

We prove that for $k \geq 0$, the least k-subtle cardinal is the least cardinality of a (k+1)-critical linear ordering.

All of the results in this paper are proved in ZFC.

1.THE BAUMGARTNER DEVELOPMENT

All results in this section are due to [Ba75] except for Lemmas 1.13, 1.22, and 1.24, (and the associated parts of Theorem 1.25), which are due to the author. Also Lemma 1.21 is well known, with references as indicated.

All of the results in this paper are proved in ZFC.

This section reproves most of the results of [Ba75], but does not include a number of additional interesting results from [Ba75] of a more specialized nature.

For sets X, we use $|X|$ for the cardinal of X. Let λ be a limit ordinal. We say that $C \subseteq \lambda$ is unbounded if and only if for all $\alpha < \lambda$ there exists $\beta \in C$ such that $\beta \geq \alpha$. We say that C is closed if and only if for all limit ordinals $x < \lambda$, if the sup of the elements of C below x is x, then $x \in C$. We say that $A \subseteq \lambda$ is stationary if and only if it intersects every closed unbounded subset of λ.

For sets A, let S(A) be the set of all subsets of A. For integers $k \geq 1$, let $S_k(A)$ be the set of all k element subsets of A.

For any set A of ordinals and $1 \leq i \leq k$, we define A_i to be the i-th least element of A. We count from 1, so that $\min(A) = A_1$.

We say that $f:S_k(\lambda) \to \lambda$ is regressive if and only if for all $A \in S_k(\lambda)$, if $\min(A) > 0$ then $f(A) < \min(A)$. We say that E is f-homogenous if and only if $E \subseteq \lambda$ and for all $B,C \in S_k(E)$, $f(B) = f(C)$.

We say that $f:S_k(\lambda) \to S(\lambda)$ is regressive if and only if for all $A \in S_k(\lambda)$, $f(A) \subseteq \min(A)$. We say that E is f-homogenous if and only if $E \subseteq \lambda$ and for all $B,C \in S_k(E)$, we have

$$f(B) \cap \min(B \cup C) = f(C) \cap \min(B \cup C).$$

NOTE: There is a slight abuse of notation here in that every map into λ can be viewed as a map into $S(\lambda)$. The context will always be made clear.

We now give the three leading definitions from [Ba75].

Let $k \geq 1$. λ is k-subtle if and only if

i) λ is a limit ordinal;
ii) For all closed unbounded $C \subseteq \lambda$ and regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \in S_{k+1}(C)$.

λ is k-almost ineffable if and only if

i) λ is a limit ordinal;
ii) For all regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous $A \subseteq \lambda$ of cardinality λ.

λ is k-ineffable if and only if

i) λ is a limit ordinal;
ii) For all regressive $f:S_k(\lambda) \to S(\lambda)$, there exists an f-homogenous stationary $A \subseteq \lambda$.

We say that λ is subtle, almost ineffable, ineffable, respectively, if and only if λ is 1-subtle, 1-almost ineffable, 1-ineffable. It is convenient to identify $S_1(\lambda)$ with λ.

Strictly speaking, [Ba75] uses "infinite cardinal" in place of "limit ordinal," which doesn't make any difference in light of Lemma 1.1 below. The use of closed unbounded sets and stationary sets in λ under either approach causes some problems in the theory until we prove that λ must be an uncountable regular cardinal, so that Fodor's theorem is applicable. This is the first order of business. Of course, much more is true of λ as we shall see. See [Fo56] and [Je78], p. 59.

LEMMA 1.1. Let $k \geq 1$ and λ be an ordinal. If λ is k-subtle, k-almost ineffable, or k-ineffable, then λ is an uncountable regular cardinal.

Proof: Let $k \geq 1$, λ be a limit ordinal, and λ not be a regular cardinal. Let x be the cofinality of λ, which must be an infinite cardinal. Let C be a closed unbounded subset of $\lambda\backslash(x+1)$ whose order type is x. Define regressive $f:S_k(\lambda\backslash x) \to S(\lambda)$ by $f(A) = \{\beta\}$, where β is the index in C of the greatest element of C that is $\leq A_1$; $\varnothing$ if $A_1 < \min(C)$. Then f provides a counterexample to λ being k-subtle, k-almost ineffable, or k-ineffable. If $\lambda = \omega$, then $f:S_k(\omega) \to S(\omega)$ given by $f(A) = \{\min(A)-1\}$ if $\min(A) > 0$; $\varnothing$ otherwise, provides the required counterexamples.

[Ba75] does not define 0-subtle, 0-almost ineffable, and 0-ineffable cardinals. From Lemma 1.1 and other considerations, it is best to define these as "uncountable regular cardinals."

LEMMA 1.2. Let $k \geq 0$ and ▯ be an ordinal. If ▯ is k-ineffable then ▯ is k-almost ineffable. If ▯ is k-almost ineffable then ▯ is k-subtle.

Proof: The first claim is trivial. The second claim is trivial for $k = 0$, and so assume $k > 0$. Let ▯ be k-almost ineffable. Let $f:S_k$(▯) ▯ S(▯) be regressive and C ▯ ▯ be closed and unbounded. Define $g:S_k$(▯) ▯ S(▯) as follows. Let A ▯ S_k(▯).

case 1. A ▯ $S_k(C)$. Set $g(A) = f(A)$.

case 2. A ▯ $C = \emptyset$. Set $g(A) = A_1$ ▯ C.

case 3. otherwise. Set $g(A) = \emptyset$.

Let E ▯ ▯ be an unbounded g-homogenous set. Suppose that there are arbitrarily large elements of E lying outside C. Then we contradict the g-homogeneity of A using case 2. Hence E ▯ C is unbounded in ▯. By case 1, E ▯ C is f-homogenous.

LEMMA 1.3. Let $k \geq n \geq 0$ and ▯ be an ordinal. If ▯ is k-ineffable then ▯ is n-ineffable. If ▯ is k-almost ineffable then ▯ is n-almost ineffable. If ▯ is k-subtle then ▯ is n-subtle.

Proof: Fix k,n,▯ as given. By Lemma 1.1 we can assume that $k,n > 0$. Let $f:S_n$(▯) ▯ S(▯) be regressive. We define $g:S_k$(▯) ▯ S(▯) by $g(A) = f(\{A_1,...,A_k\})$. Any g-homogenous set is f-homogeneous.

We now want to show that k-subtle cardinals have some stronger Ramsey properties in order to facilitate the later results.

LEMMA 1.4. Let $k \geq 1$ and ▯ be a k-subtle ordinal. Let C ▯ ▯ be closed and unbounded and $f:S_k(C)$ ▯ S(▯) be regressive. Then there exists an f-homogenous E ▯ $S_{k+1}(C)$ such that every element of E is an uncountable regular cardinal.

Proof: Let k,▯,C,f be as given. By Lemma 1.1, ▯ is an uncountable regular cardinal. Let C' be the set of all limit ordinals > □ in C. We define g:S_k(□) □ S(□) as follows. Let A □ S_k(□).

case 1. A □ S_k(C′) and every element of A is an uncountable regular cardinal. Set g(A) = (1+f(A)) □ {0}. (Here we add 1 to every element of f(A)).

case 2. A □ S_k(C′) and no element of A is an uncountable regular cardinal. Set g(A) to be an unbounded subset of A_1 of order type cf(A_1) whose least element is cf(A_1).

case 3. A □ S_k(C′) and some but not all elements of A are uncountable regular cardinals. Set g(A) = {i: A_i is an uncountable regular cardinal}.

case 4. A □ S_k(C′). Set g(A) = ∅.

Note that the outputs from these four cases are mutually disjoint.

Since □ is k-subtle, let E □ S_{k+1}(C′) be g-homogenous. If every element of E is an uncountable regular cardinal then E is as required.

Now suppose no element of E is an uncountable regular cardinal. Then g({E_1,...,E_k}) = g({E_2,...,E_{k+1}}) □ E_1, and so min(g({E_1,...,E_k})) = min(g({E_2,...,E_{k+1}})) = cf(E_1). This is a contradiction since both sets are respectively unbounded in the limit ordinals E_1 and E_2 and have order type cf(E_1).

Finally suppose some but not all elements of E are uncountable regular cardinals. Let i be such that E_i is an uncountable regular cardinal if and only if E_{i+1} is not an uncountable regular cardinal. Now g({E_1,...,E_k}) = g({E_1,...,E_{i-1},E_{i+1},...,E_{k+1}}). If k = 1 then different cases apply to the two terms, which is a contradiction. If k ≥ 2 then case 3 must apply to at least one of the two terms, and hence to both. We again have a contradiction by inspection of case 3.

It is useful to consider regressive functions on $S_{\leq k}$(□). Here $S_{\leq k}$(□) is the set of all nonempty subsets of □ of cardinality ≤ k. We say that f:$S_{\leq k}$(□) □ S(□) is regressive if and only if for all A □ $S_{\leq k}$(□), f(A) □ min(A). We say that E is f-homogenous if and only if E □ □ and for all C,D □ $S_{\leq k}$(E) of the same cardinality, we have

$$f(C) \cap \min(C \cap D) = f(D) \cap \min(C \cap D).$$

We make use of the one-one onto pairing function $< >:On^2 \rightarrow On\backslash\{0\}$ which enumerates the ordered pairs of ordinals first by the maximum term, and second lexicographically. We also use < > extended to all nonempty finite sequences of ordinals by left associativity, and $<\alpha> = 1+\alpha$. We also use < > to code finite tuples of sets of ordinals as a set of ordinals: $<A_1,...,A_n> = \{<x_1,...,x_n>: x_1 \in A_1, ... , x_n \in A_n\}$.

An ordinal x is called adequate if it is nonzero and < > maps x^2 into x. Note that for all infinite cardinals κ, the set of adequate ordinals $< \kappa$ is closed and unbounded in κ, and includes all infinite cardinals. Also, every adequate ordinal is a limit ordinal. The first adequate ordinal is ω.

LEMMA 1.5. Let $k \geq 1$ and λ be a k-subtle ordinal. Let $C \subseteq \lambda$ be closed and unbounded and $f:S_{\leq k}(\lambda) \rightarrow S(\lambda)$ be regressive. Then there exists an f-homogenous $E \in S_{k+1}(C)$ such that every element of E is an uncountable regular cardinal.

Proof: Let k,λ,C,f be as given. Then λ is an uncountable regular cardinal. We can assume that C consists of adequate ordinals. Define $g:Sk(\lambda) \rightarrow S(\lambda)$ as follows. Let $A \in Sk(\lambda)$.

case 1. A_1 is adequate. Set g(A) to code up all of the f(B), where $B \subseteq A$ and $\min(B) = A_1$.

case 2. A_1 is not adequate. Set $g(A) = \emptyset$.

By Lemma 1.4, let $E \in S_{k+1}(C)$ be g-homogenous, where every element of E is an uncountable regular cardinal. Let $1 \leq p \leq k$ and $\{E_{i\_1},...,E_{i\_p}\} \subseteq E$ be given, where $1 \leq i_1 < ... < i_p \leq k+1$. If $i_1 > 1$ then we can use $g(E_1,...,E_k\} = g(E_2,...,E_{k+1})$ to obtain $f(E_{i\_1},...,E_{i\_p}\}) = f(E_{i\_1-1},...,E_{i\_p-1}\}$. We can continue to shift the indices down until the first index is 1. Then we can shift the indices > 1 down until the second index is 2, by using $f(E_1,...,E_k\} = f(E_1,E_3,...,E_{k+1}\}$. We can continue this process and finally obtain $f(\{E_{i\_1},...,E_{i\_p}\}) = f(\{E_1,...,E_i\})$.

LEMMA 1.6. Let $k \geq 1$ and λ be a k-subtle ordinal. Let $f:S_k(\lambda) \rightarrow S(\lambda)$ be regressive and $C \subseteq \lambda$ be closed and unbounded. Then there exists an uncountable regular cardinal $\kappa \in C$ and $E \subseteq C$

□ □ which is f-homogenous and stationary in □. In particular, there are f-homogenous E □ C of every cardinality < □.

Proof: The second claim follows easily from the first claim as follows. Let □ < □ be a cardinal and let C = [□,□). Let □,E be as provided by the first claim. Then E is f-homogenous, E □ C, and |E| ≥ □.

