

THE INTERPRETATION OF SET THEORY IN PURE PREDICATION THEORY
preliminary report

by

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NOTE: This paper is in preparation, and the only proofs present are in Part II, A and B.

In fact, Gödel gave an important model of pure predication, where he showed that restricted comprehension without parameters is valid, but where restricted comprehension with parameters is not (although this invalidity was not established until Cohen). This is the model based on ordinal definability in set theory.

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PART II. WITHOUT EXTENSIONALITY.

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INTRODUCTION

This paper was referred to in the Introductions to our papers [Fr97a], "The Axiomatization of Set Theory by Separation, Reducibility, and Comprehension," and [Fr97b], "The Interpretation of Set Theory in Mathematical Predication Theory."

In [Fr97a], all systems considered include the axiom of Extensionality and made unrestricted use of parameters. Extensionality and unrestricted use of parameters is appropriate in the context of mathematical predication.

In [Fr97b], all systems considered include unrestricted use of parameters, but not Extensionality. Unrestricted use of parameters is appropriate in the context of mathematical predication, and Extensionality is inappropriate.

Here we view the underlying concept as that of "pure predication." instead of set and instead of mathematical predication. This is the concept of predication that is used outside mathematics, where it is not appropriate to assume,

e.g., that for every object x , there is a pure predicate that holds of exactly x . However, if x can be explicitly defined, then such a pure predicate exists. First order definability without parameters approximates pure predication.

Godel proposed an important model of pure predication in the context of set theory which differs from that of mathematical predication. This is his concept of ordinal definable set. Parameterless separation is provably valid in the ordinal definable sets. However ordinary separation is not provably valid in the ordinal definable sets.

The most appropriate systems in this paper for pure predication are in Part II where we discuss systems without Extensionality and without parameters. For the sake of completeness and ease of exposition, we consider systems with Extensionality and without parameters in Part I.

As in [Fr97b] we will be exclusively concerned with interpretability. In each case, we opt for the simplest and cleanest set of axioms which allow for interpretability. The interpretations will at least preserve the arithmetic of natural numbers.

There is the legitimate concern of the strength of the set theories based on ZF with the higher of the large cardinal axioms that are interpreted here. Results of Hugh Woodin about forcing over universes with very large cardinals suggest that they are very strong - probably as strong as when the axiom of Choice is added. However at the time of this writing, this has not been resolved. In any case, it is known that these systems based on ZF are at least as strong as all large cardinals for which there exists a current inner model theory. One can of course merely add the axiom of Choice in its usual formulation to the axiomatizations considered here.

We close the paper with some conceptual interpretations of the axiomatizations proposed here. Although we regard this discussion as perhaps premature at this stage, it is intended to provide motivation for the axiomatizations, and the choice of terminology: reducibility and comprehension.

PART I. WITH EXTENSIONALITY

1. ONE SUBWORLD: $ZF \setminus P$

In this section we treat the theory $T_2(W)$ in the language $L(\in, W)$, as presented in the Introduction:

1. Extensionality. $(\forall x_1)(x_1 \in x_2 \leftrightarrow x_1 \in x_3) \rightarrow (\forall x_1)(x_2 \in x_1 \leftrightarrow x_3 \in x_1)$.
2. Comprehension. $(\exists x_1)(\forall x_2)(x_2 \in x_1 \leftrightarrow (x_2 \in W \ \& \ \varphi))$, where φ is a formula in $L(\in)$ with at most the free variable x_2 .
3. Reducibility. $(x_1, \dots, x_n \in W \ \& \ \varphi) \rightarrow (\exists x_{n+1} \in W)(\varphi)$, where $n \geq 0$ and φ is a formula in $L(\in)$ whose free variables are among x_1, \dots, x_{n+1} .

Here φ has at most the free variable y and does not mention W .

We define $x = y$ if and only if x, y have the same elements. We let $x \in W$ abbreviate $W(x)$.

LEMMA IA.1. The following are provable in $T_2(W)$.

- i) $x_1, \dots, x_n \in W \rightarrow (\varphi \leftrightarrow \varphi^W)$, where $n \geq 0$ and φ is a formula in $L(\in)$ whose free variables are among x_1, \dots, x_n ;
- ii) W is nonempty;
- iii) $x = y \rightarrow (\varphi \leftrightarrow \varphi[x/y])$, where φ is a formula in $L(\in, W)$ and x, y are not free in φ .

Proof: The first claim and third claims are by induction on φ . To see that W is nonempty, apply Reducibility to the sentence $\varphi = (\exists x_1)(x_1 \in x_1 \rightarrow x_1 \in x_1)$. \square

We write $WFT(x)$ if and only if

- i) x is transitive;
- iii) for all y , if there is an element of x in y , then there is an \in -minimal element of x in y .

Here WFT abbreviates "well founded and transitive."

We write $x \in= y$ for " $x \in y$ or $x = y$."

LEMMA IA.2. Let $\varphi(x, y_1, \dots, y_n)$ be a formula with at most the free variables shown, and without W . The following is provable in $T2(W)$. $(WFT(x) \ \& \ x \in W \ \& \ \varphi^W(x, y_1, \dots, y_n)) \rightarrow (\exists z \in W) (z \in= x \ \& \ \varphi^W(z, y_1, \dots, y_n) \ \& \ (\forall w \in W) (w \in z \rightarrow \neg \varphi^W(w, y_1, \dots, y_n)))$.

Proof: Let $S = \{x \in W : (\exists y_1 \dots y_n) (\varphi^W(y_1, \dots, y_n) \ \& \ \neg (\exists z \in W) (z \in= x \ \& \ \varphi^W(z, y_1, \dots, y_n) \ \& \ (\forall w \in W) (w \in z \rightarrow \neg \varphi^W(w, y_1, \dots, y_n))))\}$. S exists by the third axiom.

It suffices to prove that S has no element x with $WFT(x)$. We suppose that $x \in S$, $WFT(x)$. Let x' be an \in -minimal element of x in S .

Let $\varphi^W(x', y_1, \dots, y_n)$, $\neg (\exists z \in W) (z \in= x' \ \& \ \varphi^W(z, y_1, \dots, y_n) \ \& \ (\forall w \in W) (w \in z \rightarrow \neg \varphi^W(w, y_1, \dots, y_n)))$. Then setting $z = x'$, we see that $\neg (\forall w \in W) (w \in x' \rightarrow \neg \varphi^W(w, y_1, \dots, y_n))$. Hence $(\exists w \in W) (w \in x' \ \& \ \varphi^W(w, y_1, \dots, y_n))$. Fix $u \in x'$ with $u \in W$, $\varphi^W(u, y_1, \dots, y_n)$. Note that $u \in x$ by the transitivity of x .

We claim that $u \in S$. To see this, suppose that $(\exists z \in W) (z \in= u \ \& \ \varphi^W(z, y_1, \dots, y_n) \ \& \ (\forall w \in W) (w \in z \rightarrow \neg \varphi^W(w, y_1, \dots, y_n)))$. Then $(\exists z \in W) (z \in= x \ \& \ \varphi^W(z, y_1, \dots, y_n) \ \& \ (\forall w \in W) (w \in z \rightarrow \neg \varphi^W(w, y_1, \dots, y_n)))$, which contradicts $x \in S$. Hence $\neg (\exists z \in W) (z \in= u \ \& \ \varphi^W(z, y_1, \dots, y_n) \ \& \ (\forall w \in W) (w \in z \rightarrow \neg \varphi^W(w, y_1, \dots, y_n)))$. Therefore $u \in S$.

We now have the desired contradiction since x' is an \in -minimal element of x in S and $u \in x', S$.

LEMMA IA.3. Let $\varphi(x, y_1, \dots, y_n)$ be a formula with at most the free variables shown, and without W . The following is provable in $T2(W)$. $(WFT(x) \ \& \ \varphi(x, y_1, \dots, y_n)) \rightarrow (\exists z \in= x) (\varphi(z, y_1, \dots, y_n) \ \& \ (\forall w \in z) (\neg \varphi(w, y_1, \dots, y_n)))$.

Proof: By Lemma IA.1 ii, $(\forall x)(W(x) \rightarrow (WFT(x) \leftrightarrow WFT^W(x)))$. By Lemma IA.2, $(\forall x, y_1, \dots, y_n \in W)(WFT^W(x) \rightarrow \varphi^W(x, y_1, \dots, y_n) \rightarrow (\exists z \in W)(z \in x \ \& \ \varphi^W(z, y_1, \dots, y_n) \ \& \ (\forall w \in W)(w \in z \rightarrow \neg \varphi^W(w, y_1, \dots, y_n)))$. By Lemma IA.1 iii, $(\forall x, y_1, \dots, y_n)(WFT(x) \rightarrow \varphi(x, y_1, \dots, y_n) \rightarrow (\exists z)(z \in x \ \& \ \varphi(z, y_1, \dots, y_n) \ \& \ (\forall w)(w \in z \rightarrow \neg \varphi(w, y_1, \dots, y_n))))$, as required.

LEMMA IA.4. Let $\varphi(x, y_1, \dots, y_n)$ be a formula with at most the free variables shown. The following is provable in $T_2(W)$.
 $(WFT(x) \ \& \ \varphi(x, y_1, \dots, y_n)) \rightarrow (\exists z \in x)(\varphi(z, y_1, \dots, y_n) \ \& \ (\forall w \in z)(\neg \varphi(w, y_1, \dots, y_n)))$.

Proof: Here we allow W to be mentioned in the property. But this is inconsequential since by Lemma IA.1 i, $\{x: W(x)\}$ exists, and therefore can be used as a value for one of the free variables y_1, \dots, y_n .

LEMMA IA.5. The following is provable in $T_2(W)$. Let $WTF(a)$. Then $b \in c \in b \in a$ is impossible.

Proof: Suppose $WFT(a)$ and $b \in c \in b \in a$. Apply Lemma IA.4 to the formula $x = a$ or $x = b$ or $x = c$. This formula holds of a and also an element of a (namely b). Hence there is an \in -minimal element d of a for which it holds. This is a contradiction.

LEMMA IA.6. The following is provable in $T_2(W)$. $(WFT(b) \ \& \ c \in b \ \& \ c \text{ is transitive}) \rightarrow WFT(c)$.

Proof: Let $WFT(b)$ and $c \in b$. Let d be given, and assume that there is an element of c in d . Apply Lemma IA.4 to the formula $x = b$ or $(x \in c \ \& \ x \in d)$. Clearly this formula holds for $x = b$. It also holds for some element of b . Hence there is an \in -minimal element y of b for which it holds. Clearly $y \neq b$. Hence y is an \in -minimal element of c lying in d .

We define $Ord(x)$ if and only if $WTF(x)$ and x is \in -connected. We say that x is an ordinal if and only if $Ord(x)$.

LEMMA IA.7. The following is provable in $T_2(W)$. Every element of an ordinal is an ordinal. The ordinals are strictly linearly ordered under \in . Transfinite induction can be

applied to the ordinals with respect to any formula with any parameters. Every transitive set of ordinals is an ordinal.

Proof: Let $x \in y$ where y is an ordinal. Obviously x is \in -connected. To see that x is transitive, let $a \in b \in x$. Then $a, b \in y$, and so $a \in x$ or $x \in a$ or $a = x$. By Lemma IA.5, $a \in x$. And by Lemma IA.6, $\text{WFT}(x)$. So x is an ordinal.

Assume the second claim is false. By Lemma IA.4, let x be an \in -minimal ordinal which is not \in -comparable with every ordinal. Thus every element of x is \in -comparable with every ordinal. Let y be an ordinal. Then every element of x is \in -comparable with y . Hence either every element of x is in y , or $y \in x$. So we can assume that $x \subseteq y$. If $x = y$ then we are done. Otherwise, by Lemma IA.4, let z be an \in -minimal element of y that is not in x . Then $z \subseteq x$. It suffices to prove that $x \subseteq z$. Let $b \in x$. Then b, z are comparable. So $b \in z$ or $z \in b$ or $b = z$. If $z \in b$ then $z \in x$, which is a contradiction. Hence $b \in z$ as required.

By Lemma IA.4, transfinite induction holds on the ordinals.

Let x be a transitive set of ordinals. Then x is \in -connected. By transfinite induction on the ordinals, we see that $\text{WFT}(x)$.

LEMMA IA.8. The following is provable in $T_2(W)$. There is a least ordinal not in W .

Proof: It suffices to prove that some ordinal is not in W . Suppose all ordinals lie in W . Since $\text{Ord}(x)$ is equivalent to $\text{Ord}(x) \cap W$ for elements of W , we see that the set S of all ordinals exists. S is obviously a transitive set of ordinals. By Lemma IA.7, S is an ordinal. Hence $S \in S$, which contradicts Lemma IA.7.

We let OW be the least ordinal not in W .

LEMMA IA.9. The following are provable in $T_2(W)$. $\text{OW} \subseteq W$. OW is a limit ordinal; i.e., it is nonempty and has no greatest element.

Proof: The first claim is by the definition of OW . For the second claim, let $x \in OW$. Then $x \in W$. Now the assertion " x is an element of an ordinal" is true. Hence " x is an element of an ordinal" W is also true by Lemma IA.1. Hence let $x \in y \in W$ and " y is an ordinal" W . By Lemma 1, $x \in y \in W$ and y is an ordinal.

We still have to prove that OW is nonempty. Note that the empty set exists by the third axiom. It is obviously an ordinal. By applying Lemma IA.1 to "the empty set exists" we obtain "the empty set exists" W , and so the empty set lies in W .

LEMMA IA.10. The following is provable in $T2(W)$. Let x be an ordinal. Then either x is empty, x is a limit ordinal, or there is a unique ordinal y such that $x = \{z: z \in y\}$. In the third case, x is called a successor ordinal, and y is called the predecessor of x .

Proof: Suppose x is an ordinal that is not empty and not a limit ordinal. Then x has a largest element, y . Let $z \in y$. Then obviously $z \in x$. On the other hand, let $z \in x$. Then z is comparable with y . So either $z = y$ or $z \in y$. The case $y \in z$ is impossible since y is the greatest element of x .

Also, if $x = \{z: z \in y'\}$, then $y \in y'$ and $y' \in y$. By Lemma IA.7, $y = y'$.

A finite ordinal is an ordinal x which is not a limit ordinal, and where no element of x is a limit ordinal. The finite ordinals obviously form an \in -connected transitive "class."

LEMMA IA.11. The following is provable in $T2(W)$. Let x be an ordinal and $y \in x$. Then $\{z: z \in y\}$ exists and is either x or a member of x . This is called the successor of y .

Proof: Let u be the least ordinal greater than y . This exists since $y \in x$. Now $y \in u$. Since u is transitive, $y \subseteq u$. Now suppose $z \in u$. Then z is not greater than y , and hence $z \in y$. We have thus shown that $u = \{z: z \in y\}$.

LEMMA IA.12. The following is provable in $T_2(W)$. Every finite ordinal lies in W . For every finite ordinal x , $\{y: y \in x\}$ exists and is a finite ordinal. The set of all finite ordinals exists and is the first limit ordinal. It is a member and a subset of W .

Proof: Suppose not every finite ordinal lies in W . Let x be the least finite ordinal that does not lie in W . By Lemma IA.9, x is not empty. Hence x is a successor ordinal, and has a unique predecessor y . So $y \in W$. Now " $\{z: z \in y\}$ exists" is a true statement. Hence " $\{z: z \in y\}$ exists" W is a true statement. Let u in W be such that $(\forall z)(z \in u \leftrightarrow z \in y)W$ holds. Then $(\forall z \in W)(z \in u \leftrightarrow z \in y)$. But by the choice of x , $(\forall z)(z \in y \rightarrow z \in W)$. Hence $(\forall z)(z \in u \leftrightarrow z \in y)$. Therefore $u = x$. Hence $x \in W$, which is the required contradiction.

Let x be a finite ordinal. By Lemma IA.9, x is not the largest ordinal in W . Hence by Lemma IA.11, $\{y: y \in x\}$ exists. It is clearly transitive, \in -connected, and well founded. It cannot be a limit ordinal since y is its greatest element. Furthermore, no elements can be limit ordinals since its elements are finite ordinals.

The set S of all finite ordinals exists since they are all in W . S is a transitive set of ordinals, and hence is itself an ordinal. By the previous paragraph, S has no greatest member, and so is a limit ordinal.

By the fourth claim of Lemma IA.1, $S \in W$. Clearly $S \subseteq W$ since all finite ordinals lie in W .

We let ω be the set of all finite ordinals. For $x \in \omega$ we write $x^+ = \{y: y \in x\}$.

The following two lemmas are painful enough that we are content to establish them in very special forms. This is all we need.

LEMMA IA.13. The following is provable in $T_2(W)$. For all $x, y, z, w \in \omega$, $\{x, y, z, w\}$ exists.

Proof: Suppose this is false. By four successive least element arguments, we obtain lexicographical least $x, y, z, w \in \omega$ such that $\{x, y, z, w\}$ does not exist. We have defined x, y, z, w without using W and without using any parameters. Since $x, y, z, w \in W$, we see that $\{x, y, z, w\}$ exists. This is the desired contradiction.

LEMMA IA.14. The following is provable in $T2(W)$. For all distinct $x, y, z, w \in \omega$, $\{x, y, z, w\} \in W$.

Proof: By Lemma IA.13, $(\forall x, y, z, w \in \omega) (\exists u) (u = \{x, y, z, w\})$ holds. Hence by Lemma 1, $(\forall x, y, z, w \in \omega) (\exists u) (u = \{x, y, z, w\})^W$ holds.

Let $x, y, z, w \in \omega$ be distinct. Then $x, y, z, w \in W$. Let $u \in W$ be such that $(u = \{x, y, z, w\})^W$ holds. We can write $u = \{x, y, z, w\}$ as

$$x \in u \ \& \ y \in u \ \& \ z \in u \ \& \ w \in u \ \& \ u \subseteq \{x, y, z, w\},$$

where we don't care about the details of how $u \subseteq \{x, y, z, w\}$ is formalized.

