A COMPLETE THEORY OF EVERYTHING: validity in the universal domain

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Extended Abstract
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1. Introduction.

Let LPC(=) be the usual language of predicate calculus with equality, and PC(=) be a standard system of axioms and rules and inference for predicate calculus with equality.

The usual interpretation of LPC(=) is the set interpretation, where domains are taken to be nonempty sets and relations and functions are taken to be set theoretic objects. The famous Gödel completeness theorem determines the formulas that are true in all set interpretations; i.e., the set validities. They are the formulas provable in PC(=).

Here we consider a number of alternative interpretations of LPC(=), and discuss the corresponding validities.

Under the domain interpretation of LPC(=), the domain is allowed to be any nonempty domain, regardless of whether that domain constitutes a set. For instance, V = the class of all sets, is a nonempty domain, but not a nonempty set. And W =

the domain consisting of everything, is even bigger, and certainly not a nonempty set.

At the time of Frege, any distinction between domains and sets - or even between classes and sets - had not been made clear. For that matter, even with regard to the set interpretation of LPC(=), we have not presently made clear whether we are talking about pure sets in the sense of set theory (the cumulative hierarchy of sets), or sets with possible urelements as elements.

Gödel’s determination of the set validities and domain validities is so robust as to be insensitive to such distinctions. We always arrive at the formulas provable in PC(=).

Why is this so convincing? Because Gödel’s completeness theorem uses only a very small collection of assumptions (facts) about these concepts.

Informally, to prove that every validity is derivable from simple axioms and rules, Gödel uses only the following:

> that there is a set or domain N of natural numbers and an appropriate successor operation, so that induction and recursion in a particularly elementary form can be performed over N with respect to relations and functions on N, which are used to create sets, relations, and functions on N and reason about them.

To prove that every sentence derivable from simple axioms and rules is a validity, Gödel only uses the following:

> that ordinary classical reasoning is valid when applied to the concepts underlying the standard interpretation of LPC(=), including elementary uses of induction.

Notice that in this context, Gödel never had to seriously consider the difficult issues as to what is really meant by the general concept of arbitrary set or domain, and relations and functions on sets or domains. Of course, one cannot always skirt fundamental issues in this way.

Let W stand for the world of objects. This is more comprehensive than the set theoretic universe V (of pure sets), and includes not only proper classes, but nonmathematical objects such as ideas, concepts, emotions, particles, points of time, etcetera.

The focus of this work is on what we call the W interpretation of LPC(=), in which the quantifiers range over W; i.e., over all objects whatsoever. The constant symbols are interpreted as arbitrary objects, the relation symbols are interpreted as arbitrary predicates on W, and the function symbols are interpreted as arbitrary functions on W. We will throughout take functions to be predicates which are everywhere defined and univalent. Equality is interpreted as identity, which we will not assume to be extensional identity.

A formula in LPC(=) is called W valid if and only if it is true in all W interpretations. This work deals with the determination of the W valid formulas of LPC(=).

We now indicate how the determination of the W valid formulas in LPC(=) includes the determination of the domain valid formulas in LPC(=), the latter being covered by the Gödel completeness theorem.

We will identify domains with extensions of unary predicates on W. It is easy to see that there is a simple syntactic transformation * such that any formula φ of PC(=) is domain valid if and only if the corresponding formula φ^* of PC(=) is W valid. Here φ^* is obtained from φ by introducing a new unary relation symbol P and taking φ^* to be

$$(\exists x)(P(x)) \to \varphi^{(P)}$$

where $\varphi^{(P)}$ is obtained from φ by relativizing all quantifers to the extension of P. (The antecedent has to be strengthened in an obvious way to accomodate constant and function symbols in φ, as well as free variables).

However, W validity cannot be reduced to domain validity in such a manner. Various new delicate issues arise in the characterization of the richer notion of W validity. The point of this work is that these issues can be fruitfully addressed in a surprisingly robust manner.

A. Notions of predication on W.

There are some fundamentally different concepts of predication on W. Many of our results will be unaffected by the choice of these concepts.

The distinctions arise naturally out of consideration of the classical doctrine of the identity of indiscernibles, advocated by Leibniz. In our terminology, the identity of indiscernibles states the following:

for all x,y, x = y if and only if
for all unary predicates P on W, $P(x) \leftrightarrow P(y)$.

This principle follows from the following singleton extensions principle:

for all x, there exists a unary predicate P on W such that
$\forall y(P(y) \leftrightarrow y = x)$.

Now you might say that the singleton extensions principle is evident anyways, and so what is the fuss about identity of indiscernibles?

Under the concept of predicate currently most useful in mathematics and set theory today, clearly the singleton extensions principle holds. In particular, in set theory and the set theoretic interpretation of predication, we use that for every x, there is a unary predicate that holds of x and x only.

However, there is another concept of predication discussed by philosophers, but not having a comparable impact on current mathematical practice, for which the singleton extensions principle is doubtful, or at least problematic. There must be some such concept, for otherwise the doctrine of identity of indiscernibles would be regarded as trivially true.

This is the notion of predicates that are given without reference to any particular objects, but only involving concepts. Or one can take a linguistic tack - relations that can be defined in a language, where language is broadly interpreted, rather than in some particular already formalized language.

We will use the term "general predicate" for the first concept of predication, where the singleton extensions principle obviously holds. And we will use the term "pure predicate" for the second concept of predication, where even the doctrine of identity of indiscernibles is problematic.

What is the relationship between these two concepts, of pure predicates and of general predicates?

It does not seem that one can hope to define the concept of pure predicate in terms of the concept of general predicate. Evidently, the pure predication concept has been much less analyzed than the general predication concept, the latter being so fundamental for mathematics.

Can one define the concept of general predicate in terms of pure predicate? Here there is a plausible construction. We can consider cross sections of pure predicates.

Specifically, consider all unary predicates that are given by a cross section of a pure binary predicate; i.e., fix the first argument to be any object. This is a reasonable proposal for the general unary predicates.

In fact, we will establish some formal results to the effect that, assuming the pure predicates obey appropriate comprehension principles without parameters, then the cross sectional predicates obey the corresponding comprehension principles with parameters.

Such results depend on a pure pairing function on the universe; i.e., we can take P(x,y,z) if and only if z is the ordered pair of x and y. We can think of this ordered pair as the concept of: x followed by y. Or we can define P(x,y,z) if and only if z is the Kuratowski order pair of x and y, which is regarded as a set and not a predicate.

Now because of this pure pairing function on W, we have a reduction of the pure predicates of several variables to the pure predicates of one variable. We also can consider the predicates of several variables given by cross sections of pure predicates; this construction will remain unchanged under various modifications.

Note that Gödel never had to consider such distinctions as between pure and general predicates, in his determination of the satisfiable sentences in LPC(=) under the usual interpretation. This is because he showed that one need only consider domains which are an initial segment of N; and in the case of N itself, one need only consider relations that are arithmetically defined. These arithmetically definable relations are clearly pure.

We now have two concepts of W validity.

1. In pure validity, the predicates are required to be pure predicates on W.
2. In general validity, the predicates are required to be general predicates on W.

One reasonable philosophical position is that the concept of pure predication on W is fundamental, whereas any concept of general predication on W is derived and not fundamental. Under this position, one may view general predicates as being defined as cross sections of pure predicates. We will not take this view here, choosing instead to view pure and general predication as separate, but related, fundamental concepts.

B. Basic formal theories of predication on W.

We now introduce the metatheories for predication on W that we use throughout the paper.

In BTP (basic theory of predication), we will have variables over W (W variables), which we write in lower case; variables over general predicates on W, which we write in upper case with a superscript g; and variables over pure predicates on W, which we write in upper case with a superscript p. We will also use = on W, the binary function symbol < > on W, and the constant symbol 0 (an object in W).