For the first claim, let k,□,f,C be as given. Without loss of generality, we may assume that every element of C is adequate. Assume that the conclusion is false. We define g:$S_{\leq k}$(□) □ S(□) as follows. Let A □ $S_{\leq k}$(□).

case 1. A □ S_k(□) and A_1 is an uncountable regular cardinal. For each 1 ≤ i ≤ k, let W_i(A) be the set of all x < A_1 such that for all B □ S_{i-1}(x),

f(B □ {x} □ {A_{i+1},...,A_k}) = f(B □ {A_i,...,A_k}).

Let W(A) be the intersection of the W_i(A). Below we verify that W(A) □ A_1 must be f-homogenous.

Since the conclusion is false, W(A) must not be stationary in A_1. Set g'(A) to be a closed unbounded subset of A_1\W(A). Finally, set g(A) = <g'(A),f(A)>.

case 2. A □ $S_{<k}$(□) and A_1 is an uncountable regular cardinal. Let |A| = 1 ≤ i < k. Set g(A) = {<B,x>: B □ S_{k-i}(A_1) and x □ f(B □ A)}.

case 3. otherwise. Set g(A) = 0.

By Lemma 1.5, let E □ S_{k+1}(C) be g-homogenous, where every element of E is an uncountable regular cardinal. So case 1 applies to every g(A), A □ S_k(E), and case 2 applies to every g(A), A □ $S_{<k}$(E).

As promised, we verify that for all A □ S_k(□), W(A) is f-homogenous. In fact, we verify that for all A □ S_k(□), f is constantly f(A) on S_k(W(A)). To see this, let Y □ S_k(W(A)). f(Y) = f({Y_1,...,Y_{k-1}} □ {Y_k} □ ∅) = f({Y_1,...,Y_{k-2}} □ {A_k}) = f({Y_1,...,Y_{k-3}} □ {Y_{k-2}} □ {A_k}) = f({Y_1,...,Y_{k-3}} □ {A_{k-1},A_k}) = ... = f({A_1,...,A_k}).

By the homogeniety of E, $g'(\{E_1,\ldots,E_k\}) = g'(\{E_2,\ldots,E_{k+1}\})$ □ E_1. Hence E_1 □ $g(\{E_2,\ldots,E_{k+1}\})$, since $g'(\{E_1,\ldots,E_k\})$ □ E_1 is closed and unbounded.

However, we claim that E_1 □ $W_i(\{E_2,\ldots,E_{k+1}\})$, $1 \leq i \leq k$. To see this, let $1 \leq i \leq k$ and B □ $S_{i-1}(E_1)$. We must verify that f(B □ $\{E_1\}$ □ $\{E_{i+2},\ldots,E_{k+1}\}$) = f(B □ $\{E_{i+1},\ldots,E_{k+1}\}$). But this follows from the g-homogeneity of E by inspection of case 2.

Thus we have shown that E_1 □ $W(\{E_2,\ldots,E_{k+1}\})$, which contradicts the disjointness of $W(\{E_2,\ldots,E_{k+1}\})$ and $g'(\{E_2,\ldots,E_{k+1}\})$.

LEMMA 1.7. Let $k \geq 1$ and □ be a k-ineffable ordinal. Then {□ < □: □ is a k-almost ineffable ordinal} is stationary in □.

Proof: Let k,□ be as given. Let C □ □ be closed and unbounded, where no elements are k-almost ineffable ordinals. We can assume that every element of C is a limit ordinal > □. By Lemma 1.1, □ is an uncountable regular cardinal.

We define g:□ □ S(□) as follows. Let □ < □.

case 1. □ □ C and □ is a cardinal. Let g'(□) code a counterexample to the k-almost ineffability of □. I.e., g'(□) codes a regressive function $h_□:S_k$(□) □ S(□) where there is no h-homogenous subset of □ of cardinality □. Set g(□) = (1+g(□)) □ {0}.

case 2. □ □ C and □ is not a cardinal. Set g(□) = |□|+1.

case 3. □ □ C and □ > min(C). Set g(□) = {□}, where □ is the greatest element of C that is strictly below □.

case 4. □ < min(C). Set g(□) = ∅.

By Lemma 1.3, □ is ineffable. Let E □ □ be g-homogenous and stationary. By examination of the cases, we see that all sufficiently large elements of E are infinite cardinals and lie in C. Let E' be the set of all infinite cardinals lying in E □ C. By case 1, the $h_□$, □ □ E', are compatible.

Let $h:S_k$(□) □ S(□) be the union of the $h_□$, where □ □ E.

By the k-ineffability of □, let B □ □ be h-homogenous and stationary. Since B is stationary and there are arbitrarily large infinite cardinals □ < □, we see that C′ = {□ < □: □ is an infinite cardinal and B □ □ is unbounded in □} is unbounded in □. (Assume otherwise, and define the obvious regresive function on B, and apply Fodor's theorem to B). Hence C′ □ □ is closed and unbounded.

Let □ □ C′ □ E′. Then B □ □ is unbounded in the infinite cardinal □, and $h_□$-homogenous, which is the desired contradiction.

LEMMA 1.8. Let k ≥ 1 and □ be a k-almost ineffable ordinal. Then {□ < □: □ is a k-subtle ordinal} is stationary in □.

Proof: Let k,□ be as given. Let C □ □ be closed and unbounded, where no element of C is a k-subtle ordinal. We can assume that C consists of adequate ordinals > □. By Lemma 1.1, □ is an uncountable regular cardinal.

We define g:□ □ S(□) as follows. Let □ < □.

case 1. □ □ C and □ is a cardinal. Set g′(□) to code a counterexample to the k-subtlety of □. I.e., g′(□) codes a regressive function $h_□$:S_k(□) □ S(□) and a closed unbounded $C_□$ □ □, where there is no h-homogenous E □ S_{k+1}($C_□$). Set g(□) = (1+g′(□)) □ {0}.

case 2. □ □ C and □ is not a cardinal. Set g(□) = |□|+1.

case 3. □ □ C and □ > min(C). Set g(□) to be the largest element of C strictly below □.

case 4. □ < min(C). Set g(□) = ∅.

By Lemma 1.3, □ is almost ineffable. Let E □ □ be g-homogenous and unbounded. By examination of cases 2 and 3, we see that all sufficiently large elements of E are infinite cardinals and lie in C. Thus we can assume that every element of E is an infinite cardinal and lies in C.

Let h:S_k(□) □ S(□) be the union of the $h_□$, where □ □ E. And let C′ be the union of the $C_□$, where □ □ E. Then h is regressive and C′ is closed and unbounded in □.

We define h′:S_k(□) □ S(□) as follows. Let A □ S_k(□).

case 1. A_1 □ C′ □ C. Set h′(A) = h(A).

case 2. A_1 □ C′ □ C and A_1 > min(C′ □ C). Set h′(A) to be the greatest element of C′ □ C that is strictly less than A_1.

case 3. A_1 < min(C′ □ C). Set h′(A) = ∅.

Since □ is k-almost ineffable, let B □ □ be h′-homogenous and unbounded. Then clearly all sufficiently large elements of B must lie in C′ □ C. Let B′ be the first k+1 elements of B □ C′ □ C and let □ □ E, where □ > max(B′). Then □ is an infinite cardinal and B′ □ S_{k+1}(C′ □ C) is $h_□$-homogenous. This is a contradiction.

LEMMA 1.9. Let k ≥ 1 and □ be a k-subtle ordinal. Then {□ < □: □ is a (k-1)-ineffable cardinal} is stationary in □.

Proof: Let k,□ be as given. Let C □ □ be closed and unbounded, where no element of C is a (k-1)-ineffable cardinal. We can assume that C consists of adequate ordinals > □. By Lemma 1.1, □ is an uncountable regular cardinal.

For each limit ordinal □ □ C, let $h_□$:S_k(□) □ S(□) be a counterexample to the (k-1)-ineffability of □.

We define g:S_k(□) □ S(□) as follows. Let A □ S_k(□).

case 1. A_1 □ C. Set g(A) = h_{A_k}({A_1,...,A_{k-1}}).

case 3. A_1 □ C. Set g(A) = ∅.

By Lemma 1.6, let □ □ C be an uncountable regular cardinal, and E □ C □ □ be g-homogenous and stationary in □. Then clearly E is $h_□$-homogenous, which is a contradiction.

LEMMA 1.10. Every subtle ordinal is a strongly inaccessible cardinal.

Proof: Let □ be a subtle ordinal. By Lemma 1.1, □ is an uncountable regular cardinal. It suffices to prove that □ is a strong limit. Suppose □ is not a strong limit. Let □ < □ and f:□ □ S(□) be one-one. Let E □ S_2((□,□)) be f-homogenous. This is a contradiction.

Let □ be a cardinal. We say that □ is totally indescribable if and only if for all B □ V(□) and all sentences □ involving □,=,B, that are $□^n_m$, for some n,m ≥ 1, if

φ holds in $(V(\kappa),\in,B)$,

then there exists $\alpha < \kappa$ such that

φ holds in $(V(\alpha),\in,B \cap V(\alpha))$.

See [Ka94], p. 59.

LEMMA 1.11. Let λ be a subtle ordinal. Then $\{\kappa < \lambda$: κ is a totally indescribable cardinal for all $n,m \geq 1\}$ is stationary in λ.

Proof: Let λ be a subtle ordinal. Let $C \subseteq \lambda$ be closed and unbounded, where no element of C is a totally indescribable cardinal. By Lemma 1.10, we can assume that C consists entirely of cardinals α such that $|V(\alpha)| = \alpha$. We define $f:\lambda \to S(\lambda)$ as follows. Let $\alpha < \lambda$.

case 1. $\alpha \in C$. Set $f(\alpha)$ to code up a counterexample to the total indescribability of α. This consists of a suitable sentence φ_α and a set $B_\alpha \subseteq V(\alpha)$.

case 2. $\alpha \notin C$. Set $f(\alpha) = \emptyset$.

Let $\{(\alpha,\beta\})\subseteq C$ be f-homogenous, where $\alpha < \beta$. Then $\varphi_\alpha = \varphi_\beta$ and $B_\alpha = B_\beta \cap \alpha$. But also φ_β holds in $(V(\alpha),\in,B_\alpha)$ and $(V(\beta),\in,B_\beta)$, which is a contradiction.

We say that κ is a 0-Mahlo cardinal if and only if κ is a strongly inaccessible cardinal. Let $k \geq 0$. We say that κ is a k+1-Mahlo cardinal if and only if $\{\alpha < \kappa$: α is a k-Mahlo cardinal$\}$ is stationary in κ. See [Ka94], p. 21.

Let $k \geq 1$ and λ be an ordinal. We say that λ is k-RP if and only if for all $f:S_2(\lambda) \to \{0,1\}$, f is constant on some subset of λ of cardinality λ. Here "RP" abbreviates "Ramsey property."

Weakly compact cardinals are defined in terms of infinitary lanaguages (see [Ka94], p. 37). For our purposes it suffices to use the following equivalence of weak compactness: κ is a weakly compact cardinal if and only if κ is an uncountable cardinal with 2-RP. See [Ka94], p. 37, 76, with references to Erdös, Tarski, Hanf, Monk, and Scott, from 1943 to 1964.

LEMMA 1.12. Every subtle cardinal is greater than the first weakly compact cardinal. Every subtle cardinal is n-Mahlo for all n ≥ 0. The first subtle cardinal is not weakly compact, whereas every almost ineffable and ineffable cardinal is weakly compact.

Proof: The first claim follows from Lemma 1.11 since the weakly compact cardinals are exactly the $□^1{}_1$-indescribable cardinals, and so every totally indescribable cardinal is weakly compact. See [HS61] and [Ka94], p. 59.

For the second claim, by Lemma 1.10, every subtle cardinal is 0-Mahlo. Now use Lemma 1.11 and that every weakly compact cardinal is n-Mahlo for all n ≥ 0. See [Ha64] and [Ka94], p. 41.

For the third claim, note that the first subtle cardinal is $□^1{}_1$-describable, and hence not weakly compact. And obviously every almost ineffable cardinal obeys the partition definition of weak compactness.

Let $S_<$ (A) be the set of all nonempty finite subsets of A. Let f:$S_<$ (□) □ □, where □ is a cardinal. We say that E □ □ is f-homogenous if and only if for all A,B □ $S_<$ (□) of the same cardinality, f(A) = f(B).

We write □ □ □ if and only if □ is a cardinal such that for all f:$S_<$ (□) □ □, there exists an infinite f-homogenous set. Note that if □' ≥ □ and □ □ □, then □' □ □.

We say that □ is a totally ineffable cardinal if and only if for all k □ □, □ is a k-ineffable cardinal.