Hence

$$x \in u \ \& \ y \in u \ \& \ z \in u \ \& \ w \in u \ \& \ (u \subseteq \{x, y, z, w\})^W.$$

Since x, y, z, w are distinct finite ordinals, we have $(x \neq y)^W$ & $(y \neq z)^W$ & $(z \neq w)^W$. Hence

$$(\exists x, y, z, w \in W) (x \in u \ \& \ y \in u \ \& \ z \in u \ \& \ w \in u \ \& \ (x \neq y)^W \ \& \ (y \neq z)^W \ \& \ (z \neq w)^W \ \& \ (u \subseteq \{x, y, z, w\})^W).$$

By Lemma IA.1,

$$(\exists x, y, z, w) (x \in u \ \& \ y \in u \ \& \ z \in u \ \& \ w \in u \ \& \ x \neq y \ \& \ y \neq z \ \& \ z \neq w \ \& \ (u \subseteq \{x, y, z, w\})).$$

Hence z is a quadruple. But since $x \in u \ \& \ y \in u \ \& \ z \in u \ \& \ w \in u$, we have $u = \{x, y, z, w\}$.

Let $x, y, z \in \omega$. We now define the ordered triple $\langle x, y, z \rangle$ in the following special way. Enumerate the small number of possible order types of triples from ω .

For distinct x, y, z , define $\langle x, y, z \rangle = \{i, x+i+100, y+i+100, z+i+100\}$, where i is the index of the order type of x, y, z . If x, y, z are not distinct, then make the same definition, except make sure that the set has exactly four distinct terms by including $z+i+101$, etcetera.

LEMMA IA.15. The following is provable in $T2(W)$. For all $x, y, z \in \omega$, $\langle x, y, z \rangle \in W$. For all x, y, z, a, b, c , $\langle x, y, z \rangle = \langle a, b, c \rangle \rightarrow (x = a \ \& \ y = b \ \& \ z = c)$.

Proof: To decode a triple, first look for the least element, i . This tells you the order type of the intended triple. In particular, it tells you how many of the next three terms to look at, and how to put them together in the right order.

LEMMA IA.16. Let $\varphi(x)$ be any formula. The following is provable in $T2(W)$. $(\varphi(0) \ \& \ (\forall x \in \omega) (\varphi(x) \rightarrow \varphi(x+))) \rightarrow (\forall x \in \omega) (\varphi(x))$.

Proof: Suppose φ fails at some $x \in \omega$. Let x be the least such. Since x is a nonzero finite ordinal, write $x = y+$, where y is a finite ordinal. Then $\varphi(y)$. Hence $\varphi(x)$, which is the desired contradiction.

LEMMA IA.17. The following is provable in $T2(W)$. There are functions $S, +, \times$ (successor, addition, and multiplication) on ω which obey the usual quantifier free axioms of Peano Arithmetic, and which are given by formulas without parameters and without W .

Proof: We handle $+$ only. We construct $+$ as a set of ordered triples from ω , defined without parameters and without W . Using the induction principle of Lemma IA.16, we see that for each $x \in \omega$, there can be at most one set of triples that defines $+$ according to its recursion equations on exactly the pairs of arguments from x . Furthermore, the different functions (sets of triples) corresponding to the various $x \in \omega$ all agree on their common domains. We caution that it is

not automatic, however, that if the appropriate set of triples exists for $x \in \omega$, then it exists for all $y \in x$.

Now if these appropriate sets of triples exist for every $x \in \omega$, then we can put them together into a single set of triples that defines the $+$ we are looking for on all of ω . This is because we are taking the set of all triples from ω obeying a condition without parameters and without W , and that the triples from ω all lie in W .

So it remains to show that these sets of triples exist for every $x \in \omega$. Suppose this is false, and let $x \in \omega$ be the least counterexample. Then x is not zero, and we can let $x = y+$. Let S be the set of triples corresponding to y . Since S is unique, and since x is defined without parameters and without W , we see that S is defined without parameters and without W , as well as y . We now have to extend the set of triples to correspond to $x = y+$. This amounts to extending the set of triples so as to work with pairs of arguments, one of which is y and the other of which is $\in = y$. But this is no problem since we can refer to the previous set of triples corresponding to y at arguments involving the predecessor of y ; if y is zero then the use of y in the triples is trivial.

This extension can be performed without parameters and without W , so that the set of triples exists. This is the desired contradiction.

LEMMA IA.18. The following is provable in EFA (exponential function arithmetic). There is an interpretation of the system pZ2 of parameterless second order arithmetic into $T2(W)$.

Proof: We can interpret the natural numbers of pZ2 as elements of ω , and the $S, +, x$ of pZ2 as the $S, +, x$ on ω as given by Lemma IA.17. We can interpret the subsets of ω in pZ2 as the subsets of ω in $T2(W)$. The interpretation of the induction axiom scheme in pZ2 is obtained from Lemma IA.7. The interpretation of the comprehension axiom scheme in pZ2 is obtained from the third axiom of $T2(W)$.

Here pZ2 = parameterless second order arithmetic is as defined and discussed in my paper, On the necessary use of

abstract set theory, *Advances in Mathematics*, vol. 41, 1981, 209-280.

In that paper, we showed that $pZ2$ is equiconsistent with $Z2 =$ ordinary second order arithmetic, which differs only in that the comprehension scheme is stated with all parameters.

(Strictly speaking, the development there is for ω -models only, but it works for general models and equiconsistency).

From the proof in that paper, it follows that there is an interpretation of $Z2$ in $pZ2$. And it is well known that there is an interpretation of $ZFC-P$ (ZFC without the powers set axiom) in $Z2$.

Let CST (countable set theory) be $ZFC-P +$ "all sets are countable." It is also well known that CST is interpretable into $ZFC-P$.

THEOREM IA.19. The following is provable in EFA (exponential function arithmetic).

- i) CST is interpretable into $T2(W)$;
- ii) $T2(W)$ is consistent if and only if CST is consistent;
- iii) every theorem of $T2(W)$ without W is a theorem of ZF .

Proof: This is immediate from the preceding lemmas and discussion. For iii), let φ be a sentence without W that is provable in $T2(W)$, and let it be provable in S , which is a finite fragment of $T2(W)$. We can interpret S in ZF by setting W to be a transitive set which is an elementary substructure of the universe with respect to the formulas that appear in S . This construction is standard within ZF , but cannot be done if the power set is dropped.

We can identify a well known subsystem of ZF that is equiconsistent with $ZFC-P$ which we can use in iii) above instead of ZF . We drop power set and replacement and use the reflection scheme

$$(\exists x)(x \text{ is nonempty and transitive} \ \& \ (\forall y_1, \dots, y_k \in x)(\varphi^x \leftrightarrow \varphi)),$$

where φ is any formula whose free variables are among y_1, \dots, y_k . In the presence of the other axioms of $ZF-P$, replacement follows immediately from reflection.

THEOREM IA.20. Every theorem of $T_2(W)$ that does not mention W is a theorem of ZF-P with the reflection scheme.

B. TWO SUBWORLDS: INDESCRIBABLE AND SUBTLE CARDINALS

In this section IB, we treat the theory $T_2(W_1, W_2)$ in the language $L(\in, W_1, W_2)$, as presented in the Introduction:

1. Extensionality. $(\forall x_1)(x_1 \in x_2 \leftrightarrow x_1 \in x_3) \rightarrow (\forall x_1)(x_2 \in x_1 \leftrightarrow x_3 \in x_1)$.
2. Comprehension. $(\exists x_1)(W_2(x_1) \ \& \ (\forall x_2)(x_2 \in x_1 \leftrightarrow (W_1(x_2) \ \& \ \varphi)))$, where φ is a formula in $L(\in, W_1)$ with at most the free variable x_2 .
3. Resemblance. $W(x_1) \rightarrow (\varphi \leftrightarrow \varphi[W_1/W_2])$, where φ is a formula in $L(\in, W_1)$ with at most the free variable x_1 .

1. Transitivity

The first order of business is to interpret the more convenient system $T_3(W_1, W_2)$ in $T_2(W_1, W_2)$. The axioms of $T_3(W_1, W_2)$ extend those of $T_2(W_1, W_2)$ by the transitivity axioms

4. $(W_1(x_1) \ \& \ x_2 \in x_1) \rightarrow W_1(x_2)$;
5. $(W_2(x_1) \ \& \ x_2 \in x_1) \rightarrow W_2(x_2)$.

We define $x = y$ if and only if x, y have the same elements. We let $x \in W_1$ and $x \in W_2$ abbreviate $W_1(x)$ and $W_2(x)$ respectively. In the next lemma, viii) guarantees that we can treat $=$ just like ordinary equality.

LEMMA IB1.1. The following are provable in $T_2(W_1, W_2)$.

- i) $(\exists x \in W_2)(\forall y)(y \in x \leftrightarrow y \in W_1)$;
- ii) $(\exists x)(\forall y)(y \in x \leftrightarrow y \in W_2)$;
- iii) $(\forall x \in W_1)(\varphi \leftrightarrow \varphi[W_1/W_2])$, where φ has at most the free variable x , and does not mention W_2 ;
- iv) $\varphi \leftrightarrow \varphi[W_1/W_2]$, where φ be a sentence not mentioning W_2 ;
- v) $(\exists y)(\forall x)(x \in y \leftrightarrow (W_2(x) \ \& \ \varphi))$, where φ has at most the free variable x , and does not mention W_1 ;

- vi) $(\exists x \in W2) (\varphi) \rightarrow (\exists x \in W1) (\varphi)$, where φ has at most the free variable x , and does not mention $W1$;
- vii) $x \in W1 \rightarrow x \in W2$;
- viii) $x = y \rightarrow (\varphi \leftrightarrow \varphi[x/y])$ where x, y are not free in φ .

Proof: The first claim is from the third axiom by setting φ to be $x = x$. The third claim is obvious from the third axiom. The fourth claim is a special case of the third claim. The second claim follows from the first claim and the fourth claim. The fifth claim follows from the fourth claim and the third axiom.

For the sixth claim, suppose $(\exists x \in W2) (\varphi)$. By the third claim, $(\exists x \in W1) (\varphi[W2/W1])$. By the third claim, $(\exists x \in W1) (\varphi[W2/W1][W1/W2])$, and hence $(\exists x \in W1) (\varphi)$. For the seventh claim, let $x \in W1$. By the third claim, $x \in W1 \leftrightarrow x \in W2$, and so $x \in W2$. For the eighth claim, note that this is standard for φ not mentioning $W1, W2$ (by induction on φ). But we have to handle the basis cases $y = z \rightarrow (y \in W1 \leftrightarrow z \in W1)$, and $y = z \rightarrow (y \in W2 \leftrightarrow z \in W2)$. First let x be as given in the first claim. Then $y = z \rightarrow (y \in x \leftrightarrow z \in x)$, and so $y = z \rightarrow (y \in W1 \leftrightarrow z \in W1)$. Finally let x be as given in the second claim. Then $y = z \rightarrow (y \in x \leftrightarrow z \in x)$, and so $y = z \rightarrow (y \in W2 \leftrightarrow z \in W2)$. \square

We say that x is transitive if and only if $(\forall y, z) (y \in z \in x \rightarrow y \in x)$.

We are now ready to interpret the $W1$ and $W2$ of $T3(W1, W2)$. We define $W1'(x)$ if and only if $x \in W1$ & $(\exists y) (y \text{ is transitive} \ \& \ x \subseteq y \subseteq W1)$. Similarly, we define $W2'(x)$ if and only if $x \in W2$ & $(\exists y) (y \text{ is transitive} \ \& \ x \subseteq y \subseteq W2)$.

Let φ be a formula in $L(\in, W1, W2)$. Then the $'$ interpretation of φ , which we write as φ' , is the result of replacing $W1$ by $W1'$ and $W2$ by $W2'$, and expanding the resulting formula out with the definitions of $W1'$ and $W2'$.

LEMMA IB1.2. The following is provable in $T2(W1, W2)$.

$$(\exists x) (W1(x))'.$$

Proof: This is $(\exists x)(W1'(x))$. By axiom 3 the empty set $\emptyset \in W2$. By Lemma IB1.1 vi, $\emptyset \in W1$. Hence $W1'(\emptyset)$.

LEMMA IB1.3. The following is provable in $T2(W1, W2)$. $[(W1(x) \ \& \ y \in x) \rightarrow W1(y)]'$. $[(W2(x) \ \& \ y \in x) \rightarrow W2(y)]'$.

Proof: The second claim follows from the first claim by Lemma IB1.1 iv. The first claim is $(W1'(x) \ \& \ y \in x) \rightarrow W1'(y)$. Let $x \in W1$, $x \subseteq z \subseteq W1$, and z is transitive. Let $y \in x$. Then $y \in z$, and so $y \in W1$. Also every element of y lies in z by the transitivity of z . Hence $y \subseteq z \subseteq W1$. \square

LEMMA IB1.4. Let φ be a formula in $L(\in, W1, W2)$ with at most the free variable y which does not mention $W2$. The following is provable in $T2(W1, W2)$. $(\exists x)(W2(x) \ \& \ (\forall y)(y \in x \leftrightarrow (W1(y) \ \& \ \varphi) \leftrightarrow (W1(y) \ \& \ \varphi[W1/W2])))'$.

Proof: This is $(\exists x)(W2'(x) \ \& \ (\forall y)(y \in x \leftrightarrow (W1'(y) \ \& \ \varphi') \leftrightarrow (W1'(y) \ \& \ \varphi'[W1/W2])))$. The inside equivalence follows from axiom 3. It now suffices to prove that $\{y: W1'(y) \ \& \ \varphi'\}$ exists and has property $W2'$. It exists and lies in $W2$ by axiom 3. Now $\{y: W2'(y)\}$ exists by Lemma IB1.1 v, and is transitive by Lemma IB1.3, and is obviously a subset of $W2$. Hence $\{y: W1'(y) \ \& \ \varphi'\}$ has property $W2'$. \square

LEMMA IB1.5. Let φ be a theorem of $T3(W1, W2)$. Then φ' is a theorem of $T2(W1, W2)$.

Proof: We have shown this for all axioms φ of $T3(W1, W2)$. The result follows. \square

In light of Lemma IB1.5, we work entirely within the system $T3(W1, W2)$. We continue to use the abbreviations $x = y$, $x \in W1$, and $x \in W2$. We now know that $W1, W2$ are transitive sets, and $W1 \in W2$.

2. Ordinals and transfinite induction

We write $WFT(x)$ if and only if

- i) x is transitive;
- iii) for all y , if there is an element of x in y , then there is an \in -minimal element of x in y .

Here WFT abbreviates "well founded and transitive."

We write $x \in= y$ for " $x \in y$ or $x = y$."

LEMMA IB2.1. Let $\varphi(x, y_1, \dots, y_n)$ be a formula with at most the free variables shown, and without W_1, W_2 . The following is provable in $T_3(W_1, W_2)$. $(WFT(x) \ \& \ x \in W_1 \ \& \ \varphi(x, y_1, \dots, y_n)) \rightarrow (\exists z \in W_1) (z \in= x \ \& \ \varphi(z, y_1, \dots, y_n) \ \& \ (\forall w \in W_1) (w \in z \rightarrow \neg \varphi(w, y_1, \dots, y_n)))$.

Proof: Let $S = \{x \in W_1 : (\exists y_1 \dots y_n) (\varphi(y_1, \dots, y_n) \ \& \ \neg (\exists z \in W_1) (z \in= x \ \& \ \varphi(z, y_1, \dots, y_n) \ \& \ (\forall w \in W_1) (w \in z \rightarrow \neg \varphi(w, y_1, \dots, y_n))))\}$. S exists by the third axiom.

It suffices to prove that S has no element x with $WFT(x)$. We suppose that $x \in S$, $WFT(x)$. Let x' be an \in -minimal element of x in S .

Let $\varphi(x', y_1, \dots, y_n)$, $\neg (\exists z \in W_1) (z \in= x' \ \& \ \varphi(z, y_1, \dots, y_n) \ \& \ (\forall w \in W_1) (w \in z \rightarrow \neg \varphi(w, y_1, \dots, y_n)))$. Then setting $z = x'$, we see that $\neg (\forall w \in W) (w \in x' \rightarrow \neg \varphi(w, y_1, \dots, y_n))$. Hence $(\exists w \in W_1) (w \in x' \ \& \ \varphi(w, y_1, \dots, y_n))$. Fix $u \in x'$ with $u \in W_1$, $\varphi(u, y_1, \dots, y_n)$. Note that $u \in x$ by the transitivity of x .

We claim that $u \in S$. To see this, suppose that $(\exists z \in W_1) (z \in= u \ \& \ \varphi(z, y_1, \dots, y_n) \ \& \ (\forall w \in W_1) (w \in z \rightarrow \neg \varphi(w, y_1, \dots, y_n)))$. Then $(\exists z \in W_1) (z \in= x \ \& \ \varphi(z, y_1, \dots, y_n) \ \& \ (\forall w \in W_1) (w \in z \rightarrow \neg \varphi(w, y_1, \dots, y_n)))$, which contradicts $x \in S$. Hence $\neg (\exists z \in W_1) (z \in= u \ \& \ \varphi(z, y_1, \dots, y_n) \ \& \ (\forall w \in W) (w \in z \rightarrow \neg \varphi(w, y_1, \dots, y_n)))$. Therefore $u \in S$.

We now have the desired contradiction since x' is an \in -minimal element of x in S and $u \in x', S$. \square

LEMMA IB2.2. Let $\varphi(x, y_1, \dots, y_n)$ be a formula with at most the free variables shown, and without W_1, W_2 . The following is provable in $T_3(W_1, W_2)$. $(WFT(x) \ \& \ x \in W_2 \ \& \ \varphi(x, y_1, \dots, y_n)) \rightarrow$

$(\exists z \in W_2) (z \in x \ \& \ \varphi(z, y_1, \dots, y_n) \ \& \ (\forall w \in W_2) (w \in z \rightarrow \neg \varphi(w, y_1, \dots, y_n)))$.

Proof: By Lemmas IB2.1 and IB1.1 iv. \square

LEMMA IB2.3. Let $\varphi(x, y_1, \dots, y_n)$ be a formula with at most the free variables shown. The following is provable in $T_3(W_1, W_2)$.
 $(WFT(x) \ \& \ x \in W_2 \ \& \ \varphi(x, y_1, \dots, y_n)) \rightarrow (\exists z \in W_2) (z \in x \ \& \ \varphi(z, y_1, \dots, y_n) \ \& \ (\forall w \in W_2) (w \in z \rightarrow \neg \varphi(w, y_1, \dots, y_n)))$.