The W terms of BTP are generated by:

1. Every W variable is a W term.
2. If s,t are W terms then <s,t> is a W term.

The atomic formulas of BTP are:

3. s = t, where s,t are W terms.

4. $P^g(t)$, $P^p(t)$, where P^g is a general predicate variable, P^p is a pure predicate variable, and t is a W term.

The formulas of BTP are generated by:

5. all atomic formulas are formulas.
6. if φ,ψ are formulas then so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$.
7. if φ is a formula and α is a variable (any kind) then $(\forall\alpha)(\varphi)$ and $(\exists\alpha)(\varphi)$ are formulas.

The axioms and rules of BTP are:

A. The usual first order logical axioms and rules for this language L(BTP).
B. Pairing. $\langle x,y\rangle = \langle z,w\rangle \leftrightarrow (x = z \wedge y = w)$.
C. Zero. $\langle x,y\rangle \neq 0$.
D. General Comprehension. $(\exists P^g)(\forall x)(P^g(x) \leftrightarrow \varphi)$, where φ is a formula in L(BTP) in which P^g is not free.
E. Pure Comprehension. $(\exists P^p)(\forall x)(P^p(x) \leftrightarrow \varphi)$, where φ is a formula in L(BTP) in which P^p is not free, no W variables other than x are free in φ, and no general predicate variables are free in φ.

Let BTPp consist of the axioms of BTP which have no general predicate variables, and BTPg consist of the axioms of BTP which have no pure predicate variables. We take the language L(BTPp) to exclude general predicate variables, and the language L(BTPg) to exclude pure predicate variables.

THEOREM B.1. BTP proves that every P^p has the same extension as some P^g. BTP does not prove that every P^g has the same extension as some P^p.

THEOREM B.2. Every theorem of BTP in L(BTPp) is a theorem of BTPp. Every theorem of BTP in L(BTPg) is a theorem of BTPg. Every theorem of BTPp is a theorem of BTPg if all superscripts p are replaced by superscripts g. Not every theorem of BTPg is a theorem of BTPp if all superscripts g are replaced by superscripts p. BTPp, BTPg, BTP prove the same theorems that don’t have any predicate variables.

There is an important interpretation of L(BTP) into itself, called the cross sectional interpretation of L(BTP). This interpretation preserves W, 0, and < >. The general predicates are interepreted as cross sections of the pure predicates.

More formally, let φ be a formula in L(BTP). For each general predicate variable P^g occurring in φ, assign a pure predicate variable P'^p and a W variable $x[P^g]$, both not occurring in φ, in such a way that $P^g \neq Q^g$ implies $P'^p \neq Q'^p$ and $x[P^g] \neq x[Q^g]$.

Now replace the quantifiers $(\forall P^g),(\exists P^g)$ in φ by $(\forall P'^p)(\forall x[P^g]),(\exists P'^p)(\exists x[P^g])$, and each atomic formula $P^g(t)$ occurring in φ by the atomic formula $P'^p(<x[P^g],t>)$. Make no other alterations.

This results in a formula φ^* in L(BTP) which is called the cross sectional interpretation of φ. Note that φ^* has no general predicate variables.

THEOREM B.3. The cross sectional interpreation of L(BTP) into L(BTPp) is an interpretation of BTP in BTPp.

Is the cross sectional interpretion of L(BTP) into L(BTPp) a faithful interpretation of BTP in BTPp? I.e., if the cross sectional interpretation of L(BTP) of a sentence is provable in BTPp then is that sentence provable in BTP?

The principle of cross sections asserts that every general predicate is a cross section of a pure predicate; i.e.,

$$(\forall P^g)(\exists Q^p)(\exists x)(\forall y)(P^g(y) \leftrightarrow Q^p(<x,y>)).$$

THEOREM B.4. The cross sectional interpretation of L(BTP) into L(BTPg) is not a faithful interpretation of BTP in BTPp In particular, the principal of cross sections is not provable in BTP yet its interpretation is provable in BTPp.

The cross sectional interpretation of L(BTPg) into L(BTPp) is the restriction of the cross sectional interpretation of L(BTP) into L(BTPp).

THEROEM B.5. The cross sectional interpretation of L(BTPg) into L(BTPp) is a faithful interpretation of BTPg in BTPp. I.e., a sentence in L(BTPg) is provable in BTPg if and only if its cross sectional interpretation is provable in BTPp.

We now discuss the development of arithmetic and finite sequences of W objects in BTP. We show that there is exactly one way to development arithmetic and finite sequences of W objects in BTP, or in BTPp and BTPg, up to the appropriate kind of isomorphism.

We wish to find pure (general) predicates Nat(x), Zero(x), Suc(x,y), Add(x,y,z), Mult(x,y,z), <(x,y), FSeq(x), Lth(x,y), Val(x,y,z), such that the following holds.

1. Zero(x) holds of a unique x, and N holds of this unique x. Call this the zero.
2. Suc,Add,Mult define unary, binary, binary functions on the extension of Nat, which together with the zero, obey the usual axioms of successor and the defining equations for addition and multiplication.
3. $<(x,y) \leftrightarrow (\exists z \neq 0)(Add(x,y,z))$.
4. $Nat(x) \rightarrow \neg FSeq(x)$.
5. Lth(x,y) defines a (length) function from the extension of FSeq into the extension of Nat.
6. Val(x,y,z) defines a (value) function which is defined if and only if FSeq(x) and <(y,u)), where Lth(x,u).
7. For FSeq(x),FSeq(y), x = y if and only if x,y have the same lengths and the same values at the same places.
8. There is a unique x with Lth(x,zero).
9. For any FSeq(x) and y, there is a (necessarily unique) Fseq(z) of length one greater whose values agree with x and whose value at the length of x is y.
10. Induction holds on the extension of Nat with respect to any general predicate.

THEOREM B.6. BTP proves the existence of pure (general) predicates obeying these conditions. BTPg proves the existence of general predicates obeying these conditions. Furthermore, such a system of pure predicates can be given by formulas in L(BTPp) with no parameters. And such a system of general predicates can be given by formulas in L(BTPg) with no parameters.

THEOREM B.7. BTPp proves the existence of pure predicates obeying these conditions, where in 10 we use pure predicates. And BTPp proves that any such system of pure predicates must obey 10 with respect to all formulas in L(BTPp). Also BTP proves that any such system of pure predicates must obey the full 10. Furthermore, such a system of pure predicates can be given by formulas in L(BTPp) without parameters.

Suppose we are given two systems of predicates as above. A general (pure) isomorphism is a general (pure) predicate that defines a bijection h of W onto W which preserves Nat, Zero, Suc, Add, Mult, <, FSeq, Lth, and where Val(x,y,z) ↔ Val(h(x),h(y),z).

THEOREM B.8. BTP proves that any two systems of general predicates obeying these conditions have a general isomorphism. In fact, BTPg proves that any two systems of general predicates obeying these conditions have a general isomorphism. BTPp proves that any two systems of pure predicates obeying these conditions (with pure predicates used in 10) have a pure isomorphism.

Theorems B.6 - B.8 tell us that there is only one way to develop arithemtic and finite sequences in BTP, either with general predicates or with pure predicates. Also there is only one way to develop arithemtic and finite sequences in BTPp with pure predicates, or in BTPg with general predicates. Furthermore, all of these ways are the same.

We will henceforth assume that we have developed arithmetic and finite sequences in BTPp with pure predicates. By B.7, induction with respect to all formulas will be provable in BTP. We know that the same development can be given in BTPg with general predicates by replacing p with g. The following is useful.