LEMMA 1.13. Let □ □ □. Then there exists a totally ineffable cardinal < □.

Proof: Let □ be the least cardinal such that □ □ □. Then □ is a strongly inaccessible cardinal (see [Si66] and [Ka94], p. 82). The proof below is suggested by the known weaker result that there exists a totally indescribable cardinal < □. See [RS65] and [Ka94], p. 109.

Add countably many Skolem functions in the usual way to the structure (V(□),□). Let E be a set of indiscernibles of order type □ for the augmented structure. This is obtained as an infinite f-homogenous set where $f:S_<(□)$ □ $2^{□}$, which we can find because of [Si66] and [Ka94], p. 82.

Now let M be the subset of V(□) generated by E and the Skolem functions. We use M* to represent M,□, together with the Skolem functions (of several variables) from M into M. We also use V(□)* to represent V(□),□, together with the Skolem functions (of several variables) from V(□) into V(□). Then M* is an elementary submodel of V(□)*. Also, there is an elementary embedding j:M* □ M* which moves an ordinal and maps M into M. In fact, j moves each element of E to the next largest element of E.

By taking transitive collapses, we have

i) a countable transitive set N together with Skolem functions, written N*;
ii) an elementary embedding j:N* □ N* which moves an ordinal;
iii) N* and V(□)* are elementarily equivalent.

Since □ is strongly inaccessible, we see that N satisfies ZFC. It suffices to prove that N satisfies that there exists a totally ineffable cardinal.

Let □ be the critical point of j; i.e., the first ordinal moved by j. By a standard argument, □ is a strongly inaccessible cardinal in N. So each iterate of j at □ is a strongly inaccessible cardinal in N. Let these iterates be written $□ = □_0 < □_1 < □_2$ ... .

It will be conceptually clearer to pass to the submodel N** of N* generated by $\{□_0,□_1,...\}$ and the Skolem functions of N*. Note that j:N** □ N** since $j(t(□_0,...,□_n)) = t(j(□_0),...,j(□_n)) = t(□_1,...,□_{n+1})$. N** is an elementary submodel of N*. And then we can again go to the transitive collapse, R. Thus we have

iv) a countable transitive set R together with Skolem functions, written R*;
v) an elementary embedding j:R* □ R* with critical point $□_0$;
vi) the iterates of j at $□_0$ are $□_0 < □_1 < □_2 < ...$;
vii) every element of R is generated from $\{□_0,□_1,...\}$ and the Skolem functions;
viii) R* and V(□)* are elementarily equivalent.

We now claim that

ix) for all formulas □$(x_1,\ldots,x_n,y_1,\ldots,y_m)$, if $0 \leq i_1 < \ldots < i_m$, $0 \leq j_1 < \ldots < j_m$, and $x_1,\ldots,x_n <$ min(□$_{i\_1}$,□$_{j\_1}$), then □$(x_1,\ldots,x_n,$□$_{i\_1},\ldots,$□$_{i\_m})$ □ □$(x_1,\ldots,x_n,$□$_{j\_1},\ldots,$□$_{j\_m})$ in R*.

To see this, first note that for any formula □$(x_1,\ldots,x_n,y_1,\ldots,y_m)$, if $0 = i_1 < \ldots < i_m$, and $x_1,\ldots,x_n <$ □$_0$, then □$(x_1,\ldots,x_n,$□$_{i\_1},\ldots,$□$_{i\_m})$ □ □$(x_1,\ldots,x_n,$□$_{i\_1+1},\ldots,$□$_{i\_m+1})$. This is because j is an elementary embedding which fixes all ordinals < □$_0$.

Now for fixed □,n,$i_1,\ldots,i_m$, we can construe the above as a universal sentence. We can then apply iterates of j to the statement so construed, and obtain that for any formula □$(x_1,\ldots,x_n,y_1,\ldots,y_m)$, if $r = i_1 < \ldots < i_m$, and $x_1,\ldots,x_n <$ □$_r$, then □$(x_1,\ldots,x_n,$□$_{i\_1},\ldots,$□$_{i\_m})$ □ □$(x_1,\ldots,x_n,$□$_{i\_1+1},\ldots,$□$_{i\_m+1})$.

We can now obtain ix) as follows. Assume $i_1 \leq j_1$. First push □$_{i\_1}$ up to □$_{j\_1}$ by applying the previous paragraph $j_1 - i_1$ times. Then push the replacement for □$_{i\_2}$ up or down to □$_{j\_2}$ by applying the previous paragraph the required number of times. Continue in this way, until the left hand side coincides with the right hand side.

We now show that R satisfies that □$_0$ is totally ineffable. As a consequence, V(□) satisfies that there exists a totally ineffable cardinal, and we are done. Fix $n \geq 1$. We now show that R satisfies that □$_0$ is n-ineffable.

Suppose R satisfies that □$_0$ is not n-ineffable. Then R satisfies that □$_1$ is not n-ineffable. Accordingly, let f:S_n(□$_1$) □ S(□$_1$) be a regressive function in R, where there is no f-homogenous stationary subset of □$_1$, and where f is given by a Skolem term in □$_1$. We wish to show that in R, there is an f-homogenous set that is stationary in □$_1$.

Let $1 \leq i \leq n$. Define W_i to be the set of all x < □$_1$ such that for all B □ $S_{i-1}(x)$,

f(B □ {x} □ {□$_{i+1},\ldots,$□$_n$}) = f(B □ {□$_i,\ldots,$□$_n$}).

Let W be the intersection of the W_i, $1 \leq i \leq n$. This definition of the W_i and W is carried out in R. It is easy to see that W □ □$_1$ is f-homogenous as in the proof of Lemma 1.6.

Using clause ix), it is easy to see that $\square_0$ □ W.

It now suffices to prove that W is stationary in $\square_1$ in the sense of R. Note that W is given by a Skolem term $t(\square_1,...,\square_n)$.

Assume W is not stationary in $\square_1$. Then W is disjoint from a closed unbounded C □ $\square_1$ which is also given by a term $r(\square_1,...,\square_n)$.

Using ix), we see that $r(\square_0,...,\square_{n-1})$ is a closed unbounded C′ □ $\square_0$, and also C′ = C □ $\square_0$. Hence $\square_0$ □ C, contradicting $\square_0$ □ W. Hence W is stationary in $\square_1$.

We now consider regressive functions $f:S_k(\square)$ □ □. These versions of subtlety, almost ineffability, and ineffability are simpler, and we will show that they are equivalent up to a shift of one in the parameter k.

Let k ≥ 1. □ is k-subtle' if and only if

i) □ is a limit ordinal;
ii) For all closed unbounded C □ □ and regressive $f:S_k(\square)$ □ □, there exists E □ $S_{k+1}(C)$ such that f is constant on $S_k(E)$.

□ is k-almost ineffable' if and only if

i) □ is a limit ordinal;
ii) For all regressive $f:S_k(\square)$ □ □, there exists E □ □ of cardinality □ such that f is constant on $S_k(E)$.

□ is k-ineffable' if and only if the following holds:

i) □ is a limit ordinal;
ii) For all regressive $f:S_k(\square)$ □ □, there exists stationary E □ □ such that f is constant on $S_k(E)$.

If f is constant on $S_k(E)$ then we say that E is f-homogenous.

LEMMA 1.14. Let k ≥ 0 and □ be an ordinal. If □ is k-subtle then □ is (k+1)-subtle'; if □ is k-ineffable then □ is (k+1)-ineffable'.

Proof: The case k = 0 is proved by Fodor's theorem. Now assume k ≥ 1 and let f:S_{k+1}(□) □ □ be regressive. We handle both claims by a single construction. By Lemma 1.1, we can assume that □ is uncountable and regular.

We define a regressive function g:S_k(□) □ S(□) as follows. Let A □ S_k(□).

Case 1. A_1 is adequate. Define g(A) = {<0,x,y>: x,y < A_1 & f(A □ {x}) = y} □ {<1,y,y>: y,1 < A_1 & (□x < A_1)(f(A □ {x}) = y)} □ {0}.

Case 2. A_1 is not adequate, A1 > □. Define g(A) = {x}, where x is the largest adequate ordinal < A_1.

Case 3. A_1 < □. Define g(A) = ∅.

Claim I: Let A □ S_{k+1}(□) be g-homogenous, where A_1 is adequate. Then there exists x < A_1 such that A □ {x} is f-homogenous.

First note that case 1 applies to g({A_1,...,A_k}), and so case 1 applies to all g(B), B □ A. Secondly, note that <1,y,y> □ g({A_2,...,A_{k+1}}), where y = f(A). Hence <1,y,y> □ g({A_1,...,A_k}). So we can let <0,x,y> □ g({A_1,...,A_k}), and f({x,A_1,...,A_k}} = y, where x,y < A_1. It is now clear that f is constant on S_{k+1}(A □ {x}), with constant value y.

Suppose □ is k-subtle. To see that □ is k+1-subtle', let f:S_{k+1}(□) □ □ be regressive and C □ □ be closed and unbounded. Let C' be the set of all adequate elements of C. Let A □ S_k(C') be g-homogenous. Then by claim I, A is f-homogenous.

Claim II: Let A □ □ be g-homogenous, where A is unbounded in □. Then every element of A is adequate.

First note that every element of A is infinite. Now suppose there exists x □ A which is not adequate. Let b be the greatest adequate ordinal < x. Then g is constantly {b} on S_k(A). This contradicts the fact that there are arbitrarily large adequate ordinals < □.

Suppose □ is k-ineffable. To see that □ is k+1-ineffable', let f:S_{k+1}(□) □ □ be regressive. Let A □ □ be g-homogenous and stationary in □. By claim II, every element of A is adequate. It is now easy to see that f agrees at any two elements of $S_{k+1}(A)$ with the same minimum element. This defines a regressive function from A into □, which therefore is constant on a stationary subset A′ of A by Fodor's theorem and the fact that □ is uncountable and regular. Then A′ is the required f-homogenous set.

The proof that k-almost ineffable □ k+1-almost ineffable′ is more difficult. We begin with two Lemmas that are analogous to Lemmas by proving a result analogous to Lemmas 1.4 and 1.5 for almost ineffability.

LEMMA 1.15. Let $k \geq 1$ and □ be a k-almost ineffable ordinal. Let f:S_k(□) □ S(□) be regressive. Then there exists an f-homogenous E □ □ of cardinality □ such that every element of E is an uncountable regular cardinal.

Proof: Let k,□,f be as given. We define g:S_k(□) □ S(□) as follows. Let A □ S_k(□).

case 1. A_1 is an uncountable regular cardinal. Set g(A) = f(A).

case 2. A_1 is a limit ordinal that is not a regular cardinal. Set g(A) to be an unbounded subset of A_1 of order type cf(A_1) whose least element is cf(A_1).

case 3. A_1 is a successor ordinal. Set g(A) = {A_1-1}.

case 4. A_1 = □ or 0. Set g(A) = ∅.

Let E □ □ be unbounded and g-homogenous. By case 2, at most one element of A_1 is a limit ordinal that is not a regular cardinal. By case 3, at most one element of A_1 is a successor ordinal. Hence all sufficiently large elements of E are uncountable regular cardinals.

LEMMA 1.16. Let $k \geq 1$ and □ be a k-almost ineffable ordinal. Let f:$S_{\leq k}$(□) □ S(□) be regressive. Then there exists an f-homogenous E □ S_{k+1}(□) of cardinality □ such that every element of E is an uncountable regular cardinal.

Proof: Let k,□,f be as given. Then □ is an uncountable regular cardinal. Define g:S_k(□) □ S(□) as follows. Let A □ $S_{\leq k}$(□).

case 1. A1 is adequate. Set g(A) to code up all of the f(B), where B □ A and min(B) = A_1.

case 2. A1 is not adequate. Set g(A) = 0.

By Lemma 1.15, let E □ □ be g-homogenous, where every element of E is an uncountable regular cardinal. Then we can argue as in the proof of Lemma 1.5.

We introduce the following modification of Lemma 1.6, which is of some independent interest.

LEMMA 1.17. Let k ≥ 1 and □ be a k-almost ineffable ordinal. Let f:S_k(□) □ S(□) be regressive. Then there exists an f-homogenous E □ □ such that there are arbitrarily large uncountable regular cardinals □ < □ for which E □ □ is stationary in □.