Proof: By Lemmas IB2.2 and IB1.1 i, ii. W_1, W_2 can then be replaced by parameters. \square

LEMMA IB2.4. The following is provable in $T_3(W_1, W_2)$. Let $WTF(a)$, $a \in W_2$. Then $b \in c \in b \in a$ is impossible.

Proof: Suppose $WTF(a)$ and $b \in c \in b \in a$. Apply Lemma IB2.3 to the formula $x = a$ or $x = b$ or $x = c$. This formula holds of a and also an element of a (namely b). Hence there is an \in -minimal element d lying in W_2 of a for which it holds. This is a contradiction. \square

LEMMA IB2.5. The following is provable in $T_3(W_1, W_2)$. $(WFT(b) \ \& \ c \in b \ \& \ c$ is transitive $\ \& \ b \in W_2) \rightarrow WFT(c)$.

Proof: Let $WFT(b)$, $c \in b$, c is transitive, and $b \in W_2$. Let d be given, and assume that there is an element of c in d . Apply Lemma IB2.3 to the formula $x = b$ or $(x \in c \ \& \ x \in d)$. Clearly this formula holds for $x = b$. It also holds for some element of b lying in W_2 (namely c). Hence there is an \in -minimal element y of b for which it holds (every element of c lies in W_2). Clearly $y \neq b$. Hence y is an \in -minimal element of c lying in d . Hence $WFT(c)$. \square

We define $Ord(x)$ if and only if $WTF(x)$ and x is \in -connected. We say that x is an ordinal if and only if $Ord(x)$.

LEMMA IB2.6. The following are provable in $T_3(W_1, W_2)$. Every element of an ordinal $\in W_2$ is an ordinal $\in W_2$. The ordinals $\in W_2$ are strictly linearly ordered under \in . Transfinite induction can be applied to the ordinals $\in W_2$ with respect to

any formula with any parameters. If $x \subseteq W_2$ is a transitive set of ordinals, then x is an ordinal and $x \in= OW_2$.

Proof: Let $x \in y$ where y is an ordinal $\in W_2$. Obviously x is \in -connected and in W_2 . To see that x is transitive, let $a \in b \in x$. Then $a, b, y \in W_2$. Also $a, b \in y$, and so $a \in x$ or $x \in a$ or $a = x$. By Lemma IB2.4, $a \in x$. Since x is transitive, by Lemma IB2.5, $WFT(x)$. So x is an ordinal $\in W_2$.

For the second claim we have only to prove that for any two ordinals $\in W_2$, the first is in the second, the second is in the first, or they are equal. Assume this is false. By Lemma IB2.3, let x be an \in -minimal element of $\{x: x \text{ is an ordinal } \in W_2 \text{ which is not } \in\text{-comparable with every ordinal } \in W_2\}$. Thus every element of x is \in -comparable with every ordinal $\in W_2$. Let y be an ordinal $\in W_2$. Then every element of x is \in -comparable with y . Hence either every element of x is in y , or $y \in x$. So we can assume that $x \subseteq y$. If $x = y$ then we are done. Otherwise, by Lemma IB2.3, let z be an \in -minimal element of y that is not in x . Then $z \subseteq x$. It suffices to prove that $x \subseteq z$. Let $b \in x$. Then b, z are comparable. So $b \in z$ or $z \in b$ or $b = z$. If $z \in= b$ then $z \in x$, which is a contradiction. Hence $b \in z$ as required.

By Lemma IB2.3, transfinite induction holds on the ordinals $\in W_2$, thus establishing the third claim.

For the fourth claim, let x be as given. Then x is \in -connected. By transfinite induction on the ordinals, we see that $WFT(x)$. Hence x is an ordinal. \square

LEMMA IB2.7. Let $\varphi(x_1, \dots, x_n, y)$ be a formula in $L(\in, W_1, W_2)$ without W_1 with at most the free variables shown. The following is provable in $T_3(W_1, W_2)$. $(x_1, \dots, x_n \in OW_2) \rightarrow (\exists z)(\forall y)(y \in z \leftrightarrow (y \in W_2 \ \& \ \varphi))$.

Proof: Let φ be as given and suppose the first claim is false. By Lemma IB2.6, we can successively choose least $x_1, \dots, x_n \in OW_2$ such that the consequent is false. I.e., choose least $x_1 \in W_2$ such that for some $x_2, \dots, x_n \in W_2$, the antecedent

is false; then choose least x_2 in W_2 such that for some $x_3, \dots, x_n \in W_2$, the antecedent is false; etcetera. This is n applications of Lemma IB2.6. Then x_1, \dots, x_n are defined without W_1 . Hence the antecedent is true after all by Lemma IB1.1 v. \square

We now let OW_1 be the set of all ordinals $\in W_1$ and OW_2 be the set of all ordinals $\in W_2$.

LEMMA IB2.8. The following is provable in $T_3(W_1, W_2)$. OW_1, OW_2 are ordinals. $OW_1 \in W_2$. $OW_1 \notin W_1$. $OW_2 \notin W_2$. $OW_1 \in OW_2$.

Proof: By Lemma IB2.6, OW_1 is a transitive sets of ordinals, each of which is $\in W_1$. Hence by Lemma IB2.6, OW_1 is an ordinal. Similarly, OW_2 is an ordinal. $OW_1 \in W_2$ by axiom 3. If $OW_1 \in W_1$ then $OW_1 \in OW_1$, contradicting Lemma IB2.4. Similarly, $OW_2 \notin W_2$. By axiom 3, $OW_1 \in OW_2$ since $OW_1 \subseteq W_1$ and is defined without W_2 . The forward direction of the last claim follows immediately from the axiom 3, and the backward direction is trivial. \square

LEMMA IB2.9. The following are provable in $T_3(W_1, W_2)$. OW_1 and OW_2 are limit ordinals; i.e., they are nonempty and have no greatest element.

Proof: By axiom 3, \emptyset exists, $\emptyset \in W_2$, and \emptyset is an ordinal. By Lemma IB1.1 iv, $\emptyset \in W_1$. Hence $\emptyset \in OW_1$. Now let $x \in OW_1$. Then "x is an element of an element of OW_2 ." By Lemma 2.1, since $x \in W_1$, we see that "x is an element of an element of OW_1 ." Therefore OW_1 is a limit ordinal. Hence by Lemma 2.1, OW_2 is a limit ordinal. \square

LEMMA IB2.10. The following is provable in $T_3(W_1, W_2)$. Let x be an ordinal in OW_2 . Then either x is empty, x is a limit ordinal, or there is a unique ordinal y such that $x = \{z: z \in y\}$. In the third case, x is called a successor ordinal, and y is called the predecessor of x .

Proof: Suppose $x \in OW_2$ is not empty and not a limit ordinal. Then x has a largest element, y . Let $z \in y$. Then obviously $z \in x$. On the other hand, let $z \in x$. Then z is comparable with y . So either $z = y$ or $z \in y$. The case $y \in z$ is impossible since y is the greatest element of x .

Also, if $x = \{z: z \in y'\}$, then $y \in y'$ and $y' \in y$. By Lemma IB2.6, $y = y'$. \square

LEMMA IB2.11. The following is provable in $T3(W1, W2)$. Let $x \in OW2$. Then $\{y: y \in x\}$ exists, and lies in $OW2$. This is called the successor of y , written y^+ .

Proof: By Lemma IB2.6, let u be the least element of $OW2$ greater than x . Then every element of u is $\in x$. Let $y \in x$. Then by Lemma 2.7, y and u are comparable. Hence $y \in u$ or $u \in y$ or $u = y$. Since $x \in u$ we see by Lemma IB2.5 that $y \in u$. \square

3. Finite ordinals, strong ordinals, and arithmetic

A finite ordinal is an ordinal x which is not a limit ordinal, and where no element of x is a limit ordinal.

LEMMA IB3.1. The following is provable in $T3(W1, W2)$. The set of all finite ordinals in $OW1$ is a limit ordinal. Every finite ordinal in $OW2$ lies in $OW1$. For every finite ordinal $x \in OW1$, $\{y: y \in x\}$ exists and is a finite ordinal. The set of all finite ordinals in $OW1$ exists, and is the same as the set of all finite ordinals in $OW2$, and is the first limit ordinal in $OW1$.

Proof: By axiom 3, let A be the set of all finite ordinals in $OW1$. Then A is transitive, and hence by Lemma IB2.6, A is an ordinal. Obviously $A \in W2$. Hence $A \in OW2$. Now suppose A has a largest element, x . Then by Lemma IB2.11, x^+ exists and $x^+ \in OW2$. Also x^+ is obviously a finite ordinal. Hence $x^+ \notin OW1$. Therefore x is the largest element of $OW1$, which contradicts Lemma IB1.9. Therefore A is a limit ordinal.

Let x be a finite ordinal in $OW2$. Then x is \in -comparable with A . $A \in x$ is impossible by the definition of finite ordinal. Hence $x \in A$.

Let $x \in A$. By Lemma IB2.11, x^+ exists and lies in $OW2$. Since x is a finite ordinal, clearly x^+ is a finite ordinal.

We have shown that A is the first limit ordinal in $OW1$, and is the set of all finite ordinals in $OW2$. Suppose there is an element of A that is a limit ordinal. Then there is an element of A that is higher than a limit ordinal. But this contradicts that every element of A is a finite ordinal. \square

We let ω be the set of all finite ordinals in $OW1$ (or $OW2$). It is the first limit ordinal in $OW1$.

We say that x is a strong ordinal if and only if x is an ordinal and for all b , if there is an element of x not in b then there is an \in -minimal element of x not in b .

LEMMA IB3.2. The following is provable in $T3(W1, W2)$. Let x be a strong ordinal. Then $x \in OW2$ or $OW2 \in x$ or $x = OW2$. $OW2$ and all of its elements are strong ordinals. ω is the set of all finite strong ordinals (i.e., strong ordinals that are finite ordinals).

Proof: First suppose that $x \subseteq OW2$. We can assume that $x \neq OW2$. Let y be the least element of $OW2$ not in x . Then $y \subseteq x$. Now y is \in -comparable with every element of x . Hence $x \subseteq y$ or $y \in x$. Hence $x \subseteq y$, and so $x = y$, in which case $x \in OW2$.

Now suppose that there is an element of x not in $OW2$. Let y be an \in -minimal element of x not in $OW2$. By the transitivity of x , $y \subseteq OW2$. To see that $OW2 \subseteq y$, let $b \in OW2$. Then $b \in x$, and so b, y are \in -comparable. Hence $b \in y$. So we have shown that $y = OW2$. Hence $OW2 \in x$.

The second claim follows from Lemma IB2.6.

For the third claim, it suffices to show that every finite strong ordinal x lies in $OW2$. But x is comparable with $OW2$. Since $OW1$ is a limit ordinal in x , we see that $x \in OW2$. \square

We are not claiming the \in -comparability of strong ordinals. Just their comparability with $OW2$.

We now define $\{x_1, \dots, x_n\} = \{y: y = x_1 \text{ or } \dots \text{ or } y = x_n\}$. This definition is meant to be made for each standard integer n . Of course, we cannot prove in $T2(W1, W2)$ that, e.g., for all

x , $\{x\}$ exists; or even for all $x \in W_1$, $\{x\}$ exists. So we must tread very carefully.

The $\{\}$ -terms are inductively defined in the metatheory as follows:

- i) each variable x_1, x_2, \dots is a $\{\}$ -term;
- ii) if t_1, \dots, t_m are $\{\}$ -terms, $n \geq 0$, then $\{t_1, \dots, t_m\}$ is a $\{\}$ -term.

We do not claim that $t(x_1, \dots, x_n)$ exists for all x_1, \dots, x_n . We have to be very careful about existence.

LEMMA IB3.3. Let $t(x_1, \dots, x_n)$ be a $\{\}$ -term whose variables are among those shown. The following are provable in $T_3(W_1, W_2)$.

$(\forall x_1, \dots, x_n \in OW_2) (t(x_1, \dots, x_n) \in W_2)$. $(\forall x_1, \dots, x_n \in OW_1) (t(x_1, \dots, x_n) \in W_1)$.

Proof: The second claim follows from the first claim by Lemma IB1.1 iv. We prove the first claim by induction on t . Assume $(\forall x_1, \dots, x_n \in OW_2) (t_i(x_1, \dots, x_n) \in W_2)$, for $1 \leq i \leq m$. We must show that $(\forall x_1, \dots, x_n \in OW_2) (\{t_1(x_1, \dots, x_n), \dots, t_m(x_1, \dots, x_n)\} \in W_2)$.

Suppose this is false. By n successive applications of transfinite induction, let x_1, \dots, x_n be lexicographically least elements of OW_2 such that $\neg(\{t_1(x_1, \dots, x_n), \dots, t_m(x_1, \dots, x_n)\} \in W_2)$. Then x_1, \dots, x_n are each definable by formulas not mentioning W_1 . By Lemma 2.1 vi we see that $x_1, \dots, x_n \in W_1$. Now by induction hypothesis, each $t_i(x_1, \dots, x_n) \in W_2$. Hence again by Lemma IB1.1 vi, each $t_i(x_1, \dots, x_n) \in W_1$. Hence by axiom 3, $\{t_1(x_1, \dots, x_n), \dots, t_m(x_1, \dots, x_n)\} \in W_2$, which is the required contradiction. \square

We now define $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$. Of course $\langle x, y \rangle$ may not exist. Define $\langle x \rangle = x$. For $n \geq 2$, inductively define $\langle x_1, \dots, x_n \rangle = \langle x_1, \langle x_2, \dots, x_n \rangle \rangle$. Again, we have to be careful about existence.

LEMMA IB3.4. Let $n \geq 1$. The following is provable in $T_3(W_1, W_2)$. $(\forall x_1, \dots, x_n \in OW_2) (\langle x_1, \dots, x_n \rangle \in W_2)$. $(\forall x_1, \dots, x_n \in OW_1) (\langle x_1, \dots, x_n \rangle \in W_1)$.

Proof: This is a special case of Lemma IB3.3. \square

For sets x and $n \geq 2$ we write x_n for $\{ \langle y_1, \dots, y_n \rangle : y_1, \dots, y_n \in x \}$. This may not necessarily exist. We take $x_1 = x$.

We also write $x_{\leq n}$ for the union of the x_m , $1 \leq m \leq n$. Again, this may not necessarily exist.

LEMMA IB3.5. Let $n \geq 2$. The following is provable in $T3(W1, W2)$. $(\forall x \in OW2) (x_n, x_{\leq n} \in W2)$. $(\forall x \in OW1) (x_n, x_{\leq n} \in W1)$. $OW2^n$ and $OW_{\leq n}$ exist.

Proof: Suppose that the first claim is false and let x be the least counterexample. By Lemma IB1.1 vi we see that $x \in OW1$. By Lemma IB3.3, we can define x_n as a subset of $W1$ without $W1$. Hence by axiom 3, $x_n \in W2$, which is the required contradiction. The second claim follows. And $OW2^n$ exists by Lemma IB3.3 and Lemma IB1.1 v. The $\leq n$ exponent versions are proved analogously. \square

We say that f is a function if and only if f is a univalent set of ordered pairs. We write $f: x \rightarrow y$ if f is a function whose domain is x and all of whose values lie in y . When we write $f: x \rightarrow y$ we are committed to the existence of x, y as objects.

We now want to give some variants of Lemma IB2.7.

LEMMA IB3.6. Let $\varphi(x_1, \dots, x_n, y)$ be a formula in $L(\in, W1, W2)$ without $W1$ with at most the free variables shown, and $m \geq 1$. The following is provable in $T3(W1, W2)$. $(x_1, \dots, x_n, z \in OW2) \rightarrow (\exists w \in W2) (\forall y) (y \in w \leftrightarrow (y \in z_{\leq m} \& \varphi))$.

Proof: As in the proof of Lemma IB2.7, successively minimize $x_1, \dots, x_n, z \in OW2$ so that the conclusion is false. Then $x_1, \dots, x_n, z \in W1$, and so $x_1, \dots, x_n, z \in OW1$. Hence $z_{\leq m} \in W1$, and so we can define $w = \{y \in z_{\leq m} : \varphi\}$ as a subset of $W1$ without mentioning $W1$. Hence $w \in W2$, which is the required contradiction. \square

LEMMA IB3.7. Let $\varphi(x_1, \dots, x_n, y)$ be a formula in $L(\in, W_1, W_2)$ without W_1 with at most the free variables shown. The following are provable in $T_3(W_1, W_2)$.

- i) $(x_1, \dots, x_n, z \in OW_1) \rightarrow (\exists w \in W_1) (\forall y) (y \in w \leftrightarrow (y \in z \leq_m \& \varphi[W_2/W_1]));$
- ii) $(x_1, \dots, x_n, y \in OW_1) \rightarrow (\varphi \leftrightarrow \varphi[W_2/W_1]);$
- iii) $(x_1, \dots, x_n, z \in OW_1) \rightarrow (\exists w \in W_1) (\forall y) (y \in w \leftrightarrow (y \in z \leq_m \& \varphi));$

Proof: The first claim is immediate from Lemma IB3.6 and Lemma IB1.1 iv. For the second claim, write $\psi(x) = (\exists x_1 \dots x_n z) (x = \{x_1, \dots, x_n, z\} \& \varphi)$. By Lemma IB1.1 iii, $x \in W_1 \rightarrow (\psi \leftrightarrow \psi[W_2/W_1])$. The claim follows from $x_1, \dots, x_n, z \in OW_1 \rightarrow \{x_1, \dots, x_n, z\} \in W_1$, by Lemma IB3.5. The third claim follows immediately from the first two claims. \square

We now develop some arithmetic of ω .

LEMMA IB3.8. The following is provable in $T_3(W_1, W_2)$. The function $S: \omega \rightarrow \omega$ given by $S(x) = x^+$ exists. S is definable without W_1, W_2 . $S \in W_1$.

Proof: $S = \{\langle x, y \rangle : x, y \in \omega \& y = x^+\}$ exists and lies in W_2 by axiom 3, because it is defined without W_1, W_2 as a subset of W_1 . By Lemma IB1.1 vi, $S \in W_1$. \square

LEMMA IB3.9. The following is provable in $T_3(W_1, W_2)$. There is a unique function $f: \omega^2 \rightarrow \omega$ such that $f(\langle x, 0 \rangle) = x$ and $f(\langle x, y^+ \rangle) = f(\langle x, y \rangle)^+$. This f is definable without W_1, W_2 , and $f \in W_1$.