THEOREM B.9. BTPp proves pure comprehension strengthened to allow any natural number parameters.

Finally, we want to discuss W structures and the satisfaction relation. A general (pure) W structure is a finite system of general (pure) predicates, and this causes no problem since we can pack the information into a single general (pure) predicate using the development of finite sequences.

THEOREM B.10. BTPg proves that any general W structure has a unique general satisfaction predicate obeying the usual Tarski clauses. BTPp proves that any pure W structure has a unique pure satisfaction predicate obeying the usual Tarski clauses. BTP proves that for any pure W structure, the unique general satisfaction predicate obeying the usual Tarski clauses is extensionally eqvuivalent to a pure predicate.

We now return to our discussion of the identity of indiscernibles and the singleton extensions principle.

The identity of indiscernibles is formalized as

$$\text{IIS)}\ (\forall x,y)(x = y \leftrightarrow (\forall P)(P^p(x) \leftrightarrow P^p(y))).$$

The singleton extensions principle is formalized as

$$\text{SEP)}\ (\forall x)(\exists P^p)(\forall y)(P^p(y) \leftrightarrow y = x).$$

Obviously in BTPp, SEP implies IIS.

THEOREM B.11. The implication IIS → SEP is not provable in BTP.

C. Predication on W in class theory.

The fairly standard theory of classes with urelements provides an appropriate model of predication on W that will be familiar to set theorists.

The class theories used here can be viewed as extensions of BTPg and BTPp. We also combine them so as to extend BTP. The main thrust is that all of our results about BTPp, BTPg, and BTP still hold.

We first introduce the well known Morse Kelley theory of classes, written as MK. We use lower case variables over sets; upper case variables over classes; = between sets and sets; ∈ between sets and sets, and between a set and a class.

1. Extensionality for sets.
2. Pairing for sets.
3. Union for sets.
4. Power set for sets.
5. Infinity for sets.
6. Foundation for sets.
7. Separation for sets with respect to all formulas.
8. Replacement for sets with respect to all formulas.
9. Comprehension for classes with respect to all formulas.

In 9, we mean $(\exists A)(\forall x)(x \in A \leftrightarrow \varphi)$, where A is not free in φ.

This theory is not fully suitable for an interpretation of predication on W because it does not support urelements.

For a more suitable interpretation of predication on W, we introduce MKU, which is MK with urelements.

In MKU, we use lower case variables over objects; upper case variables over classes; = between objects and objects; $\in$ between objects and objects, and between objects and classes; and the unary predicates M,U on objects. Here M(x) means "x is a set," and U(x) means "x is a urelement."

1. $U(x) \leftrightarrow \neg M(x)$.
2. $x \in y \rightarrow M(y)$.
3. Extensionality for sets.
4. Pairing for sets.
5. Union for sets.
6. Power set for sets.
7. Infinity for sets.
8. Foundation for sets.
9. Separation for sets with respect to all formulas.
10. Replacement for sets with respect to all formulas.
11. Comprehension for classes with respect to all formulas.

In order to suggestively reflect analogies with BTPg, we let MKg and MKUg be the same as MK and MKU except that a superscript g is placed above every class variable. Thus we can think of the class variables in MKg and MKUg as being "general" class variables, just as the predicate variables in BTPg are "general" predicate variables.

The AxC is the axiom of choice for sets, which we take to mean that every set is well orderable. We consider MKg + AxC and MKUg + AxC.

We now present the system MKUp that is the analogous strong extension of BTPp. Set theorists may wish to read MKUp as "MKU without parameters."

MKp, MKUp, are respectively obtained from MK, MKU by restricting comprehension for classes so that no object parameters are allowed (but class parameters are still allowed). The only other change is to place p as a superscript on all class variables. I.e., comprehension reads $(\exists A^p)(\forall x)(x \in A^p \leftrightarrow \varphi)$, where φ is a formula in the language of MKp in which A is not free, and no object variables other than x are free in φ.

In connection with BTP, we present a combined form of MKp and MKg, and a combined form of MKUp and MKUg.

In MKpg, we use lower case variables over sets; upper case variables X^g over general classes; upper case variables X^p over pure classes; = between sets and sets; $\in$ between sets and sets, between a set and a general class, and between a set and a pure class. I.e., L(MKpg) = L(MKp) $\cup$ L(MKg).

1. Extensionality for sets.
2. Pairing for sets.
3. Union for sets.
4. Power set for sets.
5. Infinity for sets.
6. Foundation for sets.
7. Separation for sets with respect to all formulas.
8. Replacement for sets with respect to all formulas.
9. Comprehension for general classes with respect to all formulas.
10. Comprehension for pure classes with respect to all formulas, but where parameters for sets and general classes are not allowed.

In MKUpg, we use lower case variables over objects; upper case variables X^g over general classes; upper case variables Xp over pure classes; = between objects and objects; $\in$ between objects and objects, between objects and general classes, and between objects and pure classes; and the unary predicates M,U on objects. Here M(x) means “x is a set,” and U(x) means “x is a urelement.” I.e., L(MKUpg) = L(MKUp) $\cup$ L(MKUg).

1. $U(x) \leftrightarrow \neg M(x)$.
2. $x \in y \rightarrow M(y)$.
3. Extensionality for sets.
4. Pairing for sets.
5. Union for sets.
6. Power set for sets.
7. Infinity for sets.

8. Foundation for sets.
9. Separation for sets with respect to all formulas.
10. Replacement for sets with respect to all formulas.
11. Comprehension for general classes with respect to all formulas.
12. Comprehension for pure classes with respect to all formulas, but where parameters for sets and general classes are not allowed.

We have now introduced twelve class theories. Six without urelements and six with urelements.

The tripel BTPg, BTPp, BTP stands in direct analogy with any of the following four triples of class theories:

MKg, MKp, MKpg;
MKg + AxC, MKp + AxC, MKpg + AxC;

MKUg, MKUp, MKUpg;
MKUg + AxC, MKUp + AxC, MKUpg + AxC.

THEOREM C.1. MKpg and MKUpg proves that every A^p has the same elements as some A^g. MKpg + AxC and MKUpg + AxC do not prove that every A^p has the same elements as some A^g.

THEOREM C.2. Every theorem of MKpg in L(MKp) is a theorem of MKp. Every theorem of MKpg in L(MKg) is a theorem of MKg. Every theorem of MKp is a theorem of MKg if all superscripts p are replaced by superscripts g. Not every theorem of MKg is a theorm of MKp if all superscripts g are replaced by superscripts p. MKp, MKg, MKpg prove the same theorems that don’t have any predicate variables. These results hold for the corresponding systems with urelements.

We define the cross sectional interpretation of L(MKpg) into L(MKg) analogously to the cross sectional interpretation of L(BTP) into L(BTPp) that was discussed in section B. And we also define the principal of cross sections analogously.

THEOREM C.3. The cross sectional interpetation of L(MKpg) into L(MKp) is an interpretation of MKpg into MKp. It is not a faithful interpetation of MKpg into MKp, and not a faithful interpretation of MKpg + AxC inro MKp + AxC. In particular, the principal of cross sections is not provable in MKpg + AxC yet its interpretation is provable in MKp.

THEOREM C.4. The cross sectional interpreatation of L(MKg) into L(MKp) is a faithful intrpretation of MKg into MKp.

The same results hold also for the cross sectional interpretation of L(MKUpg) into L(MKUp) and of L(MKUg) into L(MKUp).

Recall the identity of indiscernibles, IIS, and the singleton extensions principle, SEP, discussed in section B.

THEOREM C.5. The implication IIS → SEP is not provable in MKpg + AxC, and not provable in MKUpg + AxC.