Proof: Let k,□,f be as given. We now define g:$S_{\leq k}$(□) □ S(□). Let A □ $S_{\leq k}$(□).

case 1. A □ S_k(□) and A_1 is an uncountable regular cardinal. For each 1 ≤ i ≤ k, let W_i(A) be the set of all x < A_1 such that for all B □ S_{i-1}(x),

f(B □ {x} □ {A_{i+1},...,A_k}) = f(B □ {A_i,...,A_k}).

Let W(A) be the intersection of the W_i(A). As in the proof of Lemma 1.6, W(A) □ A_1 is f-homogenous.

If W(A) is not stationary in A_1, then set g'(A) to be a closed unbounded subset of A_1\W(A). Finally, set g(A) = <0,g'(A),f(A)>.

If W(A) is stationary in A_1, then set g(A) = <1,f(A),f(A)>.

case 2. A □ $S_{<k}$(□) and A_1 is an uncountable regular cardinal. Let |A| = 1 ≤ i < k. Set g(A) = {<B,x>: B □ S_{k-i}(A_1) and x □ f(B □ A)}.

case 3. otherwise. Set g(A) = 0.

By Lemma 1.16, let E' □ □ be g-homogenous and unbounded, where every element of E' is an uncountable regular cardinal. Then either for all A □ S_k(E'), W(A) is stationary in A_1, or for all A □ S_k(E'), W(A) is not stationary in A_1. In the latter case, we get a contradiction exactly as in the proof of the first claim of Lemma 1.6. Hence for all A □ S_k(E'), W(A) is stationary in A_1. Since E' is g-homogenous, we see that the W(A) cohere; i.e., for all A,B □ S_k(E'), W(A) □ min(A □ B) = W(B) □ min(A □ B). Set E to be the union of the W(A), A □ S_k(E'). Then E is f-homogenous, and is as required.

Lemma 1.17 is not quite in the right form to prove Lemma 1.19. Here is the appropriate modification.

LEMMA 1.18. Let k ≥ 1 and □ be a k-almost ineffable ordinal. Let f:S_{k+1}(□) □ □ be regressive. Then there exists an f-homogenous E □ □ such that there are arbitrarily large uncountable regular cardinals □ < □ for which E □ □ is stationary in □.

Proof: Let k,□,f be as given. We define f':S_k(□) □ S(□) by f'(A) = {<x,y>: x < A_1 and y □ f(A □ {x})}. We now modify the proof of Lemma 1.17. We define g:$S_{\leq k}$(□) □ □. Let A □ $S_{\leq k}$(□).

case 1. A □ S_k(□) and A_1 is an uncountable regular cardinal. For each 1 ≤ i ≤ k, let W_i(A) be the set of all x < A_1 such that for all B □ S_{i-1}(x),

f'(B □ {x} □ {A_{i+1},...,A_k}) = f'(B □ {A_i,...,A_k}).

Let W(A) be the intersection of the W_i(A). We that W(A) □ A_1 is f-homogenous as in the proof of Lemma 1.6.

If W(A) is not stationary in A_1, then set g'(A) to be a closed unbounded subset of A_1\W(A). Set g(A) = <0,g'(A),f'(A)>.

If W(A) is stationary in A_1, then there is a stationary W'(A) □ W(A) that is f-homogenous. To see this, note that the value of f on subsets of W(A) depends only on the min. This defines a regressive function h:W(A) □ □. Then apply Fodor's theorem. Set g(A) = <1,W'(A),W'(A)>.

case 2. A □ $S_{<k}$(□), |A| = 1 ≤ i < k. Set g(A) = {<B,x>: B □ S_{k-i}(A_1) and x □ f(B □ A)}.

case 3. otherwise. Set g(A) = 0.

As in the proof of Lemma 1.17, let E′ □ □ be g-homogenous and unbounded, where every element of E′ is an uncountable regular cardinal. Then either for all A □ $S_k(E')$, W(A) is stationary in A_1, or for all A □ $S_k(E')$, W(A) is not stationary in A_1. In the latter case, we get a contradiction exactly as in the proof of the first claim of Lemma 1.6. Hence for all A □ $S_k(E')$, W′(A) is stationary in A_1 and f-homogenous. Since E′ is g-homogenous, we see that the W′(A) cohere; i.e., for all A,B □ $S_k(E')$, W′(A) □ min(A □ B) = W′(B) □ min(A □ B). Set E to be the union of the W′(A), A □ $S_k(E')$. Then E is f-homogenous, and is as required.

LEMMA 1.19. Let k ≥ 0 and □ be an ordinal. If □ is k-almost ineffable then □ is (k+1)-almost ineffable'.

Proof: Let □ be a k-almost ineffable ordinal. The case k = 0 is trivial. Assume k ≥ 1. Let f:S_{k+1}(□) □ □ be regressive. Choose E according to Lemma 1.18.

LEMMA 1.20. Let k ≥ 1. If □ is k-subtle′, k-almost ineffable′, or k-ineffable′, then □ is an uncountable regular cardinal.

Proof: Let k ≥ 1 and □ be a limit ordinal that is not a regular cardinal. Let C □ □ be a closed unbounded set of order type cf(□). Define f:S_k(□) □ □ as follows. Let A □ S_k(□). Set f(A) to be the order type of the set of all elements of C that are ≤ min(A).

There cannot be any homogenous E □ Sk+1(C), nor any homogenous E □ □ of cardinality □.

We have shown that if □ is k-subtle′, k-almost ineffable′, or k-ineffable′, then □ is an infinite regular cardinal. To complete the proof, observe that □ is not 1-subtle′, 1-almost ineffable′, or 1-ineffable′, using the map g(A) = min(A)-1 if min(A) > 0; 0 otherwise, and C = □\{0}.

LEMMA 1.21. Let k ≥ 1 and □ be an ordinal. **If** □ is k-subtle′ then □ is (k-1)-subtle; □ is k-almost ineffable′ then □ is (k-1)-almost ineffable; □ is k-ineffable′ then □ is (k-1)-ineffable.

Proof: The the case k = 1 follows from Lemma 1.20 and Fodor’s theorem.

Let k ≥ 1 and □ be an uncountable regular cardinal. Let f:S_k(□) □ S(□) be regressive. We define g:S_{k+1}(□) □ □ as follows. Let A □ S_{k+1}(□).

case 1. A_1 is adequate and A is not f-homogenous. For all 1 ≤ i,j ≤ k+1, we let g'(A,i,j) be the least element of f(A\{A_i}) □ f(A\{A_j}) < A_1 if it exists; 0 otherwise. We also let g''(A,i,j) = 0 if f(A\{A_i}) □ f(A\{A_j}) has no elements < A_1; 1 if g'(A,i,j) □ f(A\{A_i}); 2 if g'(A,i,j) □ f(A\{A_j}). Set g(A) to code up all of the g'(A,i,j) and g''(A,i,j), and then add 2.

case 2. A_1 is adequate and A is f-homogenous. Set g(A) = 1.

case 3. A_1 is not adequate and A_1 > □. Set g(A) to be the greatest adequate ordinal < A_1 .

case 4. A1 < □. Set g(A) = 0.

For the first claim, let C □ □ be closed and unbounded. We can assume that every element of C is adequate. Let E □ S_{k+2}(C) be g-homogenous. If case 2 applies to any A □ S_{k+1}(E) then we are done. So case 1 applies to every A □ S_{k+1}(E). Look at x = f({E_1,...,E_k}), y = f({E_1,...,E_i,E_{i+2},...,E_{k+1}}), and z = f({E_1,...,E_i,E_{i+3},...,E_{k+2}}), where 0 ≤ i ≤ k-1. It suffices to prove that x = y □ E_1. Assume this is false. By applying g({E_1,...,E_{k+1}}) = g({E_1,...,E_i,E_{i+2},...,E_{k+2}}), we see that x □ y and y □ z have the same least element w < A_1, and w □ x □ w □ y. This is a contradiction.

For the second and third claims, it suffices to let E □ □ be g-homogenous and unbounded, and prove that E is f-homogenous. Since the outputs of the four cases are mutually disjoint, we see that only one case can apply to all g(A), A □ S_{k+1}(E). Since E is unbounded, it cannot be case 3 or 4. Since cases 3 abd 4 do not apply to any g(A), A □ S_{k+1}(E), we see that every element of E is adequate. We can then argue as in the previous paragraph that case 1 cannot apply to all g(A), A □ S_{k+1}(E). Hence case 2 applies to all A □ S_{k+1}(E).

So every A □ S_{k+1}(E) is f-homogenous. We now claim that E is f-homogenous. To see this, let A = {A_1,...,A_k} □ E. It suffices to prove that f(A) = f(B), where B □ S_k(E) and A_k < B_1. To see this, note that we have f({A_1,...,A_k}) = f({A_1,...,A_{k-1},B_k}) = f({A_1,...,A_{k-2},B_{k-1},B_k}) = ... = f({A_1,B_2,...,B_k}) = f({B_1,...,B_k}) by applying the fhomogeneity of $\{A_1,\ldots,A_k,B_k\}$, $\{A_1,\ldots,A_{k-1},B_{k-1},B_k\}$, ..., and $\{A_1,B_1,\ldots,B_k\}$.

Let $k \geq 1$. We say that κ is k-SRP if and only if κ is a limit ordinal, and for every $f:S_k(\kappa) \to \{0,1\}$, there exists a stationary $E \subseteq \kappa$ such that f is constant on $S_k(E)$. Here SRP stands for "stationary Ramsey property."

LEMMA 1.22. Let κ be a weakly compact cardinal. Then κ is strongly inaccessible, and for all $k \geq 1$, κ is k-RP. Furthermore, let $f:S_k(\kappa) \to x$, where $x < \kappa$. Then there exists unbounded $E \subseteq \kappa$ such that f is constant on $S_k(E)$.

Proof: See [Ka94], p. 76, with references to Erdös, Tarski, Hanf, Monk, and Scott, from 1943 to 1964.

LEMMA 1.23. Let κ be a weakly compact cardinal. Let $k \geq 1$, $f:S_k(\kappa) \to S(\kappa)$ be regressive, and $E \subseteq \kappa$ be unbounded. Then there exists unbounded $X \subseteq E$ such that for all $A,B \in S_k(X)$, if $A_1 = B_1$ then $f(A) = f(B)$.

Proof: By Lemma 1.22, κ is (2k-1)-RP, and strongly inaccessible. Therefore E is of cardinality κ. Partition the $A \in S_{2k-1}(E)$ according to whether or not $f(\{A_1,\ldots,A_k\}) = f(\{A_1,A_{k+1},\ldots,A_{2k-1}\})$. Let $X \subseteq E$ be unbounded and homogenous.

We claim that X is homogenous for the first side of the partition. Suppose X is homogenous for the second side of the partition. Then we get κ many values of f at arguments whose min is min(X), contradicting the strong inaccessibility of κ.

But since X is homogenous for the first side of the partition, it is easy to see that X is as required.

LEMMA 1.24. Let $k \geq 1$ and κ be an ordinal. Then κ is (k-1)-ineffable if and only if κ is a regular cardinal and is k-SRP.

Proof: For the forward direction, by Lemma 1.14, (k-1)-ineffable implies k-ineffable', which obviously implies k-SRP.

So assume that κ is a regular cardinal with k-SRP.

If k = 1 then □ is uncountable, and hence is 0-ineffable. So we assume k ≥ 2. Since □ is a regular cardinal, stationary subsets of □ have cardinality □, and so □ is 2-RP.

Let f:S_{k-1}(□) □ S(□) be regressive. We define g:S_k(□) □ {0,1} as follows. Let A □ S_k(□).

Case 1. For all B,C □ S_{k-1}(A), if $B_1 = C_1 = A_1$ then f(B) = f(C). Define g(A) = 0.

Case 2. Otherwise. Define g(A) = 1.

Since □ is k-SRP, let E □ □ be stationary, where g is constant on S_k(E). By Lemma 1.23, let X □ E be of cardinality □, where the values of f on S_{k-1}(X) depend only on the first term. Then g is constantly 0 on S_k(X), and hence also constantly 0 on S_k(E).

We thus have stationary E □ □ such that the values of f on S_{k-1}(E) depend only on the first term. Define h:E □ □ to be the function given by h(x) = min(f({x,y_1,...,y_{k-2}}) □ (x □ f({y_1,...,y_{k-1}}))), where $y_1 < ... < y_{k-1}$ are chosen arbitrarily from E above x. If the symmetric difference is empty, then define h(x) to be . By Fodor's theorem, h is constant on a stationary set E' □ E. If h is constantly on E' then E' is homogenous for f.

Suppose h is constantly x < □ on E'. We obtain a contradiction by looking at any three elements of S_{k-1}(E') whose first terms are distinct.