Proof: By transfinite induction, we see that there is at most one such f . We now have to prove existence. We claim that for every $x \in \omega$, there exists a unique $g: \omega \rightarrow \omega$ such that for all $y \in \omega$, $g(0) = x$ and $g(y^+) = g(y)^+$. Uniqueness follows from existence. Suppose this is false, and let x be the least counterexample. Then $x \neq 0$, because we can explicitly define the desired function without W_1, W_2 as a subset of W_1 in case $x = 0$. So let $x = y^+$ and g be for y . By induction, g is the unique such function for y . Hence g is defined without W_1, W_2 since y is. Hence we can define the function $h: \omega \rightarrow \omega$ without

W_1, W_2 as a subset of W_1 by $h(z) = g(z)+$. Then $h(0) = y+ = x$ and $h(z+) = g(z+)+ = g(z)++ = h(z)+$. Then h corresponds to x . This is the required contradiction.

Finally, we define $f(\langle x, y \rangle) = g(y)$ where g corresponds to x .

For the second claim, note that f is defined as a subset of W_1 , and hence $f \in W_2$. By Lemma IB1.1 vi, $f \in W_1$. \square

We write $+: \omega^2 \rightarrow \omega$ for the unique function provided by Lemma IB3.9. We often write $x+y$, which is defined only for $x, y \in \omega$.

LEMMA IB3.10. The following is provable in $T_3(W_1, W_2)$. There is a unique function $f: \omega^2 \rightarrow \omega$ such that $f(\langle x, 0 \rangle) = 0$ and $f(\langle x, y+ \rangle) = f(\langle x, y \rangle)+x$. This function is definable without W_1, W_2 , and lies in W_1 .

Proof: Similar to the proof of Lemma IB3.9. We prove that for each $x \in \omega$ there is a unique function $g: \omega \rightarrow \omega$ such that for all $y \in \omega$, $g(0) = 0$ and $g(y+) = g(y)+x$. Again we have the least counterexample x which is defined without W_1, W_2 . But this time we don't consider the predecessor of x . Instead, we build up g for x by finite approximations. The details are left to the reader. \square

We write $x: \omega^2 \rightarrow \omega$ for the unique function provided by Lemma IB3.10.

LEMMA IB3.11. The following is provable in $T_3(W_1, W_2)$. ω under $S, +, x$ forms a model of the Peano arithmetic axioms in which induction may be applied with any formula of $T_3(W_1, W_2)$.

Proof: Immediate by transfinite induction. \square

4. Finite sequence codes

Let $f: x \rightarrow y$. We write $f|z$ for f restricted to z ; i.e., for $\{\langle u, v \rangle \in f: u \in z\}$. When we write $f|z$ we are committed to the existence of z as an object. Of course, we have to worry about the existence of $f|z$.

We define the relation $<^*$ between pairs of objects $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$. We say $\langle a_1, a_2 \rangle <^* \langle b_1, b_2 \rangle$ if and only if

- i) both a 's are elements of both b 's; or
- ii) some a equals some b , and $a_1 \in b_1$; or
- iii) $a_1 = b_1$ and $a_2 \in b_2$.

We define $<^*(\langle a, b \rangle) = \{\langle c, d \rangle : \langle c, d \rangle <^* \langle a, b \rangle\}$. We have to worry about the existence of this set.

We say that f is an $\langle a, b \rangle$ pairing function if and only if there exists a strong ordinal y such that

- i) $<^*(\langle a, b \rangle)$ exists;
- ii) $f : <^*(\langle a, b \rangle) \rightarrow y$;
- iii) the range of f is y ;
- iv) for all $x \in <^*(\langle a, b \rangle)$, $f(x) =$ the \in -least element z of y which is not equaled to any $f(z)$, $z <^* x$.

LEMMA IB4.1. The following is provable in $T3(W1, W2)$. Let $a, b \in OW2$. There is a unique $\langle a, b \rangle$ pairing function. Furthermore, this function is an element of $W2$, and its range is an element of $OW2$.

Proof: Suppose that there exists $a, b \in OW2$ such that the two claims are false. By transfinite induction, let $\langle a, b \rangle$ be $<^*$ -least such that the two claims are false. By Lemma IB1.1 vi and Lemma IB2.8, $a, b \in OW1$ and $\langle a, b \rangle \in W1$ and $<^*(\langle a, b \rangle) \in W1$. For each $y \in <^*(\langle a, b \rangle)$, let f_y be the unique $\langle a, b \rangle$ pairing function, and let y^* be the range of f_y . Each $f_y \in W2$ and each $y^* \in OW2$. Hence each $f_y \subseteq W2$.

Now each f_y and y^* is defined from y without mentioning $W1$ and without mentioning $W2$. So we can apply axiom 3 to the true statement " $y^*, f_y \in \subseteq W2$." Hence " $y^*, f_y \in \subseteq W1$." Using transfinite induction, we see that the f_y 's are comparable under \subseteq .

Now suppose that $\langle a, b \rangle$ has no $<^*$ predecessor. Then we can define the union of the f_y , $y \in <^*(\langle a, b \rangle)$, as a subset of $W1$ without mentioning $W1$. By the previous paragraph, this union will be a function with domain $<^*(\langle a, b \rangle)$, which must exist by Lemma IB2.7. The range will be the union of the y^* , $y \in x$,

which is defined as a subset of W_1 without mentioning W_1 . Hence it exists and lies in W_2 . Also the range is a transitive subset of OW_1 . Hence by transfinite induction, it is an ordinal in W_2 . So it lies in OW_2 . So the union function is an $\langle a, b \rangle$ pairing function lying in W_2 whose range is an element of OW_2 . Also by transfinite induction, the union function is the unique $\langle a, b \rangle$ pairing function. (Here we also use Lemma IB3.3). But this contradicts the choice of a, b .

Clearly $\langle a, b \rangle \neq \langle 0, 0 \rangle$. Finally suppose that $\langle a, b \rangle$ is a \langle^* successor. Let $\langle c, d \rangle$ be the greatest element of $\langle^*(\langle a, b \rangle)$. Let f be the unique $\langle c, d \rangle$ pairing function. Also let $f: \langle^*(\langle c, d \rangle) \rightarrow y^*$ be onto, where $y^* \in W_2$ and $f \in W_1$. Since y^* is an element of W_2 defined without W_1 , we have $y^* \in OW_1$.

We can obviously extend f to f' by taking $f'(\langle c, d \rangle) = y^*$, which is defined as a subset of W_1 without mentioning W_1 . Hence it lies in W_2 . Clearly f' is a $\langle a, b \rangle$ pairing function with range y^{*+} . Clearly $y^{*+} \in W_2$ since OW_2 is a limit ordinal. By transfinite induction, there is a unique $\langle a, b \rangle$ pairing function. Again this contradicts the choice of a, b , and completes the proof. \square

We say that f is a pairing function on x if and only if there exists y such that

- i) x, y are strong ordinals;
- ii) x_2 exists;
- iii) $f: x_2 \rightarrow y$;
- iv) the range of f is y ;
- v) for all $z \in x_2$, $f(z) =$ the \in -least element z of y which is not equaled to any $f(w)$, $w \langle^* x$.

LEMMA IB4.2. The following is provable in $T_3(W_1, W_2)$. Let $x \in OW_2$. There is a unique pairing function on x . Furthermore, this function is an element of W_2 , and its range is an element of OW_2 . Let $x \in OW_1$. There is a unique pairing function on x . Furthermore, this function is an element of W_1 , and its range is an element of OW_1 .

Proof: Let $x \in OW_2$. Apply Lemma IB4.1 with $a = 0$ and $b = x$. Note that $\langle^*(\langle 0, x \rangle) = x_2$.

The part about $OW1$ follows by Lemma 2.1 iv. \square

LEMMA IB4.3. The following is provable in $T3(W1, W2)$. There is a unique pairing function f on $OW2$. The range of f is $OW2$. For all $x, y \in OW1$, $f(\langle x, y \rangle) \in OW1$. There is a unique pairing function f on $OW1$. It lies in $W2$, and its range is $OW1$.

Proof: For the first claim, let f_x be the unique pairing function on $x \in OW2$. By transfinite induction, the f_x are comparable under \subseteq . Also by transfinite induction, we see that for all $x \in OW2$, $x \subseteq \text{rng}(f_x)$. We can define the union of the f_x , $x \in OW2$, as a subset of $W2$ without mentioning $W1$. Hence the union, f , exists. It is clear that f is a pairing function on $OW2$ whose range is $OW2$. Let $x, y \in OW1$. Let $z = \max(x, y) +$. Then $z \in OW1$, and so by Lemma IB3.7, $f_z \in W1$ has its range in $OW1$. Since $f_z \subseteq f$, we see that $f(\langle x, y \rangle) \in OW1$.

There is a unique pairing function f on $OW1$ because $OW1 \in OW2$. By Lemma IB4.2, it also lies in $W2$. $\text{Rng}(f)$ is an element of $OW2$ that is included in $W1$. Since $OW1$ is a limit ordinal, we see that $OW1 \subseteq \text{rng}(f)$. Hence $\text{rng}(f) = OW1$. \square

We now fix P to be the unique pairing function $P: OW2^2 \rightarrow OW2$. P has range $OW2$.

We say that x is P -closed if and only if $x \in OW2$ and for all $y, z \in x$, $P(\langle y, z \rangle) \in x$.

LEMMA IB4.4. The following is provable in $T3(W1, W2)$. $(\forall x, y \in OW2) (x, y \subseteq P(\langle x, y \rangle))$. Every P -closed x is $0, 1$, or a limit ordinal. $0, 1, \omega$ are the first three P -closed ordinals. $(\forall x \in OW2) (x \text{ is } P\text{-closed if and only if there is a pairing function on } x \text{ whose range is } x)$.

Proof: The first claim is proved by transfinite induction. For the second claim, let $1 \in x = y + \in OW2$. Then $x \subseteq P(\langle 0, y \rangle)$. Hence $x \in P(\langle 1, y \rangle)$, and so x is not P -closed.

For the third claim, clearly $0, 1$ are P -closed. To see that ω is P -closed, recall that ω is the first limit ordinal in $OW2$, lies in $OW1$, and also the set of all finite ordinals. By

transfinite induction, choose x, y to be $<^*$ -least elements of $OW2$ such that $P(\langle x, y \rangle) = \omega$. If $x, y \in \omega$ then $\langle x, y \rangle$ has an immediate predecessor in $<^*$, which contradicts $P(\langle x, y \rangle) = \omega$. Hence either x or y is an infinite ordinal. Therefore for all finite ordinals u, v , we see that $\langle u, v \rangle <^* \langle x, y \rangle$, and hence $P(\langle u, v \rangle) \in \omega$. So ω is P -closed.

For the fourth claim, let $x \in OW2$ be P -closed. Let f be the unique pairing function on x , and let y be its range. Then $f \subseteq P$. By the first claim, $x \subseteq y$. By P -closure, $y \subseteq x$, and so $x = y$ as required. On the other hand, let $x \in OW2$ be such that the unique pairing function f on x has range x . Since $f \subseteq P$, we see that x is P -closed. \square

LEMMA IB4.5. The following is provable in $T3(W1, W2)$. $OW1, OW2$ are P -closed. $(\forall x \in OW2) (\exists y \in OW2) (x \in y \ \& \ y \text{ is } P\text{-closed})$.
 $(\forall x \in OW1) (\exists y \in OW1) (x \in y \ \& \ y \text{ is } P\text{-closed})$.

Proof: $OW1$ is P -closed by Lemma IB4.3. $OW2$ is obviously P -closed. Suppose the second claim is false, and let x be the least counterexample in $OW2$. Then x is an element of $W2$ that is definable without mentioning $W1$. Hence $x \in OW1$. But $OW1$ is P -closed. This is the required contradiction.

By Lemma IB4.4, we can rewrite $(\forall x \in OW2) (\exists y \in OW2) (x \in y \ \& \ y \text{ is } P\text{-closed})$ in the form $(\forall x \in OW2) (\exists y \in OW2) (\exists f) (x \in y \ \& \ f \text{ is a pairing function on } y \text{ with range } y)$. Then $(\forall x \in OW1) (\exists y \in OW1) (\exists f) (x \in y \ \& \ f \text{ is a pairing function on } y \text{ with range } y)$. The third claim follows by Lemma IB4.3. \square

Let $g: y \rightarrow y$ be a pairing function on y with range y . We say that $z \in y$ is a strong point of g if and only if $(\forall u, v \in z) (g(\langle u, v \rangle) \in z)$.

We say that f, g is a pairclosure system on x if and only if there exists y such that

- i) $g: y \rightarrow y$ is a pairing function on y ;
- ii) x is a strong ordinal;
- iii) $f: x \rightarrow y$;

- iv) for all $z \in x$, $f(z) =$ the \in -least strong point of g which is not equaled to any $f(z)$, $z \in x$;
 v) every strong point of g is a value of f .

LEMMA IB4.6. The following are provable in $T3(W1, W2)$.

i) let $x \in OW2$. There is a unique pairclosure system f, g on x . Furthermore, $f, g, \text{dom}(g), \text{rng}(f) \in W2$;

ii) let $x \in OW1$. There is a unique pairclosure system f, g on x . Furthermore, $f, g, \text{dom}(g), \text{rng}(f) \in W1$;

Proof: For the first claim, note that uniqueness is clear by transfinite induction. Suppose that there exists $x \in OW2$ such that \neg (there is a pairclosure system f, g on x with $f, g, \text{dom}(g), \text{rng}(f) \in W2$). By transfinite induction, let x be least with this property. By Lemma 2.1 vi, $x \in OW1$. For each $y \in x$, let f_y, g_y be the unique pairclosure system on y , and let y^* be the domain of g_y . Each $f_y, g_y, \text{rng}(f) \in W2$ and each $y^* \in OW2$. Then each g_y is simply the unique pairing function on y^* ; it has range y^* .

Now each $f_y, \text{rng}(f), y^*$ is an element of $W2$ defined from y without mentioning $W1$ and without mentioning $W2$. So we can apply axiom 3 to the true statement " $f_y, \text{rng}(f), y^* \in W2$." Hence " $f_y, \text{rng}(f), y^* \in W1$." Using transfinite induction, we see that the f_y 's are comparable under \subseteq .

Now suppose that x is a limit ordinal. Then we can define the union of the f_y , $y \in x$, as a subset of $W1$ without mentioning $W1$. We can also define the union of the y^* as a subset of $W1$ without mentioning $W1$. These result in a function f^* with domain x and an element x^* of $OW2$. Let g^* be the unique pairing function on x^* . Then $g^* \in W2$. We also see that $f^*, g^*, \text{dom}(f^*), \text{rng}(f^*) \in W2$. Note that f^*, g^* is a pairclosure system on x . This is a contradiction.

Clearly $x \neq 0$. Finally suppose that x is a successor ordinal, $x = y+$. Let f, g be the unique pairclosure system on y . We have $f, g, \text{dom}(g), \text{rng}(f) \in W2$. Since these four sets are elements of $W2$ defined without $W1$, we have $f, g, \text{dom}(g), \text{rng}(f) \in W1$. Since $\text{dom}(g)$ is an ordinal in $OW2$, we see that $\text{dom}(g) \in OW1$.

We want to extend f . By Lemma IB4.5, let z be the least P -closed ordinal with $\text{dom}(g) \in z$. Let g^* be the unique pairing function on z . Then $g^*:z \rightarrow z$ and $g^* \in W_2$. Extend f to $f^*:x \rightarrow z$ by setting $f^*(y) = \text{dom}(g)$. f^* exists because it is defined as a subset of W_1 without mentioning W_1 . $\text{rng}(f^*)$ exists for the same reason. And $f^*, \text{rng}(f^*) \in W_2$. This is the desired contradiction.

The second claim follows from the first claim by Lemma 2A iv.

□

LEMMA IB4.7. The following is provable in $T_3(W_1, W_2)$. There is a unique pairclosure system f, g on OW_2 . $\text{Rng}(f)$ exists and is the set of all P -closed elements of OW_2 , and $g = P$. For all $x \in OW_1$, $f(x) \in OW_1$. There is a unique pairclosure system f, g on OW_1 . $\text{Rng}(f)$ is an unbounded subset of OW_1 , and g is the restriction of P to OW_1 . $f, g, \text{rng}(f) \in W_2$.

Proof: Using Lemma IB4.6, let f, g be the union of the pairclosure systems f_x, g_x on $x \in OW_2$. f, g exist because they are defined as subsets of W_2 without mentioning W_1 . Then f, g is a pairclosure system on OW_2 . Uniqueness follows by transfinite induction. Since g is a pairing function on OW_2 , $g = P$. Using transfinite induction, we see that every P -closed element of OW_2 is a value of f .

Now let $x \in OW_1$. Then $x^+ \in OW_1$, $f_{x^+} \subseteq f$, and $\text{rng}(f_{x^+}) \subseteq W_1$. Hence $f(x) \in W_1$ and $f(x) \in OW_2$. Therefore $f(x) \in OW_1$.

There is a unique pairclosure system f, g on OW_1 by Lemma IB4.6. The unique pairing function on OW_1 is P restricted to OW_1 . And the P -closed elements of OW_1 are unbounded in OW_1 by Lemma IB4.5. Also $f, g, \text{rng}(f) \in W_2$ by Lemma IB4.2. □

We let $P^*:OW_2 \rightarrow OW_2$ be such that for the unique pairclosure system f, g on OW_2 , $f = P^*$.

LEMMA IB4.8. The following is provable in $T_3(W_1, W_2)$. There exist unique functions $I, J:OW_2 \rightarrow OW_2$ such that for all $x \in OW_2$, $P(\langle I(x), J(x) \rangle) = x$. For all $x \in OW_2$, $I(x), J(x) \in W_2$.

Proof: Since $P:OW22 \rightarrow OW2$ is one-one onto, the I, J are unique if they exist. But I, J can be defined as subsets of $W2$ without $W1$. Therefore I, J exist. \square

LEMMA IB4.9. The following is provable in $T3(W1, W2)$. Let $x \in OW2$. There exists unique $f_x, g_x: \omega \rightarrow x^+$ such that

- i) $f_x(0) = I(x)$;
- ii) $g_x(0) = J(x)$;
- iii) $(\forall y \in \omega) (f_x(y^+) = I(g_x(y)))$;
- iv) $(\forall y \in \omega) (g_x(y^+) = J(g_x(y)))$.