THEOREM C.6. Each of our six class theories without urelements proves the same formulas as the corresponding theory with urelements that uses only =, and ∈ between objects and classes.

We can view any formula in the language of BTPg, BTPp, BTP, respectively, that does not mention 0 or < >, as a formula in the language of any triple of the above class theories, respectively, provided we replace general predicate variables by general class variables, pure predicate variables by pure class variables, $P^g(x)$ by $x \in P^g$, and $P^p(x)$ by $x \in P^p$.

THEOREM C.7. Every theorem of BTP without 0 or < > is a theorem of MKpg and MKUpg. Hence all theorems of BTP without 0 or < > are provable in any of our twelve class theories whose language includes them.

D. Completeness of PC, PC(=), and PC(=,inf).

This section concerns what we consider to be degenerate situations, where we get the familiar validities from classical logic. We show that Wg (Wp) validity degenerates for formulas without equality. We also show that Wg (Wp) validity degenerates if we assume that Wg (Wp) can be linearly ordered - a most unlikely state of affairs.

We let LPC be the usual language of single sorted predicate calculus without equality, and LPC(=) be the usual language of single sorted predicate calculus with equality. Let PC and PC(=) be the usual associated axioms and rules of inference which are complete under the set interpretation as in the Gödel completeness theorem.

Let PC(=,inf) be PC(=) augmented with the infinitely many axioms

$$(\exists x_1,\ldots,x_n)(x_1 \neq \ldots \neq x_n),$$

where $n \geq 1$.

Here we always use variables $v_1,v_2,\ldots$, constant symbols c_m, relation symbols $R^n{}_m$, and function symbols $F^n{}_m$, where $n,m \geq 1$, and n indicates the arity.

General (pure) domain validity is formulated in BTPg (BTPp) as validity in all general (pure) structures whose domain is the extension of a general (pure) predicate on W which holds somewhere.

The following is proved just by formalizing a standard proof of the Gödel completeness theorem.

THEOREM D.1. BTPg (BTPp) proves that a formula of LPC(=) is general (pure) domain valid if and only if it is provable in PC(=). BTPg (BTPp) proves that any general (pure) predicate of sentences of LPC(=) is general (pure) domain satisfiable if and only if it is consistent in PC(=).

Here a “general (pure) predicate of sentences” is a general (pure) predicate on W which holds exclusively of sentences.

We say that a formula in PC(=) is Wg (Wp) valid if and only if it holds under all assignments in every general (pure) W structure.

We also say that a formula in PC(=) is Ng (Np) valid if and only if it holds under all assignments in every general (pure) N structure; i.e., general (pure) structure whose domain is N.

We say that a general (pure) predicate of sentences is Wg (Wp) satisfiable if and only if there exists a Wg (Wp) structure in which all sentences in the extension hold.

The following is proved by a straightforward formalization of the usual completeness theorem for predicate calculus with equality.

THEOREM D.2. BTPg (BTPp) proves that a formula of LPC(=) is Ng (Np) valid if and only if it is provable in PC(=,inf). BTPg (BTPp) proves that any general (pure) predicate of sentences of LPC(=) is Ng (Np) satisfiable if and only if it is consistent in PC(=,inf).

The following completeness result for formulas without equality is proved simply by formalizing a standard proof of the upward Skolem Lowenheim theorem for PC(=), where unlimited clones of any single element of a countable model are created.

THEOREM D.3. BTPg (BTPp) proves that a formula of LPC is Wg (Wp) valid if and only if it is provable in PC. BTPg (BTPp) proves that any general (pure) predicate of sentences of LPC is Wg (Wp) satisfiable if and only if it is consistent in PC.

We now consider LPC(=). It is obvious that there is a formula of LPC(=) which is Wg valid yet not provable in PC(=). Consider

$$(\exists x,y)(x \neq y).$$

This is Wg valid, and demonstrably so in BTPg.

Hence we can prove in BTPg (BTPp) that the Wg (Wp) validities in PC(=) do not coincide with provability in PC(=).

The highlight of this section is a discussion of the assertions

every formula in LPC(=) is Wg valid
if and only if it is N valid

every formula in LPC(=) is Wp valid
if and only if it is N valid

We now indicate how problematic these assertions are. Consider the following sentence in one binary relation symbol R:

$$(\exists x)(\exists y)(x \neq y \ \&\ (R(x,y) \leftrightarrow R(y,x))).$$

This is an instance of what we call the formal axiom scheme of binary symmetric arguments.

The general (pure) principle of binary symmetric arguments asserts that every general (pure) binary predicate on W is symmetric on a two element set; i.e., that the above formal sentence is Wg (Wp) valid.

Now obviously the above sentence is not N valid. E.g., interpret R as the usual linear ordering of the natural numbers.

There is a weaker existential sentence in R which is even more plausibly Wg (Wp) valid, yet not N valid:

R is not a linear ordering.

What are the consequences of assuming that this sentence is not Wg (Wp) valid? Obviously this sentence is not Wg (Wp) valid if and only if there is a general (pure) linear ordering of W.

Anyone who has worked hard on a hiring committee and failed to get their candidate through tends to voice suspicions of "there is a linear ordering of W." One typically says

you can't rank order these candidates.

You could if there is a linear ordering of W.

The following result demonstrates the power of the assumption that is a general (pure) linear ordering of W.

THEOREM D.4. The following are provably equivalent in BTPg (BTPp):

i) there is a general (pure) linear ordering of W;

ii) a formula in PC(=) is Wg (Wp) valid if and only if it is provable in PC(=,inf);

iii) a general (pure) set of sentences in PC(=) is Wg (Wp) satisfiable if and only if it is consistent in PC(=).

Proof: We will only handle the p case. The g case is proved analogously. We argue throughout in BTPp.

Obviously iii) implies ii). Suppose ii). Now

R is not a linear ordering

is not N valid. Hence it is not Wp valid. I.e., there is a pure linear ordering of W.

It suffices to prove that i) implies ii).

Now suppose that there is a pure linear ordering of W. It is easy to see that there must be a pure dense linear ordering of W without endpoints. This is done by surrounding each point of the linear ordering of W with a copy of the rationals.

Suppose T is a consistent pure set of sentences. By classical model theory, T has an explicitly arithmetical model M (with domain N) which is generated by explicitly arithmetical Skolem functions on an infinite set of explicitly arithmetical linearly ordered indiscernibles. Also by classical model theory, we can assume that the indiscernibles have order type the rationals.

We now consider new constants cx for each W object x. We define a structure whose domain D consists of the closed terms in these constants and the constant and function symbols of M. We use the linear ordering of W, which linearly orders the subscripts of the new constants.

The truth value of any atomic formula will be determined by making any order preserving assignment of indiscernibles to the subscripts of the new constants appearing in the atomic formula, and setting it to be the truth value of the resulting statement in M. Because of indiscernibility, this is independent of the choice of order preserving assignment.

We then prove by induction that any formula in the language of M with assigned free variables (which are closed terms) holds* in this large structure if and only if it holds in M after the closed terms are changed to corresponding elements of D. By corresponding elements of D, we mean that any order preserving map from the subscripts of the new constants appearing into the original indiscernibles in M is selected, and then the terms are evaluated in M. Part of the induction hypothesis is that the truth value in M so obtained does not depend on the choice of order preserving map.

We have placed an asterisk at the sticky point. The interpretation of = here is not taken as equality, as we want. Rather it is taken to be the equivalence relation between these closed terms, according to whether two closed terms have equal values in M when the subscripts of the constants appearing are mapped by an order preserving map into the original indiscernibles.