LEMMA 1.25. Let k ≥ 1. The least k-SRP ordinal is the same as the least (k-1)-ineffable ordinal.

Proof: The case k = 1 is trivial, since both ordinals are $□_1$. Assume k > 1.

In light of Lemma 1.24, it suffices to prove that the least k-SRP ordinal, □, is a regular cardinal. Let □ = cf(□). Then □ is a regular cardinal. If □ = □ then □ is not even 1-SRP, since every stationary subset of □ contains all sufficiently large ordinals < □. So □ is uncountable. Assume □ is embedded in □ as a closed unbounded set. Now any stationary subset of □ must be stationary in □ when intersected with □. Hence any counterexample to □ being k-SRP lifts to a counterexample to

□ being k-SRP. Hence □ is k-SRP. Therefore □ = □ and □ is a regular cardinal.

[Ba75] does not explicitly state Lemma 1.22. The result is valuable in that it justifies k-SRP as a particularly elegant way of presenting the hierarchy of large cardinals under consideration.

In [Ba75] we find the following result that is obviously in the minimalist direction.

LEMMA 1.26. Let k ≥ 1 and □ be an ordinal. Then □ is k-subtle if and only if □ is a limit ordinal such that the following holds. Let C □ □ be closed and unbounded and f:S_k(□) □ S(□) be regressive. Then there exists E □ S_{k+1}(C) such that f({E_1,...,E_k}) = f({E_2,...,E_{k+1}}) □ E_1.

Proof: The case k = 1 is immediate, and so assume k ≥ 2, and let □ be a limit ordinal such that the condition holds. Clearly □ is a 1-subtle cardinal, and hence by Lemma 1.1, □ is an uncountable regular cardinal.

Let f:S_k(□) □ S(□) be regressive and C □ □ be closed and unbounded. We can assume that every element of C is an adequate ordinal > □. We define g:S_k(□) □ S(□) as follows. Let A □ S_k(□).

case 1. A □ S_k(C). For nonempty consecutive B □ {A_1,...,A_k}, let U(B) = {(D,x): max(D) < A_1 and x □ f({D □ B})}. For nonempty consecutive B □ {A_1,...,A_{k-1}}, let B' □ {A_2,...,A_k} be the result of shifting the elements of B to the right within A. Let V(B) = the truth value of the equation U(B) = U(B') □ min(B), where we have left off the exponent of min(B), as we will in the remainder of this proof.

Set g(A) to code up the U(B), where {A_1} □ B □ {A_1,...,A_k} and B is consecutive, as well as the V(B), where B □ {A_2,...,A_{k-1}} is nonempty and consecutive. The coding up is done with respect to the indices in A of the least and greatest elements of each B.

case 2. A □ S_k(C). Set g(A) = ∅.

Let E □ S_{k+1}(C), where g({E_1,...,E_k}) = g({E_2,...,E_{k+1}}) □ E_1. Fix {E_1} □ B □ {E_1,...,E_{k-1}}, where B is consecutive. Note that U(B) = U(B') □ E_1. This is because the position of B in

$\{E_1,\ldots,E_k\}$ is the same as the position of B′ in $\{E_1,\ldots,E_{k+1}\}$. Hence V(B) is true.

Also, let $B = B_1,B_2,\ldots,B_p$ be the sequence of successive shifts of B to the right within $\{E_1,\ldots,E_k\}$, until $\max(B_p) = E_k$. Since $V(B_1)$ is true, we have $V(B_2)$ is true; this is because the position of B_1 in $\{E_1,\ldots,E_k\}$ is the same as the position of B_2 in $\{E_2,\ldots,E_{k+1}\}$. Continuing in this way, we see that $V(B_1),\ldots,V(B_p)$ are all true.

In particular, we have proved that $U(B_p) = U(B_p') \subseteq \min(B_p)$.

We can now easily see that E is f-homogenous. Let $\{E_1,\ldots,E_{i-1},E_{i+1},\ldots,E_{k+1}\}$ be given, $2 \le i \le k$. By the previous paragraph, $U(\{E_i,\ldots,E_k\}) = U(\{E_{i+1},\ldots,E_{k+1}\}) \subseteq E_i$, and so $f(\{E_1,\ldots,E_k\}) = f(\{E_1,\ldots,E_{i-1},E_{i+1},\ldots,E_{k+1}\})$. Also $f(\{E_1,\ldots,E_k\}) = f(\{E_2,\ldots,E_{k+1}\})$, since each g(A) codes up $U(\{A_1,\ldots,A_k\}) = f(\{A_1,\ldots,A_k\})$.

We now prove corresponding version for functions into κ.

LEMMA 1.27. Let $k \ge 1$ and κ be an ordinal. Then κ is k-subtle′ if and only if κ is (k-1)-subtle if and only if κ is a limit ordinal such that the following holds. Let $C \subseteq \kappa$ be closed and unbounded and $f:S_k(\kappa) \to \kappa$ be regresive. Then there exists $E \in S_{k+2}(C)$ such that $f(\{E_1,\ldots,E_k\}) = f(\{E_2,\ldots,E_{k+1}\})$.

Proof: Let $k \ge 1$ and κ be a limit ordinal. By Lemmas 1.14 and 1.21, we have only to prove that the condition implies κ is k-subtle′. Obviously, if k = 1 then the condition is equivalent to 1-subtle′. So we may assume that $k \ge 2$. By Lemma 1.20, since κ is 1-subtle′, κ is uncountable regular cardinal.

Let $C \subseteq \kappa$ be closed and unbounded, and $f:S_k(\kappa) \to \kappa$ be regressive. We can assume that every element of C is an adequate ordinal $> \omega$. We define $g:S_k(\kappa) \to \kappa$ as follows. Let $A \in S_k(\kappa)$.

case 1. $A \in S_k(C)$. For nonempty consecutive $B \subseteq \{A_1,\ldots,A_k\}$, define h_B to be the function given by $h_B(D) = f(D \cup B)$, for $|D| = k - |B|$, $\max(D) < \min(B)$. For nonempty consecutive $B \subseteq \{A_1,\ldots,A_{k-1}\}$, let $W(B) = (D,x,y)$ defined as follows. Let B′ be the result of shifting the elements of B to the right within A. If $h_B = h_{B'}$ then set $D = x = y = \emptyset$. If $h_B \neq h_{B'}$, then choose D such that $h_B(D) \ne h_{B'}(D)$, and set $x = h_B(D)$ and

$y = 1$. For nonempty consecutive $B \subseteq \{A_1,\ldots,A_{k-1}\}$, let $X(B)$ be the truth value of $h_B = h_{B'}$.

Set $g(A)$ to code up $f(\{A_1,\ldots,A_k\})$, the $W(B)$ for nonempty consecutive $\{A_1\} \subseteq B \subseteq \{A_1,\ldots,A_{k-1}\}$, and the $X(B)$ for nonempty consecutive $B \subseteq \{A_1,\ldots,A_{k-1}\}$.

case 2. $A \notin S_k(C)$. Set $g(A) = 0$.

Let $E \in S_{k+1}(C)$, where $g(\{E_1,\ldots,E_k\}) = g(\{E_2,\ldots,E_{k+1}\})$. Fix $\{E_1\} \subseteq B \subseteq \{E_1,\ldots,E_{k-1}\}$, where B is consecutive. We claim that $h_B = h_{B'}$. To see this, first note that $W(B) = W(B')$, since the position of B in $\{E_1,\ldots,E_k\}$ is the same as the position of B' in $\{E_2,\ldots,E_{k+1}\}$. Suppose $h_B \neq h_{B'}$, and let $W(B) = W(B') = (D,x,1)$. Then $h_B(D) \neq h_{B'}(D)$, and $h_B(D) = x$. Also $h_{B'}(D) \neq h_{B''}(D)$, and $h_{B'}(D) = x$. This is a contradiction. Hence $h_B = h_{B'}$ and so $X(B)$ is true.

Also, let $B = B_1,B_2,\ldots,B_p$ be the sequence of successive shifts of B to the right within $\{E_1,\ldots,E_k\}$, until $\max(B_p) = E_k$. Since $X(B_1)$ is true, we have $X(B_2)$ is true; this is because the position of B_1 in $\{E_1,\ldots,E_k\}$ is the same as the position of B_2 in $\{E_2,\ldots,E_{k+1}\}$. Continuing in this way, we see that $X(B_1),\ldots,X(B_p)$ are all true.

In particular, we have proved that $h_{B\_p} = h_{B\_p'}$.

We can now easily see that E is f-homogenous. Let $\{E_1,\ldots,E_{i-1},E_{i+1},\ldots,E_{k+1}\}$ be given, $2 \leq i \leq k$. By the previous paragraph, $h\_\{E_i,\ldots,E_k\} = h\_\{E_{i+1},\ldots,E_{k+1}\}$, and so $f(\{E_1,\ldots,E_k\}) = f(\{E_1,\ldots,E_{i-1},E_{i+1},\ldots,E_{k+1}\})$. Also $f(\{E_1,\ldots,E_k)) = f(\{E_2,\ldots,E_{k+1}\})$ since each $g(A)$ codes up $f(\{A_1,\ldots,A_k))$.

We now summarize most of the results of this section in terms of what we call the subtle, almost ineffable, and ineffable cardinal hierarchies. Some of the results in the Lemmas are sharper.

For $k \geq 0$, let STL(k) be the least k-subtle ordinal; AIN(k) be the least k-almost ineffable ordinal; IN(k) be the least k-ineffable ordinal. For $k \geq 1$, let STL'(k) be the least k-subtle' ordinal; AIN'(k) be the least k-almost ineffable' ordinal; IN'(k) be the least ineffable' ordinal. For $k \geq 1$, let SRP(k) be the least ordinal with k-SRP (stationary Ramsey property). Since we are working in ZFC, some of these may be undefined. It is convenient to use the default value   if undefined.

THEOREM 1.28. The following is provable in ZFC.

i) STL(0) = STL'(1) = AIN(0) = AIN'(1) = IN(0) = IN'(1) = SRP(1) = $□_1$;

ii) STL(1) < AIN(1) < IN(1) < STL(2) < AIN(2) < IN(2) < ..., where if any term is   then all later terms are ;

iii) for all k ≥ 0, STL(k) = STL'(k+1), AIN(k) = AIN'(k+1), IN(k) = IN'(k+1) = SRP(k+1);

iv) if STL(1) <   then STL(1) is strictly greater than the first totally indescribable cardinal, and hence the first weakly compact cardinal;

v) for all k ≥ 1, if STL(k) exists then it is n-Mahlo for all n ≥ 0; if AIN(k) exists then it is weakly compact; if IN(k) exists then it is weakly compact;

vi) if □ □ □ then every IN(k) is strictly less than □.

Proof: i) and iii) are from Lemmas 1.14, 1.19, 1.21, and 1.24. ii) is from Lemmas 1.2, 1.3, 1.7, 1.8, and 1.9. iv) is from Lemma 1.11. v) is from Lemma 1.12. vi is from Lemma 1.13.

2.THE MINIMALIST APPROACH

We now introduce a pure form of subtleness.

□ is purely k-subtle if and only if

i) □ is an ordinal;

ii) For all regressive f:S_k(□) □ □, there exists A □ S_{k+1}(□\{0,1}) such that f is constant on S_k(A).

Thus we have removed mention of closed unbounded sets. Note that this concept is upward closed in the sense that if □ is purely k-subtle and □ ≤ □, then □ is purely k-subtle.

Why do we write □\{0,1} instead of □ or □\{0}? Because then □+k would be purely k-subtle for all k ≥ 1:

LEMMA 2.1. Let $k \geq 1$ and $f:S_k(\omega+k) \to \omega+k$ be regressive. Then there exists $E \subseteq S_{k+1}(\omega+k\backslash\{0\})$ such that f is constant on $S_k(E)$.

Proof: First suppose that there exists $B \in S_k(\omega+k\backslash\{0\})$ such that $f(B) < \min(B)-1 < \omega$, and choose B with this property so that min(B) is minimized. Let $f(B) = p < \omega$. Then $\min(B) > p+1$ and f is constantly p on $S_k(B \cup \{p+1\})$. To see this, let $A \in S_k(B \cup \{p+1\})$ and suppose $p+1 \in A$. Then $\min(A) = p+1 < \min(B)$, and so $f(A) \geq \min(A)-1 = p$. Hence $f(A) = p$.