Furthermore, $f, g \in W2$.

Proof: Let x be a least counterexample in $OW2$. Then $x \in OW1$. We first claim that for all $z \in \omega$ there exists $f_z, g_z: z \rightarrow x^+$ obeying the four clauses, lying in $W2$. It is clear that if they exist then they are unique. Suppose this is false, and let $z \in \omega$ be the least counterexample. Clearly $z \neq 0$. Let $z = w^+$, and consider $f_w, g_w \in W2$. Clearly $f_w, g_w \subseteq W1$. We can appropriately extend them at w as subsets of $W1$ defined without $W1$. Hence these extensions lie in $W2$. Thus the claim is established. \square

Think of f_x as the infinite sequence coded by x . Lemma IB4.10 says that we can find a code for the result of appending any y in front of the infinite sequence coded by x .

LEMMA IB4.10. The following is provable in $T3(W1, W2)$. Let $x, y \in u \in OW2$, where u is a P -closed limit. There exists $z \in u$ such that

- i) $g_z(0) = x$;
- ii) $f_z(0) = y$.

It follows that for all $m \in \omega$, $g_z(m^+) = g_x(m)$ and $f_z(m^+) = f_x(m)$.

Proof: Let x, y, u be as given. Set $z = P(\langle y, x \rangle)$. The final part is by induction on m . For $m = 0$ note that $g_z(1) = J(g_z(0)) = J(x) = g_x(0)$. And $f_z(1) = I(g_z(0)) = I(x) = f_x(0)$. Assume true for m . Note that $g_z(m^{++}) = J(g_z(m^+)) = J(g_x(m)) = g_x(m^+)$. And $f_z(m^{++}) = I(g_z(m^+)) = I(g_x(m)) = f_x(m^+)$. \square

We now need to know that we can code the result of chopping off the front of any infinite sequence.

LEMMA IB4.11. The following is provable in $T3(W1, W2)$. Let $x \in u \in OW2$, where u is a P-closed limit. There exists $y \in u$ such that $g_y(0) = g_x(1)$. It follows that for all $m \in \omega$, $g_y(m) = g_x(m+)$ and $f_y(m) = f_x(m+)$.

Proof: Let x, u be as given. Set $y = \langle f_x(1), g_x(1) \rangle$. Then $g_y(0) = J(y) = g_x(1)$ and $f_y(0) = I(y) = f_x(1)$. Assume $g_y(m) = g_x(m+)$ and $f_y(m) = f_x(m+)$. Then $g_y(m+) = J(g_y(m)) = J(g_x(m+)) = g_x(m++)$, and $f_y(m+) = I(f_y(m)) = I(g_y(m)) = I(g_x(m+)) = f_x(m++)$. \square

With Lemmas IB4.10 and IB4.11, we have the information needed to give an appropriate coding of finite sequences. It is conceptually clearer, although not absolutely necessary, to arrange that certain elements of $OW2$ code a unique finite sequence from $OW2$, and that every finite sequence from $OW2$ have a unique code.

Accordingly, we define the set FSC of finite sequence codes as follows. FSC is the set of all $x \in OW2$ such that

- i) $I(x) \in \omega$;
- ii) $(\forall y \in x) (J(y) = J(x) \rightarrow (\exists i \in J(x)) (fI(x)(i) \neq fI(y)(i)))$.

FSC is defined as a subset of $OW2$ without mentioning $W1$, and therefore exists by Lemma 2A v.

Let $x \in FSC$. We write $lth(x) = I(x)$. $J(x)$ does the real coding, whereas $I(x)$ just indicates the (finite) length. Here ii) is the minimality property that guarantees that every finite sequence has a unique code.

Let $x \in FSC$ and $i \in lth(x)$. We write $x[i]$ for $f_x(i)$. This is the i -th term of the finite sequence coded by x , where we count from 0.

The following summarizes the essential properties of our finite sequence coding.

LEMMA IB4.12. Let $n \geq 1$. The following is provable in $T3(W1, W2)$. Let $u \in OW2$ be a P-closed limit.

- i) $(\forall x, y \in \text{FSC}) (\text{lth}(x) = \text{lth}(y) \ \& \ (\forall i \in \text{lth}(x)) (x[i] = y[i]) \rightarrow x = y)$;
- ii) 0 is the unique element of FSC such that $\text{lth}(0) = 0$;
- iii) $(\forall x \in u, \text{FSC}) (\exists y \in u, \text{FSC}) (\text{lth}(x) \neq 0 \rightarrow (\text{lth}(y) = \text{lth}(x) - 1 \ \& \ (\forall m \in \text{lth}(y)) (y[m] = x[m+1])))$;
- iv) $(\forall x \in u, \text{FSC}) (\forall y \in u) (\exists z \in u, \text{FSC}) (\text{lth}(z) = \text{lth}(x) + 1 \ \& \ z[0] = y \ \& \ (\forall m \in \text{lth}(z)) (m \neq 0 \rightarrow z[m] = x[m-1]))$;
- v) $(\forall x \in \text{FSC}) (\forall m \in \text{lth}(x)) (x[m] \in u) \rightarrow x \in u$.

Proof: All claims except v) follow from Lemmas IB4.9 - IB4.11. In order to guarantee that we stay in FSC, we have to perform a minimization. The existence before minimization comes from Lemmas IB4.9 - IB4.11.

We establish v) by induction on $\text{lth}(x)$. The case $\text{lth}(x) = 0$ is trivial since then $x = 0$. Suppose this is true for length n . Now let $\text{lth}(x) = n + 1$. By iii), we can chop off the front term of x to get $x' \in \text{FSC}$ of length n (of course we may not have $x' \in u$, but this doesn't cause a problem). We can apply the induction hypothesis to x' getting $x' \in u$. But then we can put the front term back by iv), this time getting an equivalent element of FSC that is in u . Hence we get back exactly x , and so $x \in u$. \square

We can prove all of the usual facts about finite sequences from OW2 using Lemma IB4.12 and induction (Lemma IB3.11). Two particularly useful facts are the existence of replacements of a term in a sequence by another term, and the existence of a code for a specified list of elements.

LEMMA IB4.13. The following is provable in $T_3(W_1, W_2)$. Let $u \in \text{OW2}$ be a P-closed limit. $(\forall x \in u, \text{FSC}) (\forall y \in u) (\forall n \in \omega) (\exists z \in u, \text{FSC}) (\text{lth}(x) = \text{lth}(z) \ \& \ x[n] = y \ \& \ (\forall m \in \text{lth}(x)) (m \neq n \rightarrow x[m] = z[m]))$.

Proof: Fix u, y as given. Prove by induction on n that $(\forall x \in u, \text{FSC}) (\forall y \in u) (n \in \text{lth}(x) \rightarrow (\exists z \in u, \text{FSC}) (\text{lth}(x) = \text{lth}(z) \ \& \ x[n] = y \ \& \ (\forall m \in \text{lth}(x)) (m \neq n \rightarrow x[m] = z[m])))$.

For the case $n = 0$, use Lemma IB4.12 to chop off the front term of x , put y in front, and minimize. Assume true for n and let $n + 1 \in \text{lth}(x)$. Apply the induction hypothesis to the

result of chopping off the front term of x . Then put $x[0]$ in front and minimize. \square

For $x \in \text{FSC}$, $y \in \text{OW2}$, and $n \in \text{lth}(x)$, we let $x[n/y]$ be the z given by Lemma IB4.13. Thus $x[n/y]$ is the code for the finite sequence obtained by replacing the term of the finite sequence coded by x at position n with y .

LEMMA IB4.14. Let $n \geq 1$. The following is provable in $\text{T3}(W1, W2)$. Let $u \in \text{OW2}$ be a P -closed limit, and $x_1, \dots, x_n \in u$. Then there exists $y \in u$ such that $\text{lth}(y) = n$ & $y[0] = x_1$ & ... & $y[n-1] = x_n$. Here $1, \dots, n$ are given by the closed terms $S(0), \dots, S \dots S(0)$, where there are $1, \dots, n$ S 's.

Proof: Obvious from Lemma IB4.12 by external induction on n . \square

5. Arithmetization, structures, and satisfaction relations

Armed with Lemma IB3.11, we can arithmetize formulas of set theory. For this purpose, we will use $L(\in) =$ first order predicate calculus without equality and with only the binary relation symbol \in , variables x_0, x_1, \dots , \forall , $\&$, and \neg . In the usual way, we explicitly define certain syntactic operations which provably obey the appropriate properties. This is standard material.

In particular, one develops $\text{Fmla} \subseteq \omega$, $\text{Var}:\omega \rightarrow \omega$, $\text{Neg}:\text{Fmla} \rightarrow \text{Fmla}$, $\text{And}:\text{Fmla}^2 \rightarrow \text{Fmla}$, $\text{Or}:\text{Fmla}^2 \rightarrow \text{Fmla}$, $\text{Imp}:\text{Fmla}^2 \rightarrow \text{Fmla}$, $\text{Iff}:\text{Fmla}^2 \rightarrow \text{Fmla}$, $\text{Eps}:\omega^2 \rightarrow \text{Fmla}$, $\text{All}:\omega \times \text{Fmla} \rightarrow \text{Fmla}$, $\text{Ex}:\omega \times \text{Fmla} \rightarrow \text{Fmla}$, $\#:\text{Fmla} \rightarrow \omega$, and $\rho:\text{Fmla} \rightarrow \omega$. Informally, $\text{Var}(n)$ is the index of the n -th variable. Fmla is the set of all indices of formulas in $L(\in)$, $\text{Neg}(n)$ is the index of the negation of the formula with index n . $\text{And}(n, m)$ is the index of the conjunction of the formulas with indices n, m . $\text{Or}(n, m)$ is the index of the disjunction of the formulas with indices n, m . $\text{Imp}(n, m)$ is the index of the implication of the formula with index n to the formula with index m . $\text{Iff}(n, m)$ is the index of the if and only if of the formula with index n to the formula with index m . $\text{Eps}(n, m)$ is the index of the formula $\text{Var}(n) \in \text{Var}(m)$. $\text{All}(n, m)$ is the result of universally quantifying the formula with index m by $\text{Var}(n)$.

$Ex(n,m)$ is the result of existentially quantifying the formula with index m by $Var(n)$. Also $\#(n)$ is the greatest m such that the m -th variable appears in the formula with index n . Finally $\rho(n)$ be the number of occurrences of variables in the formula with index n .

LEMMA IB5.1. The following is provable in $T3(W1,W2)$. No element of $OW2$ is an ordered pair.

Proof: Suppose $\langle x,y \rangle = \{\{x\},\{x,y\}\} \in OW2$. Then $\langle x,y \rangle$ is transitive and also $x,y \in OW2$. Hence $x = \{x\}$ or $x = \{x,y\}$. In either case, $x \in x$, violating transfinite induction. \square

An $OW2$ -structure is a set R consisting of elements of $OW2$ and $OW2^2$, where $(\forall x,y) (\langle x,y \rangle \in R \rightarrow (x,y \in R \ \& \ x,y \in OW2))$. Note that by Lemma IB5.1, R is really an arbitrary subset of $OW2$ together with a subset of its Cartesian square - if we have enough comprehension, which we don't.

It turns out to be important at one point in this proof to stay close to $OW2$. Ordered pairs from $OW2$ is quite close to $OW2$, and sets of ordered pairs are close to $OW2$. However, ordered pairs of subsets of $OW2$ are too far away.

For $OW2$ -structures R , we write $|R|$ for the least P -closed limit $u \in OW2$ such that every element of R is either in u or in u^2 .

Also it will be convenient to write $x \in' R$ to indicate that $x \in R \ \& \ x \in OW2$. We also speak of the elements' of R .

Let R be an $OW2$ -structure. We say that x is adequate for b in R if and only if

- i) $b \in Fmla$;
- ii) $x \in FSC$;
- iii) $\#(b) \in lth(x)$;
- iv) $(\forall i \in lth(x)) (x[i] \in' R)$.

We say that y is a satisfaction relation for R if and only if y is the set of all $\langle b,x \rangle$ such that x is adequate for b , and

- i) $b = Eps(n,m)$, $n,m \in \omega$, and $\langle x[n],x[m] \rangle \in R$; or

- ii) $b = \text{Neg}(n)$, $n \in \text{Fmla}$, and $\langle n, x \rangle \notin y$; or
- iii) $b = \text{And}(n, m)$, $n, m \in \text{Fmla}$, and $\langle n, x \rangle, \langle m, x \rangle \in y$; or
- iv) $b = \text{Or}(n, m)$, $n, m \in \text{Fmla}$, and $\langle n, x \rangle \in y$ or $\langle m, x \rangle \in y$; or
- v) $b = \text{Imp}(n, m)$, $n, m \in \text{Fmla}$, and either $\langle n, x \rangle \notin y$ or $\langle m, x \rangle \in y$; or
- vi) $b = \text{Iff}(n, m)$, $n, m \in \text{Fmla}$, and either $(\langle n, x \rangle, \langle m, x \rangle \in y)$ or $(\langle n, x \rangle, \langle m, x \rangle \notin y)$; or
- vii) $b = \text{All}(n, m)$, $n \in \omega$, $m \in \text{Fmla}$, and for all $u \in x$, $\langle m, x[n/u] \rangle \in y$; or
- viii) $b = \text{Ex}(n, m)$, $n \in \omega$, $m \in \text{Fmla}$, and there exists $u \in x$ such that $\langle m, x[n/u] \rangle \in y$.

The idea here is to view R as the relational structure $\langle R \cap \text{OW2}, R \cap \text{OW2} \rangle$ and do the Tarski definition of truth.

We define $\text{SAT}(R, n, x)$ if and only if R is an OW2 structure and there exists a satisfaction relation y for R such that $\langle n, x \rangle \in y$.

LEMMA IB5.2. The following is provable in $T3(W1, W2)$. Let R be an OW2-structure. There is at most one satisfaction relation y for R . Furthermore, $y \subseteq |R|^2$. If there is a satisfaction relation for R then $\text{SAT}(R, n, x)$ obeys the usual inductive clauses for satisfaction (provided x is adequate for n in R).

Proof: By induction on the number of occurrences of variables. I.e., one shows that for all $n \in \omega$, any two satisfaction relations on R have the same elements whose first component m has $\rho(m) \in n$. For the second claim, note that condition i) implies that x lies in every P-closed strict upper bound on the elements' of R , by Lemma IB4.12 v. For the third claim, use the uniqueness in Lemma IB5.1 \square

It will be convenient to have the following version of Russell's Paradox in this context.

LEMMA IB5.3. The following is provable in $T3(W1, W2)$. Let R be an OW2 structure which has a satisfaction relation. Then there exists $n \in \text{Fmla}$ such that the following fails for all $x \in R$ and $y \in \text{FSC}$: for all $a \in R$, $\langle a, x \rangle \in R \leftrightarrow \text{SAT}(R, n, y[0/a])$. In particular, this is provable in $T3(W1, W2)$

when n is taken to be the numeral (closed S -term) which is the index of the formula $\neg x_0 \in x_0$.

Proof: Left to the reader.

We now give a sufficient condition for the existence of the satisfaction relation for an OW_2 -structure R .

LEMMA IB5.4. Let φ be a formula in $L(\in, W_1, W_2)$ with at most the free variable x , not mentioning W_1 . The following is provable in $T_3(W_1, W_2)$. Suppose φ has a unique solution, and it is an OW_2 -structure R .

- i) R has a unique satisfaction relation;
- ii) the satisfaction relation is $\subseteq |R|_2$;
- iii) if $|R| \in OW_2$ then R and its satisfaction relation lie in W_1 , and $|R| \in OW_1$.

Proof: Let φ, R be as given. First assume $|R| \in OW_2$. Then $|R| \in OW_2$. Now $|R|$ is definable without mentioning W_1 . By Lemma IB1.1 vi, $|R| \in W_1$, and hence in OW_1 . Hence R is a subset of W_1 definable without mentioning W_1 . By axiom 3, $R \in W_2$. Therefore by Lemma IB1.1 vi, $R \in W_1$.

Still assuming $|R| \in OW_2$, the satisfaction relation of R , if it exists, is a subset of $|R|_2$ by inspection. So if it exists, it is a definable subset of W_1 without mentioning W_1 . So the satisfaction relation is also $\in W_1$, if it exists.

We also see by inspection that the satisfaction relation of R , if it exists, is a subset of $|R|_2$ even in the case $|R| = OW_2$.

It remains to show that R has a satisfaction relation.

Uniqueness is from Lemma IB5.1. We no longer assume $|R| \in OW_2$.

For $n \in \omega$, we can define the n -satisfaction relations for R ; we use the same 6 clauses as in the definition of satisfaction relation for R , except that we strengthen clause ii) by insisting that the number of occurrences of variables in the formula of index b is at most n ; i.e., $\rho(b) \leq n$. Again

we have that there is at most one n -satisfaction relation for R .

Now there is a direct construction of a 2-satisfaction relation for R since the only formulas with at most 2 occurrences of symbols are atomic formulas.

We then prove that for each $n \in \omega$, there exists an n -satisfaction relation for R . Assume this is false, and let n be least. Then $2 \in n$. Let y be the $(n-1)$ -satisfaction relation for R . Now we can extend y to a (the) n -satisfaction relation for R rather explicitly, using Lemma IB1.1 v.

We also argue by induction that the various n -satisfaction relations for R cohere in the obvious way. Hence we can take the union of the various n -satisfaction relations and see that this union is a (the) satisfaction relation for R . Here we again use Lemma IB1.1 v. \square

Let R be an OW2-structure with a satisfaction relation y . We say that v is a preferred name over R if and only if

- i) v is of the form $\langle b, x \rangle$ where x is adequate for b in R ;
- ii) there is no $\langle b', x' \rangle \in \langle b, x \rangle$ such that for all $a \in A$, $\text{SAT}(R, b', x' [0/a]) \leftrightarrow \text{SAT}(R, b, x [0/a])$;
- iii) there is no $c \in' R$ such that for all $a \in' R$ $\text{SAT}(R, b, x [0/a]) \in y \leftrightarrow \langle a, c \rangle \in R$.

The idea here is that the preferred names 'name' the subsets of $R \cap \text{OW2}$ that are first order definable over (the relational structure associated with) R by means of the satisfaction relation for R , although we are careful not to commit to the existence of these subsets of R as objects.