Normally, this is remedied by calling the large structure a weak model of the sentences true in M, and then proving an additional lemma. The additional lemma asserts that in any weak structure whose interpretation of equality respects the relations and functions of that structure, one can factor out by that equivalence relation and get the same sentences to hold - but this time in a strong structure so that = is interpreted properly.

The problem here is that the ensemble of equivalence classes is not - at least directly - like an extension of a pure predicate. E.g., the type is too high. One possibility is to choose representatives from each equivalence class - but this is dubious.

It suffices to find a pure function H from the domain of the large structure - i.e., the set of those terms - into W such that two terms are equivalent if and only if their values under H are equal. The size of the image of H must be the same as W because the new constants all lie in different equivalence classes.

Two terms s and s’ in D are said to have the same type if and only if they read the same from left to right except for the new constants, and the new constants appear in the same order of subscripts. There are only countably many types. We can explicitly order these types, which are basically integers.

Let t be any term in D. To define H at t, we consider the terms s in [t] of any given type. Suppose there are n occurrences of new constants in s. We consider those n-tuples (of subscripts) that can be used to make a term of the given type which lies in [t].

The new constants appearing in t partition the dense linear ordering of W without endpoints into finitely many nonempty open intervals and points, and we can classify n-tuples according to the relative order of their coordinates and according to where the coordinates fit into this partition. This classification determines whether that n-tuple can be used to make a term of the given type which lies in [t].

Thus the set of terms in [t] of a given type can be completely described in terms of the new constants appearing in t. This is true for each type.

Thus for each type, we get an apporopriate kind of canoncial presentation of the set of terms of that type that lie in [t]. And all of the subscripts of constants mentioned in these canoncial presentations are among the subscripts mentioned in t. Thus the set of subscripts mentioned in these canonical presentations must be a finite set, E.

We know that E is included in the new constants used in t. So we can consider this countable amount of information with reference not to E but to the set of new constants in t. Now let t’ be any term of M of the same type as t, and look at the equivalence class [t’] of terms in M that give the same value in M as t’. The countable amount of information with reference to the indiscernibles in t’ and M must be the same as the countable amount of information with reference to t and D. But this correspondence in M must be arithmetically given because M was arithmetically given. Hence the countable amount of information with reference to the set of new constants in t is arithmetical. Therefore this information must be arithmetical when given with reference to E.

Thus our representation of [t] is simply E (which is listed according to the pure ordering of W), together with the least possible arithmetical description of the countable amount of information.

We can now factor D by the equivalence relation using these representatives to obtain a pure structure whose domain is large; i.e., such that there is a one-one pure map from W into the domain. But then we use the Shroeder Bernstein theorem in this context, which can be proved in its pure form in BTPp.

THEOREM D.5. The following is consistent with BTP. In PC(=), the Wg validities and the Wp validities are the same as the N validities.

BTP obviously proves that every Wg valid formula of PC(=) is Wp valid. However, BTP does not prove the converse (Theorem E1.3).

E. Existential and universal sentences.

1. Principle of symmetric arguments.

We begin by dispensing with the validity problem for universal sentences in PC(=).

A formula in LPC(=) is said to be existential (universal) if and only if it begins with a block of zero or more existential (universal) quantifiers followed by a quantifier free formula.

We write LPC(∃,=) for the set of existential sentences in LPC(=), and LPC(∀,=) for the set of universal sentences in LPC(=).

THEROEM E1.1. It is provable in BTPg (BTPp) that every sentence in LPC(∀,=) is Wg (Wp) valid if and only if it is provable in PC(=). It is provable in BTPg (BTPp) that every general (pure) predicate of sentences in LPC(∃,=) is Wg (Wp) satisfiable if and only if it is consistent in PC(=).

Note that provability for universal sentences is particularly elementary. It just means that the matrix follows tautologically from the atomic equality axioms involving only the symbols in the formula. If the formula has no equality symbols then this just means that the matrix is a tautology.

The most basic sentence of PC(=) whose Wg (Wp) validity is in question is

$$(\exists x)(\exists y)(x \neq y \wedge (R(x,y) \leftrightarrow R(y,x))),$$

as alluded to in section D.

There is the corresponding general (pure) principle of binary symmetric arguments which asserts, in effect, that this sentence is Wg (Wp) valid.

More generally, the general (pure) principle of symmetric arguments asserts that for all $k \geq 1$, every general (pure) predicate holds or fails of all permutations of some k-tuple of distinct objects.

Note that this is trivial for $k = 1$. Also, if it holds for $k \geq 2$ then it holds for $k-1$.

We remark that the general principle of symmetric arguments immediately implies the pure principle of symmetric arguments in BTP.

THEOREM E1.2. The following are provably equivalent in BTPg (BTPp).

i) The general (pure) principle of symmetric arguments;

ii) Let $k,n,r \geq 1$ and $P_1,\ldots,P_r$ be general (pure) predicates. There is a finite sequence of distinct objects $(x_1,\ldots,x_n)$ such that each P_i holds or fails of all k-tuples of distinct terms from $(x_1,\ldots,x_n)$.

THEOREM E1.3. The general and pure principles of symmetric arguments are neither provable nor refutable from BTPg (BTPp). BTP + general and pure principles of symmetric arguments is consistent. BTP + pure principle of symmetric arguments does not prove the general principle of symmetric arguments.

In fact, Theorem E1.3 holds for our class theories.

THEOREM E1.4. The general and pure principles of symmetric arguments are neither provable nor refutable from MKpg + AxC. MKpg + AxC + general and pure principles of symmetric arguments is consistent. MKpg + pure principle of symmetric arguments does not prove the general principle of symmetric arguments. These results also hold for the systems with urelements.

We say that a general (pure) predicate P is finite if and only if there is a finite sequence x such that

$$(\forall y)(P(y) \rightarrow y \text{ is a term in } x).$$

We say that a general (pure) predicate P is infinite if and only if it is not finite.

This is a relatively weak kind of finiteness and relatively strong kind of infiniteness. For the sake of clarity, we also make the following definitions.

We say that a general (pure) predicate P has generally (purely) finite extension if and only if there is a general (pure) bijection between its extension and a bounded initial segment of natural numbers.

THEOREM E1.5. BTP proves that any general predicate is finite if and only if it has generally finite extension. BTP does not prove that all finite pure predicates have purely finite extensions.

A minimally infinite general predicate is an infinite general predicate P such that for any general predicate Q, either

i) there is a finite sequence x such that $(\forall y)((P(y) \wedge Q(y)) \rightarrow y$ is a term in x); or
ii) there is a finite sequence x such that $(\forall y)((P(y) \wedge \neg Q(y)) \rightarrow y$ is a term in x).

THEOREM E1.6. BTPg proves the following. If there is a minimally infinite general predicate then the general principle of symmetric arguments holds. However, the converse is not provable in BTPg.

There is a formulation of Theorem E1.6 that is appropriate for our class theories. A set E is called Dedekind finite if and only if every subset of E is either finite or cofinite in E.

THEOREM E1.7. MKg and MKUg prove the following. If there is an infinite Dedekind finite set then the general principle of symmetric arguments holds. Neither can prove the converse. MKg is consistent with the existence of an infinite Dedekind finite set. MKg + AxC and MKUg + AxC are consistent with the general principle of symmetric arguments.

On the pure side, we don’t know of any one dimensional condition like minimally infinite general predicate which suffices to derive the pure principle of symmetric arguments. However, there is the following very natural strong condition. An absolute POI (predicate of indiscernibles) is a pure predicate P such that any pure predicate holds or fails of any two equal length finite sequences of distinct objects from the extension of P.

THEOREM E1.8. BTP is consistent with the existence of an infinite absolute POI.