Now suppose that there does not exist $B \in S_k(\omega+k\backslash\{0\})$ such that $f(B) < \min(B)-1 < \omega$. Then for all $B \in S_k(\omega+k\backslash\{0\})$, if $0 < \min(B) < \omega$ then $f(B) = \min(B)-1$. Let $f(\{\omega,\ldots,\omega+k-1\}) = p < \omega$. Then f is constantly p on $S_k(\{p+1,\omega,\ldots,\omega+k-1\})$.

We will need the following technical lemma.

LEMMA 2.2. There exists regressive $h:\omega^2 \to \omega$ such that

i) for all $0 < x,y < \omega$, $h(x,y) < \min(x,y)$;

ii) rng(h) has no even elements > 2;

iii) for all $1 < x < y < z < \omega$, $h(x,y) \neq h(y,z)$.

Proof: Define h by cases, where only the first case that applies is operative. Let $x,y < \omega$.

Case 1. $x \geq y$. Define $h(x,y) = 0$.

Case 2. $x \leq 1$. Define $h(x,y) = 0$.

Case 3. $y = x+1$ and x is even. Define $h(x,y) = 0$.

Case 4. $y = x+1$ and x is odd. Define $h(x,y) = 2$.

Case 5. x is even. Define $h(x,y) = x-1$.

Case 6. x is odd. Define $h(x,y) = x-2$.

Clearly rng(h) has no even integers > 2. Now let $1 < x < y < z$. We verify that $h(x,y) \neq h(y,z)$. Suppose that $h(x,y) = h(y,z)$.

By comparing the parity of h(x,y) with that of h(y,z), we see that either x,y,z are consecutive or no two of these are consecutive. The former is impossible by cases 3 and 4.

Assume the latter. Then cases 5 - 6 apply. If x and y do not have the same parity then $y \geq x + 3$, which contradicts $h(x,y) = h(y,z)$. On the other hand, if x and y have the same parity, then obviously $h(x,y) \neq h(y,z)$.

LEMMA 2.3. Let $k \geq 2$. The least purely k-subtle ordinal (if it exists) = the least k-subtle' ordinal. The least purely 1-subtle ordinal is $\omega+\omega+1$.

Proof: We first show that the least purely 1-subtle ordinal is $\omega+\omega+1$. A counterexample for $\omega+\omega$ is afforded by the predecessor map, which is taken to be 0 where undefined. On the other hand, $\omega+\omega+1$ is purely 1-subtle by the following argument. Let $f:\omega+\omega+1 \to \omega+\omega+1$ be regressive, and assume by way of contradiction that f is one-one on $[2,\omega+\omega+1)$.

Then the range of f on $\omega\backslash\{0,1\}$ omits at most one value $< \omega$. It must omit exactly one value, p, for otherwise there is no room for $f(\omega)$. Hence $f(\omega) = p$. Therefore f maps $\omega+\omega$ onto $\omega+\omega$. Hence $f(\omega+\omega) = f(x)$ for some $x < \omega+\omega$, which is the required contradiction.

We now assume $k \geq 2$. Let λ be the least purely k-subtle ordinal. From the previous argument we see that $\lambda > \omega+\omega$. We now wish to prove that λ is a limit ordinal. Suppose λ is a successor ordinal.

Let $H:S_k([\omega,\lambda-1)) \to [\omega,\lambda-1)$ be a regressive function which is a counterexample to the pure k-subtlety of $[\omega,\lambda-1)$, in the appropriate sense; here ω, the first point in the interval, is treated like 0.

We now wish to define a regressive function $g:S_k(\lambda) \to \lambda$ by cases, where only the first case that applies is operative. Let $A \in S_k(\lambda)$.

Case i. $A_2 < \omega$. Define $g(A) = h(A_1,A_2)$.

Case ii. $A_1 < 5$. Define $g(A) = |A_1-1|$.

Case iii. $5 \leq A_1 < \omega$. Define $g(A) = 4$.

Case iv. $A_1 = \omega$. Define $g(A) = 6$.

Case v. $A_1 = \omega+1$. Define $g(A) = 8$.

Case vi. $A_k = \square-1$. Define $g(A) = 10$.

Case vii. Otherwise. I.e., $A \square [\square+2,\square-1)$. Define $g(A) = H(A)$.

We claim that g is a counterexample to the pure k-subtlety of □. To see this, let $E \square S_{k+1}(\square\backslash\{0,1\})$, and assume g is constant on $S_k(E)$.

If $E_2 < \square$ then $h(E_1,E_2) = h(E_2,E_3)$. Hence $E_2 \geq \square$. Now suppose $E_1 < \square$. Then $g(E_1,...,E_k) \leq 4$ yet $g(E_2,...,E_{k+1}) \geq 6$. Hence $E_1 \geq \square$. Also, since $g(E_1,...,E_k) = g(E_2,...,E_{k+1})$, we immediately see that $E_1 \neq \square,\square+1$. Also $E_{k+1} \neq \square-1$. Hence $E \square [\square+2,\square-1)$, and so we are entirely in case vii. Since g is constant on $S_k(E)$, we see that H is constant on $S_k(E)$. This is a contradiction. Hence g is a counterexample to the pure k-subtlety of □.

We have thus shown that □ is a limit ordinal $> \square+\square$.

We now prove that $cf(\square) = \square$. Suppose $cf(\square) = \square < \square$. Let C be a closed unbounded subset of □ of length □, where the first two elements of C are □ and $\square+\square$. A C-interval is an interval $[x,y)$, where x,y are consecutive elements of C. The C-intervals partition all of $[\square,\square)$.

For each C-interval $[x,y)$ we associate a regressive $F(x,y):S_k[x,y) \square [x,y)$ which is a counterexample to the pure k-subtlety of $[x,y)$.

Also let G be a one-one function from the C-intervals into $[\square,\square+\square) \square \{16\}$, where $G([\square,\square+\square)) = 16$.

We now define a regressive $g:S_k(\square) \square \square$ by cases, where only the first applicable clause is operative. Let $A \square S_k(\square)$.

Case a. $A_2 < \square$. Define $g(A) = h(A_1,A_2)$.

Case b. $A_1 < 5$. Define $g(A) = |A_1-1|$.

Case c. $5 \leq A_1 < \square$. Define $g(A) = 4$.

Case d. For some but not all i, A_i and A_{i+1} lie within the same C-interval. Let i be least such that A_i and A_{i+1} lie within the same C-interval. Let j be least such that A_j and

A_{j+1} do not lie within the same C-interval. Define g(A) = $10<i,j>$.

Case e. A is wholly contained in some C-interval whose left endpoint is A_1. Define g(A) = 12.

Case f. A is wholly contained in some C-interval whose left endpoint is A_1-1. Define g(A) = 14.

Case g. A □ [x+2,y) for some C-interval [x,y). Define g(A) = F(x,y)(A).

Case h. The elements of A lie in distinct C-intervals. Let A_1 belong to the C-interval [x,y). Define g(A) = G([x,y)).

Let g be constant on S_k(E), where E □ S_{k+1}(□\{0,1}). As in the treatment above of cases i - vii, we see that E_1 ≥ □, and so only cases d-h come up for g on S_k(E).

It is clear that cases d-f yield mutually disjoint outputs which are in turn disjoint from the outputs of cases g,h. Hence if any of d-f apply to any g(A), A □ S_k(E), then that same case must apply to all g(A), A □ S_k(E). If it is case d, then consider g({E_1,...,E_k}) and the associated i,j. Either i or j is > 1, in which case we get a contradiction from g({E_1,...,E_k}) = g({E_2,...,E_{k+1}}). Clearly it cannot be case e or case f.

Hence for all A □ S_k(E), case g or case h must apply to g(A). If E_1 ≥ □ then the outputs from case g are disjoint from the outputs from case h, and hence either case g applies to all g(A), A □ S_k(E), or case h applies to all g(A), A □ S_k(E). The former is ruled out because of the choice of F. The latter is obviously ruled out since G is one-one.

So □+2 ≤ E_1 < □. If E_2 ≥ □ then g({E_1,...,E_k}) = 16. But then case h must apply to all g(A), A □ S_k(E). This is a contradiction since G is one-one.

Thus we have shown that □ is a regular cardinal > □+□, and hence an uncountable regular cardinal. We now dignify a = the least purely k-subtle ordinal by calling it □.

We now modify the above construction to prove that □ is k-subtle'. Let C □ □ be closed unbounded and let H:S_k(□) □ □

be regressive. We will prove that H is constant on some $S_k(E)$, where B □ $S_{k+1}(C)$. Without loss of generality, we can assume that C consists of powers of □ with exponent ≥ 2, and that H maps $S_k(C)$ into [w,l), by adding □ to values.

Here the C-intervals are defined as the intervals [x,y), where x < y are consecutive elements from C □ {□}.

As before, for each C-interval [x,y) we associate a regressive $F(x,y):S_k([x,y))$ □ [x,y) which is a counterexample to the pure k-subtlety of [x,y).

We now define a regressive function $g:S_k$(□) □ □ as follows.

Cases a-f. Same as above.

Case g′. A □ [x+2,y) for some C-interval [x,y). Define g(A) = F(x,y)(A).

Case h′. The elements of A lie in distinct C-intervals and A_1 < min(C). Define g(A) = 16.

Case i′. The elements of A lie in distinct C-intervals and A_1 ≥ min(C). Let E be the set of left endpoints of these distinct C-intervals. Define g(A) = H(E).

(Note that case g′ is the same as case g; we repeat it for convenience).

Now let g be constant on E □ S_{k+1}(□\{0,1}). As before, we see that E_1 ≥ □, and also that only cases g,h′,i′ can apply to any g(A), A □ $S_k(E)$.

If case h′ applies to some g(A), A □ $S_k(E)$, then it must apply to all A □ $S_k(E)$, which is a contradiction. So case h′ applies to no g(A), A □ $S_k(E)$.

Suppose case g′ applies to g(A) for some A □ $S_k(E)$. If k ≥ 3 then case i′ cannot apply to any g(A), A □ $S_k(E)$, and so case g′ applies to all g(A), A □ $S_k(E)$. But this contradicts the choice of F. If k = 2 then either a) E_1,E_2 lie in the same C-interval and E_3 does not; or b) E_2,E_3 lie in the same C-interval and E_1 does not; or c) E_1,E_2,E_3 lie in the same C-interval. In case a), $g(\{E_1,E_2\}) > g(\{E_1,E_3\})$; in case b),

$g(\{E_1,E_3\}) < g(\{E_2,E_3\})$; and case c) contradicts the definition of F.

So case i' applies to all g(A), A □ $S_k(E)$. Now let W be the set of left endpoints of the distinct C-intervals that meet E. Then W □ $S_{k+1}(C)$ and H is constant on $S_k(W)$. This completes the proof that □ is k-subtle'.

We now distill pure k-subtlety down even further.

□ is k-large if and only if

i) □ is an ordinal;
ii) For all regressive f:$□^k$ □ □, there exists 1 < $□_1$ < ... < $□_{k+1}$ such that f($□_1$,...,$□_k$) = f($□_2$,...,$□_{k+1}$).

Note that this concept is also upward closed. Obviously, every purely k-subtle ordinal is k-large.

To provide continuity of notation with the preceding development, we can trivially restate k-large as follows. X is k-large if and only if for all regressive f:S_k(□) □ □, there exists 1 < $□_1$ < ... < $□_{k+1}$ such that f({$□_1$,...,$□_k$}) = f({$□_2$,...,$□_{k+1}$}).

Recall the definition of < > and adequate ordinals given in section 1 right before Lemma 1.5. It is easy to see □ is adquate if and only if <0,□> = □ and □ > 0. Also if □ > 0 then <□,□> > max(□,□). We define u_1,u_2:On\{0} □ On as the unique functions such that <u_1(□),u_2(□)> = □. Thus □ is adequate if and only if □ > 0 and u_2(□) = □. And if □ is adequate then u_1(□) = 0.

LEMMA 2.4. Let k ≥ 1. An ordinal is k-large if and only if it is purely k-subtle.

Proof: Suppose k = 1. Obviously 1-large is the same as purely 1-subtle.

Now assume k ≥ 2. Let □ be k-large. We now prove that □ is purely k-subtle. By using h in Lemma 2.2, we see that □ > □.

Let f:S_k(□) □ □ be regressive. We define regressive g:$□^k$ □ □ by cases, where only the first case that applies is operative. If x □ $□^k$ is not strictly increasing, then we define $g(x) = 0$. We can think of any strictly increasing x □ □^{k} as an element A of S_k(□). This will conform the construction to the constructions in the proof of Lemma 2.3. We also use the construction in our proof of Lemma 1.27. Accordingly, let A □ S_k(□).