(The subset of $R \cap \text{OW2}$ 'named' by $\langle b, x \rangle$ is the set of all $a \in' R$ that $\text{SAT}(R, b, x [0/a])$). We don't even commit to the existence of $R \cap \text{OW2}$. The preferred names are those that are minimal among those names that 'name' the same subset, and we also require of a preferred name that it not 'name' any set of the form $\{a: \langle a, c \rangle \in R\}$, $c \in' R$, where we again are careful not to commit to the existence of this $\{a: \langle a, c \rangle \in R\}$. It will eventually turn out that we can prove the existence of all these relevant subsets of R in the contexts

we are interested in, and we need this existence only considerably later.

We introduce a second absolute value on OW2-structures R . We write $||R||$ for the least $d \in \text{OW2}$ such that for all $x \in R$, $I(x) \in d$. Note that $||A|| \in |A|$.

Let R be an OW2-structure, $|R| \in \text{OW2}$, $||R|| = d$, and γ be the satisfaction relation for R . We define the OW2-structure $R^* = S$ as follows. The elements of S consist of

- i) the elements' of R together with the $P(\langle d, v \rangle)$ such that v is a preferred name over R ;
- ii) the elements of R which are pairs together with the pairs $\langle a, P(\langle d, v \rangle) \rangle$ such that $v = \langle b, x \rangle$ is a preferred name over R , $a \in R$, and $\text{SAT}(R, b, x[0/a])$.

Note that the $P(\langle d, v \rangle)$ in i) are new (i.e., are not in R) because $||R|| = d$.

We say that an OW2-structure R is extensional if and only if for all $x, y \in R$, if $(\forall z \in R) (\langle z, x \rangle \in R \leftrightarrow \langle z, y \rangle \in R)$ then $x = y$.

LEMMA IB5.5. Let φ be a formula in $L(\in, W1, W2)$ with at most the free variable x , not mentioning $W1$. The following is provable in $T3(W1, W2)$. Suppose φ has a unique solution, and this solution is R , where R is an OW2-structure such that $|R| \in \text{OW2}$. Then $R, R^* \in W1$. If R is extensional then R^* is extensional. And $||R^*|| = ||R|| +$.

Proof: By Lemma IB5.4, R and its satisfaction relation are both in $W1$. Then R^* can be defined as a subset of $W1$ without mentioning $W1$, and hence exists by axiom 3. By Lemma IB1.1 vi, $R^* \in W1$. The extensionality claim is obvious by construction. For the final claim, note that by Lemma IB5.3, a new element $P(\langle ||R||, v \rangle)$ in clause i) survives clause ii).

□

LEMMA IB5.6. Let R be an OW2-structure with a satisfaction relation. Let $n \in \text{Fmla}$ and $x \in \text{FSC}$, where x is adequate for n in R . There exists $x \in R^*$ such that for all $a \in R$, $\langle a, x \rangle \in R^*$ if and only if $\text{SAT}(R, n, x[0/a])$.

Proof: By construction of R^* . Left to the reader.

6. Constructible universe structure

We are now ready to complete the construction of the constructible universe on $OW2$. Recall the function $P^*:OW2 \rightarrow OW2$ which enumerates the P -closed elements of $OW2$.

The most obvious definition of an L -function is a function f with domain some $x \in OW2$ such that the following holds.

- i) every value of f is a structure;
- ii) if $0 \in x$ then $F(0) = \emptyset$;
- iii) if $y^+ \in x$ then $f(y^+) = f(y)^*$;
- iv) if $y \in x$ is a limit then $f(y)$ is the union of the $f(z)$, $z \in y$.

However, these are set valued functions and some technical problems crop up when we get too far from $OW2$. So we instead make the following definitions. Let T, x be arbitrary. We write $T_x = \{y: \langle x, y \rangle \in T\}$. Of course, we have to worry about whether this set exists.

We say that T is an L -system on x if and only if

- i) $x \in OW2$;
- ii) $T \subseteq x^2$;
- iii) for all $y \in x$, T_y exists and is an $OW2$ -structure;
- iv) if $0 \in x$ then $T_0 = \emptyset$;
- v) if $y^+ \in x$ then $T_{y^+} = T_y^*$;
- vi) if $y \in x$ is a limit then $f(y)$ is the union of the T_z , $z \in y$.

Note that T is a set of ordered pairs and triples from $OW2$, and every T_y is a set of elements and ordered pairs from $OW2$.

Let R and S be $OW2$ -structures. We write $R \subseteq^* S$ if and only if $R \subseteq S$, and $(\forall x, y \in R) (\langle x, y \rangle \in R \leftrightarrow \langle x, y \rangle \in S)$.

LEMMA IB6.1. The following is provable in $T_3(W_1, W_2)$. Let $x \in y \in \text{OW}_2$, T be an L-system on x , and T' be an L-system on y . Then for all $z \in x$, $Tz = T'z$.

Proof: By transfinite induction, using the fact that clauses iii) - vi) in the definition of L-systems are entirely deterministic. \square

LEMMA IB6.2. The following are provable in $T_3(W_1, W_2)$. Let $x \in \text{OW}_2$ and T be an L-system on x .

- i) for all $y \in x$, $||Ty|| = y$ and $|Ty| \in \text{max}(\omega, P^*(y))$;
- ii) for all $z \in y \in x$, $Tz \subseteq^* Ty$;
- iii) $T \subseteq \text{max}(\omega, P^*(x)) \leq 3$.

Proof: Let $x \in \text{OW}_2$ and T be an L-system on x . Now if $0 \in x$ then $|T0| = |\emptyset| = \omega$, and $||T0|| = ||\emptyset|| = 0$. So by transfinite induction, $y \in x \rightarrow ||Ty|| = y$. Also $|f(1)| = |\emptyset^*| = \omega$ and $|T_2(W_1, W_2)| = |\emptyset^{**}| = \omega$. And by starting the transfinite induction at 2, we obtain $1 \in y \in x \rightarrow |Ty| \in P^*(y)$.

For the second claim, show that the following holds for all $y \in x$: for all $z \in y$, $Tz \subseteq^* Ty$. The basis case $y = 0$ is trivial. Suppose $y = w+$. Clearly $Tw \subseteq^* Ty$. If $z \in w$ then $Tz \subseteq^* Tw$, and so $Tz \subseteq^* Ty$. Finally suppose y is a limit. Then the Tz , $z \in y$, are ordered under \subseteq^* . Let $z \in y$. Then clearly $Tz \subseteq Ty$. Also let $u, v \in Tz$. If $\langle u, v \rangle \in Tz$ then $\langle u, v \rangle \in Ty$. If $\langle u, v \rangle \notin Tz$ then $\langle u, v \rangle$ is outside every Tv , $v \in y$. Hence $\langle u, v \rangle \notin Ty$.

The third claim is immediate from the first claim. \square

LEMMA IB6.3. The following is provable in $T_3(W_1, W_2)$. Let $x \in \text{OW}_2$ and T be the L-system on x . If $v \in x$ then $Tv \in W_2$. If $v \in x$ and $v \in \text{OW}_1$ then $Tv \in W_1$. If $x \in \text{OW}_2$ then $T \in W_2$. If $x \in \text{OW}_1$ then $T \in W_1$.

Proof: For the first claim, first note that by Lemma IB6.1, Tv does not depend on T . So let $v \in \text{OW}_2$ be least such that there exists an L-system on some x , $v \in x \in \text{OW}_2$, such that

$T_v \notin W_2$. Then T_v is an OW_2 -structure with $|T_v| \in OW_2$ (Lemma IB6.2) which is definable without mentioning W_1 , and so according to Lemma IB5.3 iii, $T_v \in W_1$. This is a contradiction.

For the second claim, we see from Lemma IB6.2 that $T_v \subseteq \max(\omega, P^*(v)) \leq 2$. Hence T_v is defined as a subset of $\max(\omega, P^*(v)) \leq 2$ without mentioning W_1 but with parameters for elements of W_1 . And by Lemma IB4.7, $P^*(v) \in OW_1$. Hence Lemma IB3.7 iii applies. So $f(v) \in W_1$.

For the third claim, let $x \in OW_2$ be least such that there exists an L-system T on x such that $T \notin W_2$. Then as usual, $x \in OW_1$. By Lemma IB6.2, $T \subseteq \max(\omega, P^*(x)) \leq 3$. By Lemma IB4.7, $P^*(x) \in OW_1$. Hence T is defined as a subset of W_1 without mentioning W_1 , and so exists as an element of W_2 by axiom 3. This is a contradiction.

For the fourth claim, let $x \in OW_1$. As above, $T \subseteq \max(\omega, P^*(x)) \leq 3$ and $P^*(x) \in OW_1$. Hence t is defined as a subset of $\max(\omega, P^*(x)) \leq 3$ without mentioning W_1 , but with parameters from W_1 (namely x). Then $T \in W_1$ by Lemma IB3.7 iii.

LEMMA IB6.4. The following is provable in $T_3(W_1, W_2)$. For all $x \in OW_2$ there exists a unique L-system T on x . For all $y \in x$, T_y is an extensional structure.

Proof: Note that uniqueness follows from Lemma IB6.2. So it suffices to prove that for all $x \in OW_2$, there exists an L-system on x all of whose cross sections are extensional. Let $x \in OW_2$ be least such that there is no L-system T on x , all of whose values are extensional. Clearly $x \neq 0$ since the empty function is the L-system on 0. First suppose x is a limit. Now for each $y \in x$ let T^y be the unique L-system with domain y . By Lemma IB6.1, these T^y are all comparable in the strong sense that they have the same appropriate cross sections. Now the union of the T^y , $y \in x$, if it exists, must have the same appropriate cross sections as the T^y , $y \in x$. So this union, if it exists, must be the L-system on x . So in the case of x being a limit, it suffices to prove the

existence of this union. By Lemma IB6.2, for all $y \in x$, $T^y \subseteq \max(\omega, P^*(x)) \leq 3$. Hence this union can be defined as a subset of W_2 without mentioning W_1 . Therefore the union exists by Lemma IB1.1 v, and we have the required contradiction.

Finally suppose $x = y^+$, and let T be the L-system on y all of whose values are extensional. We wish to extend T to T' so that T' is the L-system on x , all of whose cross sections are extensional, thereby obtaining the required contradiction. If $y = 0$ then define $T' = \emptyset$. It is easy to see that T' is the L-system on 1 (and also the L-system on 0). Assume first that $y = z^+$. Then $y, z \in OW_1$ by definability considerations. Also the extensional OW_2 -structure T_z is definable without mentioning W_1 and $|T_z| \in OW_2$, by Lemma IB6.2. Hence by Lemma IB5.5, $T_z^* \in W_1$ and T_z^* is extensional. And by Lemma IB6.3, $T \in W_1$. Now take $T' = T \cup \{ \langle y, u \rangle : u \in T_z^* \}$. This union exists by axiom 3 since it defines a subset of W_1 without mentioning W_1 . This is the required contradiction.

Finally, assume that $x = y^+$ where y is a limit. We again have $T \in W_1$ and $x, y \in OW_1$. By Lemma IB6.3, for all $z \in y$, $T_z \in W_1$ and T_z is extensional. By Lemma IB6.2 ii, these T_z are comparable under \subseteq^* . Therefore the union R of the T_z , $z \in y$, if it exists, is an extensional OW_2 -structure. Now by Lemma IB6.3, each T_z , $z \in y$, lies in W_1 . Therefore this union, R , is defined as a subset of W_1 without mentioning W_1 , and so exists in W_2 by axiom 3. By Lemma IB1.1 vi, $R \in W_1$. Now take $T' = T \cup \{ \langle y, u \rangle : u \in R \}$. This union exists by axiom 3 since it defines a subset of W_1 without mentioning W_1 . It is easy to check that T' is the L-system on x all of whose cross sections are extensional. This is the required contradiction.

□

We now introduce some notation. For $A \subseteq OW_2$ write $|A|$ for the least P -closed limit $x \in OW_2$ such that $A \subseteq x$, and $||A||$ for the least $x \in OW_2$ such that $A \subseteq x$. $|A|$ exists if and only if $||A|| \in OW_2$.

For $x \in OW_2$ we define $A[x] = \{ u : u \in' f(x) \}$ and $R[x] = \{ \langle u, v \rangle : \langle u, v \rangle \in f(x) \}$, where f is the L-system for x^+ . We define $A[OW_2] = \{ u : (\exists x \in OW_2) (u \in A[x]) \}$ and $R[OW_2] = \{ v : (\exists x$

$\in OW2$) ($v \in R[x]$)}. But we need to know that these sets exist. The definition is well formed because it is independent of the choice of f by Lemma IB6.1.

LEMMA IB6.5. The following are provable in $T3(W1, W2)$.

- i) for all $x \in OW2$, $A[x], R[x], A[x]^2, A[x] \cup R[x]$ exist and are $\in W2$;
- ii) for all $x \in OW1$, $A[x], R[x], A[x]^2, A[x] \cup R[x] \in W1$;
- iii) for all $x \in OW2$, $A[x] \subseteq OW2$ and $R[x] \subseteq A[x]^2$;
- iv) for all $x \in OW2$, $|A[x]| \in \max(\omega, P^*(x))$;
- v) for all $x \in OW2$, $||A[x]|| = x$;
- vi) $A[0] = R[0] = A[1] = R[1] = \emptyset$;
- vii) for all $x \in OW2$, $A[x+] \cup R[x+] = (A[x] \cup R[x])^*$;
- viii) for all limits $x \in OW2$, $A[x] = \{y: (\exists z \in x)(y \in A[z])\}$, $R[x] = \{y: (\exists z \in x)(y \in R[z])\}$;
- ix) for all $x \in OW2$, there is a unique satisfaction relation on $A[x] \cup R[x]$, and it lies in $W2$;
- x) for all $x \in OW1$, the satisfaction relation on $A[x] \cup R[x]$ lies in $W1$;
- xi) for all $x \in OW2$ and w , $\{a: \langle a, w \rangle \in R[x]\}$ exists and is $\in W2$;
- xii) for all $x \in OW1$ and w , $\{a: \langle a, w \rangle \in R[x]\} \in W1$;
- xiii) for all $x \in OW2$ and $y \in OW2$, $A[x] \subseteq A[y]$ and $R[x] \subseteq R[y]$;
- xiv) for all $x \in OW2$ and a, b , $\langle a, b \rangle \in R[x] \leftrightarrow \langle a, b \rangle \in R[OW2]$;
- xv) for all $x \in OW2$ and $a \in b \in OW2$, if $b \in A[x]$ and $a \in A[OW2]$ then $a \in A[x]$;
- xvi) for all $x \in OW2$ and $y, z \in A[x]$, if for all $a \in A[x]$, $\langle a, y \rangle \in R[x]$ if and only if $\langle a, z \rangle \in R[x]$, then $y = z$.

Proof: We first prove that for all $x \in OW2$ and L-systems T on $x+$, the two components of Tx exist; i.e., the elements of Tx in $OW2$ and the elements of Tx in $OW2^2$, separately exist. This is proved by taking the least counterexample $x \in OW2$ and then constructing these two sets as subsets of $W1$, using Lemma IB6.3. Thus for all $x \in OW2$, $A[x], R[x]$ exist and are just the components of Tx , where T is the L-system on $x+$. Also by Lemma IB1.1 v, $A[OW2], R[OW2]$ exist. The rest of this is straightforward, using Lemmas IB5.4, IB5.5, IB3.6 and IB3.7. \square

7. Axioms of ZFC

We now aim to prove in $T3(W1, W2)$ that $(A[OW2], R[OW2])$ satisfies the axioms of ZFC, formulated in $L(\in)$. This fact can easily be stated in $T3(W1, W2)$ using Lemma IB3.11.

We let $SAT(A, R, n)$ be $(\forall x \in FSC)$ (if x is adequate for n in A, R then $SAT(A, R, n, x)$).

Externally, let φ be a formula in $L(\in)$ and let t be the closed S -term representing the index of φ ; i.e., $S...S(0)$, where the number of S 's is the index of φ . Also let $x \in= OW2$. We define the formula $\varphi^{(x)}$ in $L(\in, W1, W2)$ to be the result of relativizing the quantifiers to $A[x]$ and replacing each $x \in y$ with $\langle x, y \rangle \in R[x]$ and then expanding into primitive notation. In general, $W1$ will never appear in $\varphi^{(x)}$, but $W2$ will.

We say that w is x -adequate for n if and only if

- i) $n \in Fmla$;
- ii) $w \in FSC$;
- iii) $\#n \in lth(w)$;
- iv) $(\forall m \in lth(w)) (w[m] \in A[x])$.

This is a focusing of the previous definition of adequacy made after Lemma IB5.1 to fit the present context. Recall the previous definition of SAT within $T3(W1, W2)$.

LEMMA IB7.1. Let $\varphi(x_0, \dots, x_k)$ be a formula in $L(\in)$ and t be the closed S -term representing the index of φ . The following is provable in $T3(W1, W2)$. Let $x \in= OW2$ and w be adequate for t in $A[x]$. Then $SAT(A[x], R[x], t, w) \leftrightarrow \varphi^{(x)}(w[0], \dots, w[k])$.

Proof: By induction on the complexity of φ .

LEMMA IB7.2. Let φ be a sentence in $L(\in)$ and t be the closed S-term representing the index of φ . The following is provable in $T3(W1, W2)$. $x \in OW2 \rightarrow SAT(A[x], R[x], t) \leftrightarrow \varphi^{(x)}$.

Proof: From Lemma IB7.1.

Lemma IB7.2 supports the following way of proving that a specific sentence holds in $A[x], R[x]$, formulated in terms of SAT. One simply informally argues for the relativization of the sentence within $T3(W1, W2)$.

LEMMA IB7.3. Let φ be a formula in $L(\in)$ and t be the closed S-term representing the index of φ . The following is provable in $T3(W1, W2)$. Let $x \in OW2$ and w be adequate for t in $A[x]$. Then $SAT(A[x], R[x], t, w) \leftrightarrow \varphi(x)$.

Proof: By induction on the number of occurrences of variables in φ . \square

LEMMA IB7.4. The following is provable in $T3(W1, W2)$. $x \in A[OW2] \rightarrow (x \in A[I(x)+] \ \& \ x \notin A[I(x)])$. $\langle x, y \rangle \in R[OW2] \rightarrow I(x) \in I(y)$.