In class theory, an absolute SOI (set of indiscernibles) is a set E such that for any set A, any two equal length finite sequences of distinct elements of E either both lie in A or both lie outside A.

THEOREM E1.9. MKp + AxC and MKUp + AxC are consistent with the existence of an absolute SOI.

2. Completeness.

We say that τ is a relational type if and only if τ is a (at most countable) set of constant, relation, and function symbols. Here we take all constant, relation, and function symbols to be indexed by natural numbers. We will use “rel” for the important type of all relation symbols.

Let LPC(=,τ), LPC($\exists$,=,τ), LPC($\forall$,=,τ) be, respectively, the restriction of LPC(=), LPC($\exists$,=), LPC($\forall$,=) to formulas within type τ. Here equality is always allowed even if τ is empty. Recall that PC(=,inf) is the usual axioms and rules of inference for predicate calculus with equality augmented by the axioms asserting that there are infinitely many objects.

The formal axioms of symmetric arguments for LPC(=) consist of the following sentences. Let $k \geq 1$ and φ be a formula in LPC(=) with at most the free variables $x_1,\ldots,x_k$. Take

$$(\exists x_1 \neq \ldots \neq x_k)(\text{the conjunction of } (\varphi(x_1,\ldots,x_k) \leftrightarrow \varphi(x_{\pi 1},\ldots,x_{\pi k}))),$$

where the conjunction ranges over all permutations π of $\{1,\ldots,k\}$.

We write SYM(=) for PC(=,inf) augmented with the formal axioms of symmetric arguments for LPC(=).

We also consider the formal quantifier free axioms of symmetric arguments, which are as above, except that the φ

are additionally required to be quantifier free. We write QFSYM(=) for LPC(=) augmented with the formal quantifier free axioms of symmetric arguments for LPC(=).

More generally, we write SYM(=,τ), QFSYM(=,τ) for the fragments of SYM(=), QFSYM(=) whose axioms lie in LPC(=,τ).

THEOREM E2.1. Let τ be a finite relational type and φ be a sentence in LPC(∃,=,τ). The following are equivalent.
i) φ is provable in SYM(=);
ii) φ is provable in SYM(=,τ);
iii) φ is provable in QFSYM(=);
iv) φ is provable in QFSYM(=,τ).

The axioms of SYM(=,τ) and QFSYM(=,τ) can also be given in the following multiple form. Let $k,n,r \geq 1$. Let $\varphi_1,...,\varphi_r$ be formulas in LPC(=,τ) with at most the free variables $x_1,...,x_k$. Take

$$(\exists x_1 \neq ... \neq x_n)(\text{the conjunction of } (\varphi_i(x_1,...,x_k) \leftrightarrow \varphi_i(x_{p1},...,x_{pk}))),$$

where the conjunction ranges over all choices of k-tuples $(p_1,...,p_k)$ of distinct integers from $\{1,...,n\}$, and $1 \leq i \leq r$. We follow the normal convention that an empty conjunction is vacuously true.

THEOREM E2.2. Let τ be a relational type. Each SYM(=,τ) derives SYM(=,τ) in multiple form. Each QFSYM(=,τ) derives QFSYM(=,τ) in multiple form.

THEOREM E2.3. Let φ be a formula in LPC (i.e., predicate calculus with no =) or in LPC(∀,=). Then φ is provable in SYM(=) if and only if φ is provable in PC(=,inf).

THEOREM E2.4. The following are provable in BTPg (BTPp).
i) Every Wg (Wp) valid sentence in LPC(∃,=) is provable in SYM(=);
ii) Every general (pure) predicate of sentences in LPC(∀,=) that is consistent in SYM(=) is Wg (Wp) satisfiable.

Thus we see that the Wg (Wp) valid sentences in LPC(∃,=) include those provable in PC(=,inf), and are included in those provable in SYM(=).

This validity problem is settled if we assume the general (pure) principle of symmetric arguments.

THEOREM E2.5. The following are provable in BTPg + the general (pure) principle of symmetric arguments.

i) Every sentence in PC(∃,=) is Wg (Wp) valid if and only if it is provable in SYM(=);

ii) Every general (pure) predicate of sentences in PC(∀,=) that is consistent in SYM(=) is Wg (Wp) satisfiable.

In fact, the principle of symmetric arguments is equivalent to completeness and soundness.

THEOREM E2.6. The following are provably equivalent in BTPg (BTPp).

i) the general (pure) principle of symmetric arguments;

ii) every sentence in PC(∃,=) that is provable in SYM(=) is Wg (Wp) valid;

iii) every sentence in PC(∃,=,rel) that is provable in SYM(=,rel) is Wg (Wp) valid;

iv) every general (pure) predicate of sentences in PC(∀,=) is Wg (Wp) satisfiable if and only if it is consistent in SYM(=).

Thus if you accept the principle of symmetric arguments, then you have completely determined the sentences in PC(∃,=) that are W valid. In fact, you have determined the ones in PC(∃,=,rel) in a very concrete sense.

THEOREM E2.7. The set of sentences in PC(∃,=,rel) provable in PC(=,rel) is co-nondeterministic exponential time complete. The set of sentences in PC(∃,=) provable in SYM(=,rel) is in co-nondeterministic polynomial time, and therefore co-nondeterministic polynomial time complete. However, the set of sentences in PC(∃,=) provable in PC(=) and the set of sentences in PC(∃,=) provable in SYM(=) is complete r.e.

We now discuss how our determination of the Wg (Wp) valid sentences in PC(∃,=) is distinguished from alternative determinations.

Let B be a set of sentences in LPC(=). A determination of the Wg (Wp) valid sentences in B is a set X of sentences in B such that for some consistent extension T of BTPg (BTPp) obtained by adding finitely many new axioms, X is exactly the set of sentences φ in B such that T proves "φ is Wg (Wp) valid." We are interested in the class of determinations under inclusion.

THEOREM E2.8. There is a smallest determination of the Wg (Wp) valid sentences in LPC(=). It is the set of sentences provable in PC(=,inf), and is realized by BTPg (BTPp). There is a largest determination of the Wg (Wp) valid sentences in LPC(∃,=,rel). This is the set of sentences in LPC(∃,=,rel) that are provable in SYM(=,rel), and is realized by BTPg (BTPp) together with the general (pure) principle of symmetric arguments. However, there is no largest determination of the Wg (Wp) valid sentences in LPC(∃,=).

Suppose we modify the definition of determination above to allow T to be a consistent recursively axiomatized extension of BTPg (BTPp), or even an arbitrary consistent extension of BTPg (BTPp). Then Theorem E2.8 would still hold.

A faithful determination of the Wg (Wp) valid sentences in B is a set X of sentences in B such that for some extension T of BTPg (BTPp) obtained by adding finitely many new axioms, which does not prove any false sentences of arithmetic, X is exactly the set of sentences φ in B such that T proves "φ is Wg (Wp) valid."

THEOREM E2.9. There is a smallest faithful determination of the Wg (Wp) valid sentences in LPC(=). This is the set of sentences provable in PC(=,inf), and is realized by BTPg (BTPp). There is a largest faithful determination of the Wg (Wp) valid sentences in LPC(∃,=). This is the set of sentences in LPC(∃,=) that are provable in SYM(=), and is realized by BTPg (BTPp) together with the general (pure) principle of symmetric arguments.

Theorem E2.9 holds even if we weaken the condition on T in the definition of faithful determination to “which does not prove any false universal sentence of arithmetic.”

We can carry over this discussion of determinations into the context of class theory. For this purpose, let Sg, Sp, Spg be theories in the language L(MKg), L(MKp), L(MKpg), respectively, and B be a set of sentences in LPC(=).