Case 1. A_2 < □. Define $g(A) = h(A_1,A_2)$.

Case 2. $A_1 < 5$. Define $g(A) = |A_1-1|$.

Case 3. $5 \leq A_1$ < □. Define $g(A) = 4$.

Case 4. some elements of A are adequate and others are not adequate. Let i be least such that A_i is adequate and j be least such that A_j is not normal. Define $g(A) = 12\langle i,j\rangle$.

Case 5. all elements of A are adequate. We import case 1 from the proof of Lemma 1.24. For nonempty consecutive B □ $\{A_1,\ldots,A_k\}$, define h_B to be the function given by $h_B(D) = f(D$ □ $B)$, for $|D| = k - |B|$, $\max(D) < \min(B)$. For nonempty consecutive B □ $\{A_1,\ldots,A_{k-1}\}$, let $W(B) = (D,x,y)$ defined as follows. Let B′ be the result of shifting the elements of B to the right within A. If h_B □ $h_{B'}$ then set $D = x = y = \emptyset$. If h_B □□ $h_{B'}$, then choose D such that $h_B(D) \neq h_{B'}(D)$, and set $x = h_B(D)$ and $y = 1$. For nonempty consecutive B □ $\{A_1,\ldots,A_{k-1}\}$, let X(B) be the truth value of h_B □ $h_{B'}$.

Let $g'(A)$ to code up $f(\{A_1,\ldots,A_k\})$, the W(B) for nonempty consecutive $\{A_1\}$ □ B □ $\{A_1,\ldots,A_{k-1}\}$, and the X(B) for nonempty consecutive B □ $\{A_1,\ldots,A_{k-1}\}$. Set $g(A) = g'(A)$ if infinite; $12g'(A) + 2$ otherwise.

[We have now covered all cases except where no elements of A are adequate.]

Case 6. For some but not all $i < k$, we have $u_1(A_i) = u_1(A_{i+1})$. Let i be least such that $u_1(A_i) = u_1(A_{i+1})$ and j be least such that $u_1(A_j) \neq u_1(A_{j+1})$. Define $g(A) = 12\langle i,j\rangle + 4$.

Case 7. For some but not all $i < k$, we have $u_2(A_i) = u_2(A_{i+1})$. Let i be least such that $u_2(A_i) = u_2(A_{i+1})$ and j be least such that $u_2(A_j) \neq u_2(A_{j+1})$. Define $g(A) = 12\langle i,j\rangle + 6$.

Case 8. For no $i < k$ is $u_1(A_i) = u_1(A_{i+1})$. Define $g(A) = u_1(A_1)$ if infinite; $12u_1(A_1) + 8$ otherwise.

Case 9. For no $i < k$ is $u_2(A_i) = u_2(A_{i+1})$. Define $g(A) = u_2(A_1)$ if infinite; $12u_2(A_1) + 10$ otherwise.

[We have now covered all cases since $u_1(A_i) = u_1(A_{i+1})$ & $u_2(A_i) = u_2(A_{i+1})$ is impossible].

Now assume that $1 < \square_1 < ... < \square_{k+1} < \square$ be such that the equality $g(\square_1,...,\square_k) = g(\square_2,...,\square_{k+1})$ holds. It suffices to prove that f is constant on $S_k(\{\square_1,...,\square_{k+1}\})$. We first prove that case 5 applies to $g(\square_1,...,\square_k)$ and $g(\square_2,...,\square_{k+1})$.

By the same argument used to handle cases i-vii in the proof of Lemma 2.3, we see that $\square_1 \geq \square$. Thus only cases 4-9 can apply to $g(\square_1,...,\square_k)$ and $g(\square_2,...,\square_{k+1})$.

Note that cases 4,6,7 have mutually disjoint outputs, which are also disjoint from the outputs of cases 5,8,9. Thus if any of cases 4.6,7 apply to $g(\square_1,...,\square_k)$ or $g(\square_2,...,\square_{k+1})$, then that case applies to $g(\square_1,...,\square_k)$ and $g(\square_2,...,\square_{k+1})$.

Suppose case 5 applies to $g(\square_1,..., \square_k)$ or $g(\square_2,...,\square_{k+1})$. Then case 5 or 7 applies to the other. Hence case 5 applies to the other.

Suppose case 8 applies to $g(\square_1,..., \square_k)$ or $g(\square_2,...,\square_{k+1})$. Then case 6 or 8 applies to the other. Hence case 8 applies to the other.

Suppose case 9 applies to $g(\square_1,...,\square_k)$ or $g(\square_2,...,\square_{k+1})$. Then case 7 or 9 applies to the other. Hence case 9 applies to the other.

Thus we have shown that exactly one of cases 4-9 applies to $g(\square_1,...,\square_k)$ and $g(\square_2,...,\square_{k+1})$.

Suppose case 4 is the single case. If the i for $g(\square_1,...,\square_k)$ is not 1 then it differs from the i for $g(\square_2,...,\square_{k+1})$. The same holds for the j's. But i is never j. This is a contradiction. The same argument shows that the single case is not case 6 or 7.

Obviously the single case cannot be case 8 or 9 because the equality is immediately violated.

Hence the single case must be case 5. Then we argue exactly as in the proof of Lemma 1.27 that $\{\alpha_1,\ldots,\alpha_{k+1}\}$ is f-homogenous.

We now go a step further and consider linearly ordered sets (X,<). We say that $f:X^k \to X$ is regressive if and only if for all $y \in X^k$, if min(y) is not X-minimum then f(y) < min(y).

Let (X,<) be a linear ordering and $k \geq 1$. We say that X is k-large if and only if the following holds:

> For all regressive $f:X^k \to X$, there exists $b_1 < \ldots < b_{k+3}$ such that $f(b_3,\ldots,b_{k+2}) = f(b_4,\ldots,b_{k+3})$.

This concept is compatible with the concept of k-large ordinals. I.e., α is k-large if and only if $(\alpha,<)$ is k-large. This is because we can set $b_1 = 0$ and $b_2 = 1$.

Note that this concept is upward closed in the sense that if (X,<) is k-large forms an initial segment of (Y,<), then (Y,<) is k-large.

To provide continuity of notation with the preceding development, we can trivially restate k-large as follows. X is k-large if and only if for all regressive $f:S_k(X) \to X$, there exists $b_1 < \ldots < b_{k+3}$ such that $f(\{b_3,\ldots,b_{k+2}\}) = f(\{b_4,\ldots,b_{k+3}\})$.

Let X be a linearly ordered set. An interval in X is defined to be a subset Y of X such that for all x < y < z from X, if $x,z \in Y$ then $y \in Y$. An initial segment of X is a $Y \subseteq X$ such that for all x < y from X, if $y \in Y$ then $x \in Y$. A proper initial segment of X is an initial segment of X that is not X. A tail in X is a $Y \subseteq X$ such that for all x < y from X, if $x \in Y$ then $y \in Y$. For intervals I,J in X, we write I < J if and only if every element of I is < every element of J.

We say that X is compressed if and only if

i) X is infinite;

ii) There is a (unique) $n < \omega$ such that X has an initial segment of cardinality n but no initial segment of cardinality n+1;

iii) All infinite initial segments of X are of the same cardinality.

The number n in ii) is called the initial integer of X. We always identify the first n elements of compressed X with 0,1,...,n-1. We write X\n for the set of all elements of X that are > n-1.

LEMMA 2.5. Every infinite linear ordering has an initial segment of type ω or a compressed initial segment.

Proof: Suppose (X,<) does not have an initial segment of type w. Inductively define x_i as the least element of I greater than all x_j, j < i. Then there exists i such that x_i is undefined. Let n be least such that x_n is undefined. Let I be an infinite initial segment of X of minimal cardinality. Then all infinite initial segments of I are of the same cardinality. Also I has an initial segment of cardinality n but not of cardinality n+1.

We need to prove an analog of Lemma 2.2 for compressed I.

LEMMA 2.6. Let I be compressed with initial integer n. Then there exists one-one regressive functions $h(I):I\backslash n \to I\backslash n$ and $H(I):(I\backslash n)^2 \to I\backslash n$, and one-one $G(I):\omega^3 \to I\backslash n$, such that the three ranges are disjoint, and there are |I| many elements of I\n that are in none of the three ranges.

Proof: Well order the elements of I\n with length |I|. Define h(I) and H(I) by transfinite induction. There's plenty of room because I is compressed.

We fix h(I),H(I), and G(I) as above.

LEMMA 2.7. Let $k \geq 2$ and (X,<) be a k-large linear ordering. Let X be partitioned into three nonempty intervals I < J < K. Assume I has order type ω. Assume K is compressed with initial integer 0, or $|K| \leq |I \cup J|$. Then J is k-large.

Proof: Let k,X,I,J,K be as given. We identify I with ω. Let $f:S_k(J) \to J$ be regressive. We now define a regressive $g:S_k(X) \to X$ by cases, where only the first applicable case is operative. Let $A \in S_k(X)$. Recall the function h from Lemma 2.2.

Case 1. $A_2 \in \omega = I$. Define $g(A) = h(A_1,A_2)$.

Case 2. $A_1 < 5$. Define $g(A) = |A_1-1|$.

Case 3. $5 \leq A_1 \square \square$. Define $g(A) = 4$.

Case 4. some but not all elements of A are in J. Let j be least such that $A_j \square K$. Define $g(A) = 8j$.

Case 5. $A \square J$ and $A_1 = \min(J)$. Define $g(A) = 10$.

Case 6. $A \square J$ and A_1 is the second element of J. Define $g(A) = 12$.

Case 7. $A \square J$. Define $g(A) = f(A)$.

Case 8. $A \square K$ and $|K| \leq |I \square J|$. We use a one-one map $u:K \square I \square J$ such that all values of u lying in $\square$ are congruent to 6 mod 8. Define $g(A) = u(A_1)$.

Case 9. $A \square K$ and K is compressed with initial integer 0. Define $g(A) = H(K)(A_1,A_2)$.

Since X is k-large, let $b_1 < \ldots < b_{k+3}$ and $g(\{b_3,\ldots,b_{k+2}\}) = g(\{b_4,\ldots,b_{k+3}\})$. As in the proof of Lemma 2.4, we see that $b_3 \square I$. Inspection shows that a single case must apply to both of these applications of g. All cases can be ruled out to be the single case except case 7. So the single case is case 7, and hence $b_3,\ldots,b_{k+3} \square J$, and no b_i is the first or second element of J. This establishes that J is k-large.

LEMMA 2.8. Let $k \geq 2$ and $(X,<)$ be a k-large linear ordering. Let X be partitioned into three intervals $I < J < K$. Assume I is compressed. Assume K is compressed with initial integer 0, or $|K| \leq |I \square J|$. Then J is k-large.

Proof: Let k,X,I,J,K be as given. Let n be the initial integer of I. Let $f:S_k(J) \square J$ be regressive. We now define a regressive $g:S_k(X) \square X$ by cases, where only the first applicable case is operative. Let $A \square S_k(X)$.

Case 1. $A_1,A_2 \square I\backslash n$. Define $g(A) = H(I)(A_1,A_2)$.

Case 2. $A_1 = 0 < n$. Define $g(A) = 0$.

Case 3. $A_1 < n$. Define $g(A) = A_1-1$.

Case 4. A_1 □ I. Define g(A) = h(I)(A_1).

Case 5. Some but not all A_i lie in J. Let j be least such that A_j □ K. Define g(A) = G(I)(0,1,j).

Case 6. A □ J and A_1 = min(J). Define g(A) = G(I)(1,0,0).

Case 7. A □ J and A_1 is the second element of J. Define g(A) = G(I)(2,0,0).

Case 8. A □ J. Define g(A) = f(A).

Case 9. A □ K and |K| ≤ |I □ J|. We use a one-one map u:K □ I\n □ J such that all values of u lying in I\n are not in the range of h(I),H(I),G(I). Define g(A) = u(A_1).

Case 10. A □ K and K is compressed with initial integer 0. Define g(A) = H(K)(A_1,A_2).

Since X is k-large, let $b_1 < ... < b_{k+3}$ and g({b_3,...,b_{k+2}}) = g({b_4,...,b_{k+3}}). Inspection shows that a single case must apply to both of these applications of g. All cases can be ruled out at a single glance except cases 5 and 8. A second glance rules out case 5. So the single case is case 8, and hence b_3,...,b_{k+3} □ J and are not the first or second element of J. This establishes that J is k-large.