Proof: The first claim is proved by transfinite induction. Suppose this is true for all $x \in A[y]$, $y \in OW2$. Let $x \in A[y+]$. By the construction of $A[y+]$, either $x \in A[y]$, in which case we are done, or $I(x) = y$. In the latter case, $x \in A[I(x)+]$. It remains to see that $x \notin A[I(x)]$. Suppose $x \in A[I(x)] = A[y]$. Then by induction hypothesis, $x \notin A[I(x)]$, which is the required contradiction. The second claim is clear by transfinite induction. \square

1. Extensionality. Let Ext be the closed S-term representing the index of this sentence.

2. Pairing. Let $Pair$ be the closed S-term representing the index of this sentence.

LEMMA IB7.5. The following is provable in $T3(W1, W2)$. $SAT(A[OW2], R[OW2], Ext)$. $SAT(A[OW2], R[OW2], Pair)$.

Proof: The first claim follows from Lemma IB6.5 xvi. For the second claim, let $x, y \in A[z]$. Then the pair in $(A[OW2], R[OW2])$ lies in $A[z+]$.

3. Union. Let Union be the closed S-term representing the index of this formula.

4. Foundation. Let Found be the closed S-term representing the index of this formula.

LEMMA IB7.6. The following is provable in $T3(W1, W2)$.
 $SAT(A[OW2], R[OW2], Union)$. $SAT(A[OW2], R[OW2], Found)$.

Proof: Let $x \in A\{OW2\}$. By Lemma IB7.4, in $([A[OW2], R[OW2]])$, the elements of the elements of x all lie in $A[I(x)]$, and so the union lies in $A[I(x)+]$. The second claim follows from IB7.4. \square

5. Infinity. Let Inf be the closed S-term representing the index of this formula.

LEMMA IB7.7. The following is provable in $T3(W1, W2)$.
 $SAT(A[OW2], R[OW2], Inf)$.

Proof: Show by induction on $n \in \omega$ that there is a unique map $f: n \rightarrow OW2$ such that

- i) if $0 \in x$ then $f(0)$ is the empty set of $(A[OW2], R[OW2])$;
- ii) for all $m+ \in n$, $f(m+)$ is the $f(m) \cup \{f(m)\}$ of $(A[OW2], R[OW2])$.
- iii) for all $m \in n$, $f(m) \in A[m+]$.

Then argue by induction that for all n , $f(n)$ is \in -connected and transitive in $A[\omega], R[\omega]$. Next show, using Lemma IB7.6, that for all $n \in \omega$, the \in -connected transitive sets of $([A[\omega], R[\omega])$ lying in $A[n]$ are exactly the $f(m)$, $m \in n$. Hence the \in -connected transitive sets of $(A[\omega], R[\omega])$ are exactly the range of f . Therefore the ω of $(A[OW2], R[OW2])$ lies in $A[\omega+1]$. \square

6. Bounded separation. We make the following definition in $T3(W1, W2)$. Let m be the index of a formula φ of $L(\in)$ and let n be the smallest index of a variable other than x_0, x_1 that does not appear in φ . Let $B(m)$ be the index of the formula φ' resulting from relativizing all quantifiers to membership in x_n . Let $Bsep(m)$ be the index of the formula $(\forall x_1) \dots (\forall x_n) (\exists x_{n+1}) (\forall x_0) (x_0 \in x_{n+1} \leftrightarrow (x_0 \in x_n \ \& \ \varphi'))$.

LEMMA IB7.8. The following is provable in $T3(W1, W2)$. $m \in Fmla \rightarrow SAT(A[OW2], R[OW2], Bsep(m))$.

Proof: We argue in $T3(W1, W2)$. Let $m \in Fmla$. Let w be $OW2$ -adequate for $Bsep(m)$. We must show $SAT(A[OW2], R[OW2], Bsep(m), w)$. It suffices to show $SAT(A[OW2], R[OW2], Bsep'(m), w)$, where $Bsep'(m)$ is the result of deleting the block of universal quantifiers in front of $Bsep(m)$. Let n be the length of this block of universal quantifiers. Without loss of generality we can assume $lth(w) = n+$. It suffices to show that there exists $y \in A[OW2]$ such that for all $z \in A[OW2]$, $\langle z, y \rangle \in R[OW2]$ if and only if $\langle z, w[n] \rangle \in SAT(A[OW2], R[OW2], B(m), w[0/z])$. We can then put this in a form so that y can be chosen in $A[u+]$ where for all $m \in n+$, $w[m] \in A[u]$. \square

Since we have used bounded separation instead of separation, we need to verify the following strong form of replacement.

7. Collection. Let φ be a formula of $L(\in)$ and let n be the smallest index of a variable other than x_0, x_1 that does not appear in φ . $(\forall x_1) \dots (\forall x_{n-1}) (\exists x_n) (\forall x_0) ((x_0 \in x_n \ \& \ (\exists x_1) (\varphi)) \rightarrow (\exists x_1) (x_1 \in x_n \ \& \ \varphi))$. Define $Coll: Fmla \rightarrow Fmla$ by $Coll(m) =$ the index of the above formula where φ is the formula with index m .

LEMMA IB7.9. Let φ be a formula in $L(\in, W1, W2)$ without $W1$, in which x_0 is not free. The following is provable in $T3(W1, W2)$. $x_2, \dots, x_n \in OW2 \rightarrow (\exists x_0 \in OW2) (\forall x_1 \in x_2) ((\exists x_3 \in OW2) (\varphi) \rightarrow (\exists x_3 \in x_0) (\varphi))$.

Proof: Argue in $T3(W1, W2)$. Suppose this is false for some choice of $x_2, \dots, x_n \in OW2$, and let $x_2, \dots, x_n \in OW2$ be minimized.

Then x_2, \dots, x_n are defined without mentioning W_1 , and so $x_2, \dots, x_n \in OW_1$. According to Lemma IB3.7 ii, $(x_1 \in x_2 \ \& \ (\exists x_3 \in OW_2) (\varphi)) \leftrightarrow (x_1 \in x_2 \ \& \ (\exists x_3 \in OW_1) (\varphi[W_2/W_1])) \leftrightarrow (x_1 \in x_2 \ \& \ (\exists x_3 \in OW_1) (\varphi))$. The second equivalence is more easily seen by reversing the order. Thus we can take $x_0 = OW_1$ and obtain the required contradiction. \square

LEMMA IB7.10. The following is provable in $T_3(W_1, W_2)$. $m \in Fml_a \rightarrow SAT(A[OW_2], R[OW_2], Coll(m))$.

Proof: Use Lemma IB7.10. We can argue semiformaly as in Lemma IB7.8, starting with an OW_2 -adequate w for $Coll(m)$. we omit the details. \square

8. Power set. Let Pow be the closed term representing the index of this sentence.

This is the most interesting case, and we draw on a new idea. We need to develop a strong kind of absoluteness of the L-systems. In particular, we need to generalize L-systems to strong ordinals without mentioning W_1 or W_2 . Recall the important comparability result, Lemma IB3.2, which says that strong ordinals are \in -comparable with OW_2 . We use this to establish that general L-systems on strong ordinals are comparable with $A[OW_2], R[OW_2]$.

We say that x is a very strong ordinal if and only if

- i) x is a strong ordinal;
- ii) x_2 exists;
- iii) the ordered pair of any two elements of x exists;
- iv) there is a unique pairing function f on x ; and $f: x \rightarrow x$;
- v) there is a unique pairclosure system f, g on x ; and $g: x \rightarrow x$;
- vi) no element of x is an ordered pair;
- vii) finite sequence coding relative to the unique pairclosure system exists and has all of the relevant properties;
- viii) there is no \in -greatest element of x .

Let x be a very strong ordinal. An x -structure is a set R consisting of elements of x and x_2 , where $(\forall a, b) \langle a, b \rangle \in R \rightarrow$

$(a, b \in R \ \& \ a, b \in x)$). We define x^* in the appropriate way as in the case of OW_2 -structures, using the finite sequence coding given by vii) above.

We say that T is a generalized L-system on x if and only if

- i) x is a very strong ordinal;
- ii) $T \subseteq x^2$;
- iii) for all $y \in x$, T_y exists and is an x -structure;
- iv) if $0 \in x$ then $T_0 = \emptyset$;
- v) if $y \in x$ is a limit then $f(y)$ is the union of the T_z , $z \in y$;
- vi) if $y^+ \in x$ then $T_{y^+} = T_{y^*}$;
- vii) the sets $T/1 = \{y \in x : (\exists z \in x) (y \in T_z)\}$ and $T/2 = \{y \in x^2 : (\exists z \in x) (y \in T_z)\}$ exist, $(T/1)^2$ exists, and $T/2 \subseteq (T/1)^2$.

LEMMA IB7.11. The following is provable in $T_3(W_1, W_2)$.

- i) let $x \in OW_2$ be 0 or a limit. The L-system on x is the unique generalized L-system T on x . $T/1 = A[x]$ and $T/2 = R[x]$;
- ii) let x be a very strong ordinal such that $OW_2 \in x$, and let T be a generalized L-system on x . Then for all $y \in OW_2$, $T_y = A[y] \cup R[y]$.

Proof: Obviously every $x \in OW_2$ that is 0 or a limit is a very strong ordinal. The rest of the first claim is by transfinite induction. The second claim is also proved by transfinite induction. The unique pairclosure system on x is used to control the L-systems. \square

LEMMA IB7.12. The following is provable in $T_3(W_1, W_2)$.
 $SAT(A[OW_2], R[OW_2], Pow)$.

Proof: Let $x \in OW_2$ be least such that x has no power set in $A[OW_2]$ in the sense of $A[OW_2], R[OW_2]$. Then as usual, $x \in OW_1$, and so $rk(x) \in OW_1$. We write $y \subseteq' x$ to indicate that $y \subseteq x$ in the sense of $A[OW_2], R[OW_2]$, which is the same as saying that for all a , if $\langle a, y \rangle \in R[OW_2]$ then $\langle a, x \rangle \in R[OW_2]$. This is also the same as saying that for all $a \in OW_1$, if $\langle a, y \rangle \in R[OW_2]$ then $\langle a, x \rangle \in R[OW_2]$.

Now we claim that there exists $w \in OW2$ such that $w \subseteq' x$ and $w \notin OW1$. For otherwise, we would get the power set of x in the sense of $A[OW2], R[OW2]$ in $A[OW1+]$, contradicting the choice of x .

Now fix w to be the least such $w \in OW2$. Note that w can be alternatively defined as the unique w such that

there exists a very strong ordinal u and a
generalized L-system T on u such that
 $OW1 \in u$ and w is the least element of u
such that $w \subseteq x$ in the sense of $(T/1, T/2)$,
and $w \notin OW1$.

That this uniquely defines the same w follows from Lemmas IB7.13 and IB3.2.

Note that this definition of w does not mention $W2$. Instead, it uses x as a parameter and mentions $W1$. Therefore by Lemma IB1.1 iv there is also a unique w' such that

there exists a very strong ordinal u and a
generalized L-system T on u such that
 $OW2 \in u$ and w is the least element of u
such that $w \subseteq x$ in the sense of $(T/1, T/2)$,
and $w' \notin OW2$.

Recall that $x \in OW1$ and so $\langle a, x \rangle \in R[OW2] \rightarrow a \in OW1$. Hence by Lemma IB3.7 ii, we see that for all a , if $\langle a, x \rangle \in R[OW2]$ then $a \in w \leftrightarrow a \in w'$. Hence $w = w'$. But $w \in OW2$ and $w' \notin OW2$. This is the required contradiction. \square

We make the following definition in $T3(W1, W2)$. Let m be the index of a formula φ of $L(\in)$ and let n be the smallest index of a variable other than x_0, x_1 that does not appear in φ . Let $B(m)$ be the index of the formula φ' resulting from relativizing all quantifiers to membership in x_n . Let $RF(m)$ be the index of the formula $(\exists x_n)(\forall x_0) \dots (\forall x_{n-1})(\varphi \leftrightarrow \varphi')$. (RF for "reflection.")

LEMMA IB7.13. The following is provable in $T3(W1, W2)$. $\#(m) \rightarrow \text{SAT}(A[OW2], R[OW2], RF(m))$.

Proof: This is not the strongest result of this kind, but it suffices for our purposes. Argue by induction on the number of occurrences of variables in m . Lemma IB7.9 is used ω times to obtain a suitable ω -sequence. Then Lemma IB7.9 is used again to get an upper bound. \square

9. Separation. Let φ be a formula of $L(\in)$ and let n be the smallest index of a variable other than x_0, x_1 that does not appear in φ . $(\forall x_1) \dots (\forall x_n) (\exists x_{n+1}) (\forall x_0) (x_0 \in x_{n+1} \leftrightarrow (x_0 \in x_n \ \& \ \varphi))$. Define $\text{Sep}: \text{Fmla} \rightarrow \text{Fmla}$ by $\text{Sep}(m) =$ the index of the above formula where φ is the formula with index m .

LEMMA IB7.14. The following is provable in $T3(W1, W2)$. $m \in \text{Fmla} \rightarrow \text{SAT}(A[OW2], R[OW2], \text{Sep}(m))$.

Proof: From Lemmas IB7.13 and IB7.8. \square

LEMMA IB7.15. The following is provable in $T3(W1, W2)$. $(A[OW2], R[OW2])$ satisfies ZF.

Proof: By Lemmas IB7.5, IB7.6, IB7.7, IB7.8, IB7.10, IB7.12, and IB7.14. Strictly speaking, Replacement is used instead of Collection in ZF. But the triviality that Replacement follows from Collection in the presence of Separation can be lifted to this context and formalized within $T3(W1, W2)$.

We now come to the axiom of choice, which is the most complicated of the verifications of the axioms of ZFC. But now that we have ZF in $(A[OW2], R[OW2])$, we can rely heavily on previous developments of the constructible universe within ZF, due to Gödel.

This is a good point to talk about the soundness theorem, which we have avoided doing up to now.

We have been using the notions of satisfaction relation and SAT for OW2-structures. Here the domain lives within OW2. In this context, we have no trouble stating and proving the following soundness theorem:

LEMMA IB7.16. The following is provable in $T_3(W_1, W_2)$. Let R be an OW_2 -structure with a satisfaction relation. Then R satisfies every valid sentence of the usual Hilbert style axioms and rules of inference for $L(\in)$.

Proof: By suitably formalizing the standard proof of soundness of predicate calculus. \square

Lemma IB7.16 will allow us to argue that if a sentence is provable in ZF then it is satisfied by $(A[OW_2], R[OW_2])$, as well as related facts.

LEMMA IB7.17. The following is provable in $T_3(W_1, W_2)$. For each $x \in OW_2$ there is a unique $o(x) \in A[OW_2]$ such that $I(o(x)) = x$ and $(A[OW_2], R[OW_2])$ satisfies that $o(x)$ is an ordinal. Furthermore, $o(x) \in A[x^+]$, $o(x) \notin A[x]$.

Proof: Uniqueness follows from the fact that $(A[OW_2], R[OW_2])$ satisfies ZF and therefore $(A[OW_2], R[OW_2])$ satisfies that the ordinals are \in -connected. Suppose existence is true for all $x \in y$, where $y \in OW_2$ is a limit. Now use "being an ordinal" over $A[y]$ in order to obtain $o(y) = z$ such that $I(z) = y$ and $(A[OW_2], R[OW_2])$ satisfies z is an ordinal. \square

We will use the notation $o(x)$ below.

We now make the following standard definition within ZF. We define $L(0) = \emptyset$, $L(\alpha+1) =$ the set of all subsets of $L(\alpha)$ that are first order definable over $L(\alpha)$ with parameters allowed for elements of $L(\alpha)$, $L(\lambda) =$ the union of the $L(\beta)$, $\beta < \lambda$.

LEMMA IB7.18. The following is provable in $T_3(W_1, W_2)$. Let $x \in OW_2$, $y \in A[OW_2]$, and $(A[OW_2], R[OW_2])$ satisfies " $y = L[o(x)]$," Then $(\forall v) (\langle v, y \rangle \in R[OW_2] \leftrightarrow v \in A[y])$. $(A[OW_2], R[OW_2])$ satisfies ZF + $V=L$.

Proof: Argue by transfinite induction on $x \in OW_2$ within $T_3(W_1, W_2)$. The crucial case is the successor case, $x = u^+$. By induction hypothesis, the $L(u)$ in $(A[OW_2], R[OW_2])$ is the element representing the set $A[u]$. Let $L(o(u)^+)$ be the $L(o(u)^+)$ of $(A[OW_2], R[OW_2])$. Then the elements of $L(o(u)^+)$ in the sense of $(A[OW_2], R[OW_2])$ are exactly the elements of $A[OW_2]$ whose extension is first order definable over

$(A[u], R[u])$. This is because the formalization of "first order definable over" in ZF is accurate. But these are exactly the elements of $A[u^+]$. The last claim is now obvious.

10. Choice. Let Choice be the index of the axiom of choice.

LEMMA IB7.19. The following is provable in $T_3(W_1, W_2)$.
 $SAT(A[OW_2], R[OW_2], \text{Choice})$.

Proof: By Lemma IB7.18. \square

LEMMA IB7.20. The following is provable in $T_3(W_1, W_2)$.
 $(A[OW_2], R[OW_2])$ satisfies ZFC + $V=L$.

Proof: By Lemmas IB7.15, IB7.18, and IB7.19. \square

8. Indescribable cardinals, subtle cardinals, and \in -models

We will first show that $o(OW_1)$ is a strongly inaccessible cardinal in $(A[OW_2], R[OW_2])$.

LEMMA IB8.1. The following is provable in $T_3(W_1, W_2)$. $o(OW_1)$ is an uncountable regular cardinal in $(A[OW_2], R[OW_2])$.

Proof: If $o(OW_1)$ is a regular cardinal in $(A[OW_2], R[OW_2])$ then surely it is an uncountable regular cardinal in $(A[OW_2], R[OW_2])$, because $\omega \in OW_2$ and $o(OW_1)$ serves as the first infinite ordinal in $(A[OW_2], R[OW_2])$.

Now suppose $o(OW_1)$ is not a regular cardinal in $(A[OW_2], R[OW_2])$. Since $(A[OW_2], R[OW_2])$ satisfies ZFC, there exists $w \in A[OW_2]$ such that in $(A[OW_2], R[OW_2])$, w is satisfied to be an unbounded map from an element of $o(OW_1)$ into $o(OW_1)$. I.e., $(\exists b \in OW_1) (\exists w \in A[OW_2]) ((A[OW_2], R[OW_2]) \text{ satisfies that } w \text{ is an unbounded map from } o(b) \text{ into } o(OW_1))$. Choose α to be least, and then choose w to be least.