A determination of the Wg (Wp) valid sentences in B over Sg (Sp) is a set X of sentences in B such that for some consistent extension T of Sg (Sp) obtained by adding finitely many new axioms, X is exactly the set of sentences φ in B such that T proves “φ is Wg (Wp) valid.”

A faithful determination of the Wg (Wp) valid sentences in B over Sg (Sp) is a set X of sentences in B such that for some extension T of Sg (Sp) obtained by adding finitely many new axioms, which does not prove any false sentences of arithmetic, X is exactly the set of sentences φ in B such that T proves “φ is Wg (Wp) valid.”

THEOREM E2.10. There is a smallest determination of the Wg valid sentences in LPC(=) over any of MKg, MKg + AxC, MKUg, MKUg + AxC, MKpg, MKpg + AxC, MKUpg, MKUpg + AxC. It is the set of sentences provable in PC(=,inf). There is a largest determination of the Wg valid sentences in LPC(∃,=,rel) over any of MKg, MKg + AxC, MKUg, MKUg + AxC, MKpg, MKpg + AxC, MKUpg, MKUpg + AxC. It is the set of sentences in LPC(∃,=,rel) that are provable in SYM(=,rel). There is no greatest determination of the Wg valid sentences in LPC(∃,=) over any of MKg, MKg + AxC, MKUg, MKUg + AxC, MKpg, MKpg + AxC, MKUpg, MKUpg + AxC. These results hold with g and p interchanged.

THEOREM E2.11. There is a smallest faithful determination of the Wg valid sentences in LPC(=) over any MKg, MKg + AxC, MKUg, MKUg + AxC, MKpg, MKpg + AxC, MKUpg, MKUpg + AxC. It is the set of sentences provable in PC(=,inf). There is a largest determination of the Wg valid sentences in LPC(∃,=) over any of MKg, MKg + AxC, MKUg, MKUg + AxC, MKpg, MKpg + AxC, MKUpg, MKUpg + AxC. It is the set of sentences in LPC(∃,=) that are provable in SYM(=). These results hold with g and p interchanged.

3. Alternative axiomatizations.

Here we consider some axiomatizations other than SYM(=) and QFSYM(=), which give rise to the same provable sentences in PC(∃,=).

We start with the most rudimentary axioms of universal symmetry. Let φ be an atomic formula in LPC(=) with at most the free variables $x_1,\ldots,x_k$, $k \geq 1$. Take

$$(\forall x_1 \neq \ldots \neq x_k)(\forall y_1 \neq \ldots \neq y_k)(\varphi(x_1,\ldots,x_k) \leftrightarrow \varphi(y_1,\ldots,y_k)),$$

where $x_1,\ldots,x_n,y_1,\ldots,y_n$ are distinct variables.

We write USYM(=) for PC(=,inf) together with these axioms of universal symmetry. If the atomic formulas φ above are restricted to τ, then we write USYM(=,τ).

We can formulate USYM(=) where φ is an arbitrary formula in LPC(=) subject to the free variable condition. It is easy to see that this formulation is equivalent.

THEOREM E3.1. SYM(=) and USYM(=) prove the same sentences in LPC(∃,=). In fact, for each relational type τ, SYM(=,τ) and USYM(=,τ) prove the same sentences in LPC(∃,=,τ).

Thus we can restate most of the completeness results of section E2 in terms of USYM(=). Of course, the axioms of USYM(=) are demonstrably not Wg or Wp valid. The axioms of SYM(=) are Wg (Wp) valid according to the general (pure) principle of symmetric arguments.

Also note that USYM(=) is a universal theory whereas SYM(=) is not. We now present another universal theory.

We introduce constants $d_1,d_2,\ldots$ that are new to LPC(=). Let φ be an atomic sentence in LPC(=) with at most the free variables $x_1,\ldots,x_k$ and π be a permutation of $(1,\ldots,k)$. Take

$$\varphi(d_1,\ldots,d_k) \leftrightarrow \varphi(d_{\pi 1},\ldots,d_{\pi k}).$$

We let ATSYMC(=) be the system based on these axioms together with the axioms $d_i \neq d_j$, $i \neq j$. Here ATSYMC is read "atomic symmetry with constants." If the formulas φ are resricted to a relational type τ, then we write SYMC(=,τ).

We also let SYMC(=) be the extension of ATSYMC(=) where the φ are allowed to be arbitrary formulas in LPC(=) subject to the free variable condition. We also consider SYMC(=,τ).

THEOREM E3.2. SYM(=) and SYMC(=) prove the same formulas in LPC(=). In fact, for each relational type τ, SYM(=,τ) and SYMC(=,τ) prove the same formulas in LPC(=,τ).

THEOREM E3.3. QFSYM(=) and ATSYMC(=) prove the same formulas in LPC(=). In fact, for each relational type τ, QFSYM(=,τ) and ATSYMC(=,τ) prove the same formulas in LPC(=,τ).

Thus we can restate most of the completeness results of section E2 in terms of SYMC(=) and the universal theory ATSYMC(=).

4. Alternative determinations.

We may deny the general (pure) principle of symmetric arguments. Then the Wg (Wp) valid sentences in (∃,=,rel) will not include the axioms of QFSYM(=,rel). Can we adopt some of the general (pure) principle of symmetric arguments without adopting all of it?

THEOREM E4.1. Let $k \geq 2$. The following are jointly consistent with MKg + AxC, and with MKUg + AxC.

i) there exists a general class E, where is no sequence of length k of distinct objects from E such that all permutations lie in E or all permutations lie outisde E;

ii) for all $n,r \geq 1$ and general classes $E_1,\ldots,E_r$, there exists a set A of cardinality n such that for each $1 \leq i \leq r$, all (k-1)-tuples of distinct elements of A lie in E_i or all (k-1)-tupes of distinct elements of A k-1 lie outside E_i.

Note that i) and ii) for $k \geq 2$ implies that the set of Wg valid sentences in LPC(∃,=,rel) lies strictly between those provable in PC(=,inf) and SYM(=,rel).

THEOREM E4.2. Let $k \geq 2$. The following are jointly consistent with MKp + AxC, and with MKUp + AxC.

i) there exists a pure class E, where there is no sequence of length k of distinct elements from E such that all permutations lie in E or all permutations lie outside E;

ii) there exists an infinite pure class E such that for all pure classes A, either all (k-1)-tuples of distinct elements of E lie in A, or all (k-1)-tuples of distinct elements of E lie outside A.

Note that i) and ii) for $k \geq 2$ implies that the set of Wp valid sentences in LPC(∃,=,rel) lies strictly between those provable in PC(=,inf) and SYM(=,rel). Of course, we could get this result directly from i) and ii) of Theorem E4.1 stated for p instead of g; this implication follows by Theorem B.2.

It is likely that these results lead to alternative determinations of the Wg (Wp) valid sentences in PC(∃,=rel) in low computational complexity, that are consistent with MKpg + AxC. In addition, in MKpg + AxC, there should be no relationship between the Wg and the Wp valid sentences in PC(∃,=,rel) other than the trivial one that every Wg valid sentence is Wp valid. We have not gone into these further matters in detail.

F. ∀∃ and ∃∀ sentences.

We can extend our completeness theorem to ∀∃ sentences by allowing parameters in SYM(=). Thus we write SYM'(=) for the formal axioms of symmetric arguments for LPC(=) with parameters, which consists of the following formulas (interpreted universally).

Let $k \geq 1$ and φ be a formula in LPC. Take

$$(\exists x_1 \neq \ldots \neq x_k)(\text{the conjunction of } (\varphi(x_1,\ldots,x_k) \leftrightarrow \varphi(x_{\pi 1},\ldots,x_{\pi k}))),$$

where the conjunction ranges over all permutations π of $(1,\ldots,k)$.