We now fix k ≥ 2 and □ to be the minimum cardinality of a k-large linear ordering. Obviously □ is infinite. Fix (X,<) to be a k-large linear ordering of cardinality □. We are going to prove that □ is purely k-subtle.

To this end, we fix A to be the union of all initial segments of X of cardinality < □.

LEMMA 2.9. (A,<) has the following properties:
i) |A| = □ ≥ □;
ii) Every proper initial segment of A has cardinality < □ and is not k-large;
iii) Some tail of (A,<) is k-large.

Proof: We claim that A is infinite. If A is finite then X is compressed. But then X is not k-large, because we can easily construct a regressive function which is the predecessor function on A\{min(A)}, and which is one-one on X\A by a simple construction by transfinite induction over a well ordering of X\A.

Suppose |A| < □. Then X\A is compressed with initial integer 0. Now by Lemma 2.7, A has an initial segment of type □ or a compressed initial segment. Hence by Lemmas 2.7 and 2.8, we see that A has a tail which is k-large. This contradicts that □ is the least cardinality of any k-large linear ordering.

Hence |A| = □. We have established i) and ii).

Note that |X\A| ≤ |A|. By Lemma 2.5, let I be an initial segment of A that is either of type □ or compressed. By Lemmas 2.7 and 2.8, A\I is k-large, as required.

We now let B = A\I, where I is an initial segment of A that is either of type □ or compressed. We summarize the properties of B, according to the last paragraph in the proof of Lemma 2.9.

LEMMA 2.10. (B,<) has the following properties:

i) |B| = □ ≥ □;

ii) Every proper initial segment of B has cardinality < □ and is not k-large;

iii) (B,<) is k-large.

LEMMA 2.11. Suppose B has an initial segment of order type □. Then □ is k-large.

Proof: We identify the initial segment of B of order type □ with □. We use the fact that there is no k-large linear ordering of cardinality < □ in a crucial way. Since □ is not k-large, clearly B\□ is nonempty. Also B\□ has no greatest element.

We can now let C □ B\□ be well ordered and unbounded in B.

The C-intervals are the intervals I of B\□ that are maximal with respect to the following property: C □ I is either empty or consists of just the left endpoint of I. It is easy to see that the set I(C) of all C-intervals is well ordered, and its order type is ≤ □.

We wish to prove that I(C) is k-large. Once we establish this, we see that I(C) has order type □, and so □ is k-large, which completes the proof. Note that no C-interval is k-large, because the cardinality of every C-interval is < □.

We fix a regressive function $f:S_k(I(C))$ □ $I(C)$. We define a regressive function $g:S_k(B)$ □ B by cases. We use the function h from Lemma 2.2. We also use a one-one < >:□^{2} □ □\{0}. Let A □ $S_k(B)$. For each C-interval J let $F_J:S_k(J)$ □ J be a counterexample to the k-largeness of J.

Case 1. A_2 □ □. Define $g(A) = h(A_1,A_2)$.

Case 2. $A_1 < 5$. Define $g(A) = |A_1-1|$.

Case 3. $5 \leq A_1$ □ □. Define $g(A) = 4$.

Case 4. Some but not all adjacent pairs of elements of A lie in the same C-interval. Let i be least such that A_i,A_{i+1} lie in the same C-interval. Let j be least such that A_j,A_{j+1} do not lie in the same C-interval. Define g(A) = 10<i,j>.

Case 5. A is contained in a C-interval J where $A_1 = \min(J)$. Define $g(A) = 6$.

Case 6. A is contained in a C-interval J where A_1 is the second smallest element of J. Define $g(A) = 8$.

Case 7. A is contained in a C-interval J. Define $g(A) = F_J(A)$.

Case 8. The elements of A lie in distinct C-intervals, the first of which is the first C-interval. Define $g(A) = 12$.

Case 9. The elements of A lie in distinct C-intervals, the first of which is the second C-interval. Define $g(A) = 14$.

Case 10. The elements of A lie in distinct C-intervals. Let these C-intervals be $C_1,\ldots,C_k$ in strictly increasing order. Define g(A) to be any element of $f(\{C_1,\ldots,C_k\})$.

Since B is k-large, let $b_1 < \ldots < b_{k+3}$, where $g(\{b_3,\ldots,b_{k+2}\}) = g(\{b_4,\ldots,b_{k+3}\})$. As in the proof of Theorem 2.4, we see that b_3 □ □.

Inspection shows that a single case must apply to these two applications of g, with the possible exception that case 7 and case 10 may apply. But this exception is clearly impossible since the outputs lie in different C-intervals. The single case must be case 10. This establishes that I(C) is k-large.

LEMMA 2.12. □ is k-large.

Proof: According to Lemma 2.5, B has an initial segment of type □ or a compressed initial segment. By Lemma 2.11, we can assume that B has the compressed initial segment K, with initial integer n. The argument is very similar to that of Lemma 2.11, where we use K instead of □.

Clearly B is not compressed, because |B| = □ and proper initial segments of B have cardinality < □. Hence K is a proper initial segment of B.

Let C □ B\K be well ordered and unbounded in B. The C-intervals are the intervals I of B\K that are maximal with respect to the same property used in Lemma 2.13. Also I(C) = the set of all C-intervals is again well ordered of type ≤ □. And again it suffices to prove that I(C) is k-large.

We fix a regressive function $f:S_k(I(C))$ □ I(C). We define a regressive function $g:S_k(B)$ □ B by cases. We use the functions h(K),H(K),G(K) from Lemma 2.6. Let A □ $S_k(B)$.

Let $F_J:S_k(J)$ □ J be a counterexample to the k-largeness of J.

Case 1. A_1,A_2 □ K\n. Define g(A) = H(K)(A_1,A_2).

Case 2. $A_1 = 0 < n$. Define g(A) = 0.

Case 3. $A_1 < n$. Define g(A) = A_1-1.

Case 4. A_1 □ K. Define g(A) = h(K)(A_1).

Case 5. Some but not all adjacent pairs of elements of A lie in the same C-interval. Let i be least such that A_i,A_{i+1} lie in the same C-interval. Let j be least such that A_j,A_{j+1} do not lie in the same C-interval. Define g(A) = G(K)(0,i,j).

Case 6. A is contained in a C-interval J where A_1 = min(J). Define g(A) = G(K)(1,0,0).

Case 7. A is contained in a C-interval J where A_1 is the second smallest element of J. Define g(A) = G(K)(2,0,0).

Case 8. A is contained in a C-interval J. Define g(A) = F_J(A).

Case 9. The elements of A lie in distinct C-intervals, the first of which is the first C-interval. Define g(A) = G(K)(3,0,0).

Case 10. The elements of A lie in distinct C-intervals, the first of which is the second C-interval. Define g(A) = G(K)(4,0,0).

Case 11. The elements of A lie in distinct C-intervals. Let these C-intervals be $C_1,\ldots,C_k$ in strictly increasing order. Define g(A) to be any element of $f(\{C_1,\ldots,C_k\})$.

Since B is k-large, let $b_1 < \ldots < b_{k+3}$, where $g(\{b_3,\ldots,b_{k+2}\}) = g(\{b_4,\ldots,b_{k+3}\})$. As in the proof of Lemma 2.8, we see that b_3 □ K. By inspection, a single case must apply to these two applications of g, with the possible exception that case 8 and case 11 may apply. But this exception is clearly impossible since the outputs lie in different C-intervals. The single case must be case 11. This establishes that I(C) is k-large.

LEMMA 2.13. Let $k \geq 2$. The cardinality of every k-large linear ordering is a k-large cardinal. Every 1-large linear ordering is infinite.

Proof: Let $k \geq 2$. By Lemma 2.12, the least cardinality of a k-large linear ordering is k-large. The second claim is obvious.

We take one final step towards conceptual simplicity.

Let (X,<) be a linear ordering with no endpoints, and $k \geq 1$. We say that $f:X^k \to X$ is regressive if and only if it obeys the inequality $f(x) < \min(x)$.

We say that a linear ordering (X,<) is k-critical if and only if it has no endpoints, and:

For all regressive $f:X^k \to X$, there exists $b_1 < \ldots < b_{k+1}$ such that $f(b_1,\ldots,b_k) = f(b_2,\ldots,b_{k+1})$.

It is obvious that every k-critical linear ordering is k-large. The reverse is of course false because well orderings have endpoints.

LEMMA 2.14. Let k ≥ 1 and □ be the least k-subtle' cardinal. Then -□ + □ is k-critical.

Proof: Let k,□ be as given. Let f:S_k(-□ + □) □ (-□ + □) be regressive. Let g:S_k(□ + □) □ (□ + □) be defined as follows. For min(A) □ □, let g(A) = 0. For min(A) □ □ with f(A) □ □, let g(A) = f(A). For min(A) □ □ with f(A) □ -□, let g(A) = -f(A). Applying k-subtle' to g we obtain E □ S_{k+1}(□) such that f is constant on S_k(E). (Here + indicates disjoint union).

LEMMA 2.15. Every 1-critical linear ordering is uncountable.

Proof: Let (X,<) be countable with no endpoints. There obviously is a one-one regressive function f:X^k □ X.

We introduce the following terminology. An ordinal is said to be infinitely large if and only if it is k-large for every k. A linear ordering is said to be infinitely large if and only if it is k-large for every k. A linear ordering is said to be infinitely critical if and only if it is k-critical for every k.

The next Theorem summarizes the results of this section.

THEOREM 2.16. Let k ≥ 2 and □ be an ordinal. The following are equivalent:

i) □ is purely k-subtle;
ii) □ is k-large;
iii) there exists a k-subtle' ordinal ≤ □;
iv) there exists a k-large linear ordering with the same cardinality as □;
v) there exists a k-critical linear ordering with the same cardinality as □.

Let □ be an ordinal. The following are equivalent:

vi) □ is purely 1-subtle;
vii) □ is 1-large;
viii) □+□+1 ≤ □.

The cardinalities of 1-large linear orderings are exactly the infinite cardinalities. The cardinalities of 1-critical linear orderings are exactly the uncountable cardinalities.

Proof: i) □ iii) is by Lemma 2.3. i) □ ii) is by Lemma 2.4. iii) □ iv) is by Lemma 2.13. v) □ iii) is by Lemmas 2.4 and 2.14. vi) □ viii) is by Lemma 2.3. vii) □ vi) is by Lemma 2.4. The first of the two final claims is by Lemma 2.13 and vii) □ viii). The second of the two final claims is by Lemmas 2.14 and 2.15.

We can now state two immediate Corollaries of Theorems 1.28 and 2.16. The first concerns the characterization of certain cardinals. The second focuses on provable equivalences. These Corollaries, as well as all Theorems and Lemmas in this paper, are provable in ZFC.

COROLLARY 2.17. Let k ≥ 2. The following cardinals are equal; if any one of them is undefined then all of them are undefined.

i) the least purely k-subtle ordinal;
ii) the least k-large ordinal;
iii) the least cardinality of a k-large linear ordering;
iv) the least cardinality of a k-critical linear ordering;
v) the least k-subtle’ cardinal;
vi) the least (k-1)-subtle cardinal.

The least purely 1-subtle ordinal = the least 1-large ordinal = □+□+1. The least cardinality of a 1-critical linear ordering = the least 1-subtle’ cardinal = the least 0-subtle cardinal = $□_1$.

COROLLARY 2.18. The following fourteen statements are equivalent.

i) for all k ≥ 1 there exists a k-SRP, k-subtle, k-almost ineffable, k-ineffable, k-subtle', k-almost ineffable', or k-ineffable' cardinal;
ii) for all k ≥ 1, there exists a purely k-subtle, or k-large ordinal;
iii) for all k ≥ 1, there exists a k-large linear ordering;
iv) for all k ≥ 1, there exists a k-critical linear ordering;
v) there exists an ordinal which is purely k-subtle for all k ≥ 1;

vi) there exists a linear ordering which is k-large for all k ≥ 1;

vii) there exists a linear ordering which is k-critical for all k ≥ 1.

Proof: By Theorems 1.28 and 2.16. We need only remark that □ = the union over k of the least k-subtle cardinal, if it exists, is purely k-subtle and k-large for all k ≥ 1. Also −□ + □ is k-critical for all k ≥ 1.

For practical use in combinatorial set theoretic arguments, k-ineffable and k-ineffable' are the most convenient. For set theoretic elegance, k-SRP and k-large ordinal are best. From a general mathematical point of view, k-critical is best.

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