Then w can be defined as the unique w such that

$$(\exists u) (\exists T) (u \text{ is a very strong ordinal, } T \text{ is a generalized L-system on } u,$$

$OW1 \in u$, and $w \in u$ is least such that
 $(T/1, T/2)$ satisfies that w is an unbounded
map from $o(b)$ into $o(OW1)$.

Note that this definition of x does not mention $W2$ but uses b as a parameter and mentions $W1$. Therefore by Lemma IB1.1 iv there is also a unique w' such that

$(\exists u) (\exists T)$ (u is a very strong ordinal,
 T is a generalized L-system on u ,
 $OW2 \in u$, and $w' \in u$ is least such that
 $(T/1, T/2)$ satisfies that w' is an unbounded
map from $o(b)$ into $o(OW2)$).

We can view w as defining an unbounded map from b into $OW1$ and w' as defining an unbounded map from b into $OW2$.

According to Lemma IB3.7 ii, we see that for all $v \in \alpha$ and $v' \in OW1$, $w(b) = v' \leftrightarrow w'(b) = v'$. Hence for all $v \in b$, $w(b) = w'(b)$. This is the desired contradiction. \square

LEMMA IB8.2. The following is provable in $T3(W1, W2)$. $o(OW1)$ is a strongly inaccessible cardinal in $(A[OW2], R[OW2])$.

Proof: By Lemma IB8.1, it remains to show that $o(OW1)$ is a strong limit. Fix $x \in OW1$ and consider the cardinal of the power set of x according to $(A[OW2], R[OW2])$. This makes appropriate sense since the latter is a model of ZFC. Now this cardinal is some $y \in OW2$ which is defined without mentioning $W1$ using the parameter x . Hence by Lemma IB3.7 ii, $y \in OW1$. \square

Using this technique, we can go somewhat farther - say into higher order inaccessibles. However, to get into indescribable territory we must use $V=L$ in $(A[OW2], R[OW2])$.

We say that T is a distinguished generalized L-system on x if and only if

- i) T is a generalized L-system on x ;
- ii) $(T/1, T/2)$ has a satisfaction relation;
- iii) $(T/1, T/2)$ satisfies $ZFC + V=L$;
- iv) if T' is a generalized L'-system on x' then T and T' are comparable in the sense that:
- v) $x \in x'$ or $x' \in x$ or $x = x'$; and

vi) for all y in both x and x' , $T[y] = T'[y]$.

LEMMA IB8.3. The following is provable in $T3(W1, W2)$. The L-system on $OW2$ is the unique distinguished generalized L-system T on $OW2$.

Proof: From Lemmas IB7.20 and IB3.2. \square

In $ZFC + V=L$, there is the standard construction of a well ordering of the universe, $<c$. This ordering is defined by transfinite recursion on each $L(\alpha)$, where all elements of $L(\alpha)$ come before all other sets, and where $<c$ is defined on $L(\alpha+1)$ is defined in terms of the least finite sequence of parameters need to define the elements over $L(\alpha)$; the finite sequences of parameters are in turn ordered according to the lexicographic ordering over $<c$ defined on $L(\alpha)$. Thus in order to compare two sets under $<c$, we need only look at the first place in the constructible hierarchy where these sets appear.

LEMMA IB8.4. The following is provable in $T3(W1, W2)$. Let T be a distinguished generalized L-system for u and T' be a distinguished generalized L-system for u' . Then the $<c$ of $(T1, T2)$ and the $<c$ of $(T'1, T'2)$ are comparable.

Proof: From the above remarks, the comparability of $(T1, T2)$ and $(T'1, T'2)$ given by the definition of distinguished generalized L-systems carries over to the comparability in question. \square

LEMMA IB8.5. The following is provable in $T3(W1, W2)$. Let T be a distinguished generalized L-system on u . Furthermore let $x \in OW2$, $x \in u$, $y \in T/1$, and $y \subseteq o(x)$ hold in $(T/1, T/2)$. Then $y \in A[OW2]$.

Proof: Suppose this is false and fix $x \in OW2$ be least such that this is false for some u, T . Then x is definable without mentioning $W1$, and so $x \in OW1$. We claim that there is an ϵ -least y such that there exists a distinguished generalized L-system T on some u such that

*) $x \in u$, $y \in T/1$, $y \subseteq o(x)$ holds in
 $(T/1, T/2)$, and $y \notin A[OW2]$.

This is by the comparability property of distinguished generalized L-systems. We fix this necessarily unique y . Also fix a strong ordinal u and a distinguished generalized L-system on u such that $*$) holds. Now since $(T/1, T/2)$ satisfies $ZFC + V=L$, we see that $(T/1, T/2)$ satisfies "there is a one-one map f from $o(x)$ onto $o(OW2)$." (This is according to the main facts Gödel established in order to prove the generalized continuum hypothesis in $ZFC + V=L$). Now within $(T/1, T/2)$ we can minimize this f with respect to $<c$. According to Lemma IB8.4, the minimized internal f that we obtain in this process is independent of the choice of u and distinguished generalized L-system satisfying $*$). From this it follows that f is defined without mentioning $W1$. Now f can also be viewed as a map f' from x onto $OW2$. Therefore f' is defined without mentioning $W1$. But this contradicts Lemma IB7.9. \square

LEMMA IB8.6. The following is provable in $T3(W1, W2)$. Let T be a distinguished generalized L-system on u , where $OW2 \in u$. Let $x \in OW2$. Then the $V(o(x))$ of $(T/1, T/2)$ is the same as the $V(o(x))$ of $(A[OW2], R[OW2])$.

Proof: By transfinite recursion on $x \in OW2$. The nontrivial case is the successor case $y+$. Here we take any internal map from $V(o(y))$ one-one onto an internal ordinal $o(x)$ in order to apply Lemma IB8.5 for subsets of $o(x)$. \square

Let $n \geq 1$ and κ be a cardinal. We say that κ is n -th order indescribable if and only if for all $R \subseteq V(\kappa)$ and first order sentence φ , if $(V(\kappa+n), \in, R)$ satisfies φ , then there is an $\alpha < \kappa$ such that $(V(\alpha+n), \in, R \cap V(\alpha))$ satisfies φ .

We say that κ is totally indescribable if and only if κ is n -th order indescribable for all $n \geq 1$. This definition agrees with the one given in, say,

A. Kanamori, *The Higher Infinite*, Springer-Verlag, 1994, p. 59.

It is natural to go a bit further, and so we say that κ is extremely indescribable if and only if for all $R \subseteq V(\kappa)$ and first order sentence φ , if $(V(\kappa+\kappa), \in, R)$ satisfies φ , then there is an $\alpha < \kappa$ such that $(V(\alpha+\alpha), \in, R \cap V(\alpha))$ satisfies φ .

LEMMA IB8.7. The following is provable in $T_3(W_1, W_2)$. $o(OW_1)$ is an extremely indescribable cardinal in $(A[OW_2], R[OW_2])$.

Proof: Suppose this is false. Then there exists x , and sentence φ such that the following holds in $(A[OW_2], R[OW_2])$:

$$x \subseteq V(o(OW_1)) \text{ and } (V(o(OW_1)+o(OW_1)), \epsilon, x) \text{ satisfies } \varphi,$$

but for all $z \in o(OW_1)$,

$$(V(o(z)+o(z)), \epsilon, x \cap V(o(z))) \text{ satisfies } \neg\varphi.$$

Choose φ to have least index, and then $x \in A[OW_2]$ to be $<c$ -least with this property.

Now x can be defined as the unique x such that

$$(\forall u) (\forall T) (\text{if } T \text{ is a distinguished generalized L-system on } u \text{ then } (\exists u') (\exists T') (u \in= u' \ \& \ T' \text{ is a distinguished generalized L-system on } u', \ OW_1 \in u', \text{ and } x \in u' \text{ is } <c\text{-least such that:} \\ (T'/1, T'/2) \text{ satisfies " } x \subseteq V(o(OW_1)) \text{ and } \\ (V(o(OW_1)+o(OW_1)), \epsilon, x) \text{ satisfies } \varphi, \text{" but} \\ \text{for all } z \in OW_1, \ T'/1, T'/2) \text{ satisfies} \\ \text{" } (V(o(z)+o(z)), \epsilon, x \cap V(o(z))) \text{ satisfies } \neg\varphi \text{"}).$$

To see that this definition is correct, we first check that this statement holds of the actual x . Let T be a distinguished generalized L-system on u . If $OW_2 \in= u$ then we set $u' = u$ and $T' = T$. Otherwise, we set $u = OW_2$ and $T =$ the L-system on OW_2 . By the suitable preservation of $<c$ and the V 's (Lemma IB8.6) as well as ordinal addition (left to the reader), we see that the statement holds of the actual x .

Now suppose the above statement holds of an x . We set $u = OW_2$ and $T =$ the L-system on OW_2 . Then argue by preservation as above in order to obtain that this x is the actual x .

Note that this definition of x does not mention W_2 but uses φ as a parameter and mentions W_1 . Therefore by Lemma IB1.1 iv there is also a unique x' such that

$(\forall u)(\forall T)$ (if T is a distinguished generalized L-system on u then $(\exists u')(\exists T')(u \in u' \ \& \ T'$ is a distinguished generalized L-system on u' , $OW_1 \in u'$, and $x' \in u'$ is $\langle c$ -least such that:
 $(T'/1, T'/2)$ satisfies " $x' \subseteq V(o(OW_1))$ and
 $(V(o(OW_1)+o(OW_1)), \in, x')$ satisfies φ ," but
for all $z \in OW_1$, $(T'/1, T'/2)$ satisfies
" $(V(o(z)+o(z)), \in, x' \cap V(o(z)))$ satisfies $\neg\varphi$ ").

From the original definition of x , we have $(A[OW_2], R[OW_2])$ satisfies " $x \subseteq V(o(OW_1)) \ \& \ (V(o(OW_1)+o(OW_1)), \in, x)$ satisfies φ ."

From the above definition of x' , we can set $z = OW_1$ and obtain $(T'/1, T'/2)$ satisfies " $(V(o(OW_1)+o(OW_1)), \in, x' \cap V(o(OW_1)))$ satisfies $\neg\varphi$." By choosing $u = OW_2$, we have $OW_2 \in u'$, and so by Lemma IB8.6, we see that $(A[OW_2], R[OW_2])$ satisfies " $(V(o(OW_1)+o(OW_1)), \in, x' \cap V(o(OW_1)))$ satisfies $\neg\varphi$."

We have our required contradiction provided we can show that in $(A[OW_2], R[OW_2])$, the equation $x' \cap V(o(OW_1)) = x$ holds. Now x' is defined in exactly the same way as x except with W_1 replaced by W_2 (and there is no mention of W_2 in the definition of x). Now by the strong inaccessibility of $o(OW_1)$ in $(A[OW_2], R[OW_2])$ given by Lemma IB8.2, we see that in $(A[OW_2], R[OW_2])$, the elements of $V(o(OW_1))$ are externally elements of OW_1 . The equation now follows from Lemma IB3.7
ii. \square

THEOREM IB8.8. The following is provable in $T_3(W_1, W_2)$.
 $(A[OW_2], R[OW_2])$ satisfies $ZFC + V=L +$ "there exists an extremely indescribable cardinal." The following is provable in T_2 . $ZFC + V=L +$ "there exists an extremely indescribable cardinal" is consistent. The following is provable in EFA (exponential function arithmetic). If T_2 is consistent then $ZFC + V=L +$ "there exists an extremely indescribably cardinal" is consistent.

Proof: By Lemmas IB7.20, IB8.7, and IB7.16. We also need the interpretation of $T3(W1, W2)$ in $T2(W1, W2)$ given in section A. $T2$ also has an adequate way of discussing arithmetic. \square

We would like to replace $(A[OW2], R[OW2])$ by a set model. We can either go the route of relativizations, avoiding a discussion of satisfaction relations. Or we can consider satisfaction relations. Here we have to be careful in that it is not assumed that the underlying set is made up of ordinals. So the finite sequence apparatus we have developed cannot be used. For this reason, we prefer to go the relativization route.

LEMMA IB8.9. The following is provable in $T3(W1, W2)$. There is a unique function f such that

- i) $\text{dom}(f) = A[OW2]$;
- ii) for all $x, y \in A[OW2]$, $\langle x, y \rangle \in R[OW2] \leftrightarrow x \in y$;
- iii) $\text{rng}(f)$ exists and is a transitive set.

Proof: Uniqueness is clear by transfinite induction and the fact that $(A[OW2], R[OW2])$ satisfies extensionality. We now prove that for all $x \in OW2$ there is a function f such that

- i) $\text{dom}(f) = A[x]$;
- ii) for all $y, z \in A[x]$, $\langle y, z \rangle \in R[OW2] \leftrightarrow f(y) \in f(z)$;
- iii) $r, \text{rng}(f) \in W2$, and $\text{rng}(f)$ is transitive.

Suppose this is false and let $x \in OW2$ be least such that it is false. Then x is defined without mentioning $W1$, and so $x \in OW1$. By Lemma IB6.5 ii), for all $y \in x$, $A[y] \in W1$. Now by induction hypothesis, for all $y \in x$, the unique f satisfying I)-iii) lies in $W2$, as well as its range. By axiom 3, for all $y \in x$, the unique f satisfying I)-iii) lies in $W1$. Hence for all $y \in x$, the unique f satisfying i)-iii) lies in $W1$, as well as its range. Clearly $x \neq 0$. Suppose x is a limit. Then the union of the f 's associated with the $y \in x$ defines a subset of $W1$ without mentioning $W1$. Hence it exists, and satisfies i)-iii) for x . This is the required contradiction.

Now suppose $x = y^+$ and let f satisfy i)-iii) with y . Then $f \in W1$. We want to extend f to the domain $A[x^+]$. Now each new element w of $A[x^+]$ is to be sent to a unique corresponding subset of $\text{rng}(f)$. This unique corresponding subset B_w of

$\text{rng}(f)$ is defined in the obvious way and lies in W_1 according to Lemma IB3.7 iii. But we also need to know that each $\langle w, Bw \rangle \in W_1$. To see this, suppose w is the \in -least element of $A[x^+]$ such that $\langle w, Bw \rangle \notin W_1$. Then w is defined without mentioning W_2 . Hence by axiom 3, w is defined without mentioning W_1 . Then we can form $\{w, Bw\}$ as a subset of W_1 defined without mentioning W_1 , and hence it exists. Now consider the statement about w that $\{w, Bw\} \in W_2$. This statement does not mention W_1 . Hence $\{w, Bw\} \in W_1$. The same argument works to show that $\{w\} \in W_1$ (something we already knew). Now we can repeat the argument to obtain the existence of $\{\{w\}, \{w, Bw\}\} = \langle w, Bw \rangle$ and see that it lies in W_1 . Hence we can define the set of all such pairs, together with the pairs in f , as a subset of W_1 without mentioning W_1 . This is the extension of f that satisfies I)-iii) for $x = y^+$ and it lies in W_2 . This is the required contradiction.

Finally, we can define the union of all the f 's for the various $x \in OW_2$ as a subset of W_2 without mentioning W_1 . This is the required function, and it and its range exists by Lemma IB1.1 ii.

THEOREM IB8.10. i) let φ be any conjunction of theorems of $ZFC + V=L +$ "there exists an extremely indescribable cardinal." The following is provable in T_2 . There is a set x such that the relativization of φ to x holds. The set can be taken to be a nonempty transitive set.

ii) there is a single explicit definition $\{x: \psi\}$, where ψ is in the language of T_2 , which provably defines a set in T_2 , such that the following holds: let φ be any conjunction of theorems of $ZFC + V=L +$ "there exists an extremely indescribable cardinal." Then the relativization of φ to $\{x: \psi\}$ is provable in T_2 . Furthermore, we can insist that $\{x: \psi\}$ is provably transitive in T_2 .

Proof: Lemma IB8.9 provides an isomorphism between $(A[OW_2], R[OW_2])$ and a transitive set. Apply Theorem IB8.8. We also use the interpretation of $T_3(W_1, W_2)$ in T_2 as given in section A. \square

We make the following definitions in ZFC . Let κ be a cardinal. We write $S(\kappa)$ for the set of all subsets of κ . We

say that $f:\kappa \rightarrow S(\kappa)$ is regressive if and only if for all $\alpha < \kappa$, $f(\alpha) \subseteq \alpha$. We say that $A \subseteq \kappa$ is closed unbounded if and only if the union of any bounded subset of A with no greatest element lies in A , and A is unbounded in κ . We say that κ is a subtle cardinal if and only if for all stationary $A \subseteq \kappa$ and regressive $f:\kappa \rightarrow S(\kappa)$, there exists $\alpha < \beta \in A$ such that $f(\alpha) = f(\beta)$. This definition is due to Kunen (unpublished).

Subtle cardinals are inaccessible. The first subtle cardinal is known to be greater than the first weakly compact cardinal, and even greater than the first extremely indescribable cardinal. It is smaller than the first cardinal which arrows ω , where we partition all of the finite subsets into finitely many pieces.

THEOREM IB8.11. The following is provable in ZFC + "there exists a subtle cardinal." There are three transitive sets $x \in y \in z$ such that every theorem of T2 holds in (z, \in) with W1 interpreted as x and W2 interpreted as y . T2 is consistent. The following is provable in EFA. If ZFC + "there exists a subtle cardinal" is consistent then T2 is consistent.

Proof: Let κ be a subtle cardinal. Define $f:\kappa \rightarrow S(\kappa)$ by $f(\alpha) = \{ \langle x, n \rangle : x \in V(\alpha) \text{ \& } n \text{ is the index of a sentence true in } (V(\kappa), \in) \text{ about } x, V(\alpha) \}$ provided α is a limit ordinal; 0 otherwise. Let $A \subseteq \kappa$ be the set of all limit ordinals $< \kappa$. By the subtlety of κ , fix $\alpha < \beta < \kappa$ to be limit ordinals such that $f(\alpha) = f(\beta)$. Then set $x = V(\alpha)$, $y = V(\beta)$, and $z = V(\kappa)$. \square

PART II. WITHOUT EXTENSIONALITY