We write SYM'(=) for PC(=,inf) augmented with the formal axioms of symmetric arguments for LPC(=) with parameters.

We also consider QFSYM’(=), which is as above, except φ is required to be quantifier free.

More generally, we write SYM’(=,τ), QFSYM’(=,τ) for the fragments of SYM’(=), QFSYM’(=) whose axioms lie in LPC(=,τ).

THEOREM F.1. Let τ be a finite relational type and φ be a sentence in LPC(∀∃,=,τ). The following are equivalent.
i) φ is provable in SYM’(=);
ii) φ is provable in SYM’(=,τ);
iii) φ is provable in QFSYM’(=);
iv) φ is provable in QFSYM’(=,τ).

THEOREM F.2. Let φ be a sentence in LPC(∃,=). Then φ is provable in SYM(=) if and only if it is provable in SYM’(=).

These axioms can also be given in the following multiple form. Let $k,n,r \geq 1$. Let $\varphi_1,\ldots,\varphi_r$ be formulas in LPC(=). Take

$$(\exists x_1 \neq \ldots \neq x_n)(\text{the conjunction of } (\varphi_i(x_1,\ldots,x_k) \leftrightarrow \varphi_i(x_{p1},\ldots,x_{pk}))),$$

where the conjunction ranges over all choices of k-tuples $(p_1,\ldots,p_k)$ of distinct integers from $\{1,\ldots,n\}$, and $1 \leq i \leq r$.

THEOREM F.3. Let τ be a relational type. Each SYM’(=,τ) derives SYM’(=,τ) in multiple form. Each QFSYM’(=,τ) derives QFSYM’(=,τ) in multiple form.

THEOREM F.4. Let φ be a formula in LPC (i.e., predicate calculus with no =) or in LPC(∀,=). Then φ is provable in SYM’(=) if and only if φ is provable in PC(=,inf).

THEOREM F.5. The following are provable in BTPg (BTPp).
i) Every Wg (Wp) valid sentence in LPC(∀∃,=) is provable in SYM’(=);
ii) Every general (pure) predicate of sentences in LPC(∃∀,=) that is consistent in SYM’(=) is Wg (Wp) satisfiable.

THEOREM F.6. The following are provable in BTPg + the general principle of symmetric arguments.

i) Every sentence in PC(∀∃,=) is Wg valid if and only if it is provable in SYM'(=);

ii) Every general predicate of sentences in PC(∃∀,=) that is consistent in SYM'(=) is Wg satisfiable.

The same holds with g replaced by p and general replaced by pure.

THEOREM F.7. The following are provably equivalent in BTPg.

i) the general principle of symmetric arguments;

ii) every sentence in PC(∀∃,=) that is provable in SYM'(=) is Wg valid;

iii) every general set of sentences in PC(∃∀,=) is Wg satisfiable if and only if it is consistent in SYM'(=).

The same holds with g replaced by p and general replaced by pure.

THEOREM F.8. There is a largest faithful determination of the Wg (Wp) valid sentences in LPC(∀∃,=). This is the set of sentences in LPC(∀∃,=) that are provable in SYM'(=), and is realized by BTPg (BTPp) together with the general (pure) principle of symmetric arguments. However, there is no largest determination of the Wg (Wp) valid sentences in LPC(∀∃,=).

THEOREM F.9. There is a largest determination of the Wg valid sentences in LPC(∀∃,=) over any of MKg, MKg + AxC, MKUg, MKUg + AxC, MKpg, MKpg + AxC, MKUpg, MKUpg + AxC. It is the set of sentences in LPC(∀∃,=) that are provable in SYM'(=), and is realized by any of these theories together with the general principle of symmetric arguments. These results hold with g and p interchanged.

G. Arbitrary formulas - model theoretic determination.

Here we give a determination of the Wg and Wp validity of arbitrary formulas in LPC(=) in model theoretic terms. In particular, the formulas determined to be Wg (Wp) valid will be recursively enumerable.

Let M be a relational structure in a relational type τ and E $\subseteq$ dom(M). We say that M is symmetric over E if and only if every permutation of E extends to an automorphism of M.

We say that M is uniformizing if and only if for all $k \geq 1$ and $R \subseteq \text{dom}(M)^{k+1}$ definable in M without parameters, there exists a function $F:\text{dom}(M)^k \to \text{dom}(M)$ definable in M without parameters, such that for all $x_1,...,x_n,y \in \text{dom}(M)$, if $R(x_1,...,x_k,y)$ then $R(x_1,...,x_k,F(x_1,...,x_k))$.

Let SYMUNI(=) be the set of all sentences φ that hold in every uniformizing structure that is symmetric over an infinite set. We emphasize that the relational type of the uniformzing structure may have to contain symbols beyond those occurring in φ.

Let SYMUNI(=,τ) be the set of sentences in SYMUNI(=) and LPC(=,τ).

THEOREM G.1. SYMUNI(=) and SYMUNI(=,rel) are complete r.e. A sentence in LPC(∀∃,=) lies in SYMUNI(=) if and only if it is provable in SYM′(=). A sentence in LPC(∃,=) lies in SYMUNI(=) if and only if it is provable in SYM(=).

THEOREM G.2. The following is provable in BTPg (BTPp). Every Wg (Wp) valid sentence lies in SYMUNI(=).

THEOREM G.3. SYMUNI(=) is the largest faithful determination of the Wg (Wp) valid sentences in LPC(=). SYMUNI(=) is also the largest faithful determination of the Wg valid sentences in LPC(=) over any of MKg, MKg + AxC, MKUg, MKUg + AxC, MKpg, MKpg + AxC, MKUpg, MKUpg + AxC. This result is false for MKg + AxC. These results holds with g and p interchanged.

We now give an extension of the general principle of symmetric arguments that is equivalent to “every element of SYMUNI(=) is Wg (Wp) valid.” However, this extension doesn’t have the simplicity and plausibility of the general principle of symmetric arguments.

Let P be a general (pure) predicate and F be a function with domain A^k, where A is finite. F and A are treated in BTPg (BTPp) as objects using finite sequences.

We say that F is a k-ary finite choice function for P if and only if for all $x \in A^k$, if $(\exists y)(P(<x,y>)$ then $P(<x,F(x)>)$.

Let $E \subseteq A$ and Q be a general (pure) predicate. We say that F is n,m-symmetric over E with respect to Q if and only if for any terms $t_1,...,t_m$ involving F as a k-ary function symbol and at most n occurences of elements of E, we have

$Q(t_1,...,t_r) \leftrightarrow Q(t_1[\pi],...,t_m[\pi])$,

where every t_i is defined, and $t_i[\pi]$ is the term resulting from permuting the elements of E used in t_i by π.

The extended general (pure) principle of symmetric arguments asserts the following. For all $k,n,m,r \geq 1$ and general (pure) predicates P,Q, there exists a k-ary finite choice function for P which is n,m-symmetric over some set of cardinality r.

THEOREM G.4. The following are provably equivalent in BTPg.
i) the extended general principle of symmetric arguments;
ii) every sentence in PC(=) that is provable in SYMUNI(=) is Wg valid;
iii) every general set of sentences in PC(=) is Wg satisfiable if and only if it is consistent in SYMUNI(=).
The same holds with g replaced by p and general replaced by pure.

Theorem G.4 also holds over MKg and MKUg, and over Mkp and MKUp.

We can axiomatize SYMNUNI(=) by introducing Skolem functions autonomously for all formulas, and then writing down the formal principle of symmetric arguments without parameters. There are other relevant formalizations involving new constants $d_1,d_2,...$.