FINITE PHASE TRANSITIONS

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DRAFT

This topic has been discussed earlier on the FOM email list in various guises. The common theme is: big numbers and long sequences associated with mathematical objects. See

http://www.cs.nyu.edu/pipermail/fom/1998-July/001921.html
http://www.cs.nyu.edu/pipermail/fom/1998-October/002332.html
http://www.cs.nyu.edu/pipermail/fom/1998-October/002339.html
http://www.cs.nyu.edu/pipermail/fom/1998-October/002356.html
http://www.cs.nyu.edu/pipermail/fom/1998-October/002365.html
http://www.cs.nyu.edu/pipermail/fom/1998-October/002383.html
http://www.cs.nyu.edu/pipermail/fom/1998-October/002395.html
http://www.cs.nyu.edu/pipermail/fom/1998-October/002407.html
http://www.cs.nyu.edu/pipermail/fom/1998-October/002409.html
http://www.cs.nyu.edu/pipermail/fom/1998-October/002410.html
http://www.cs.nyu.edu/pipermail/fom/1998-November/002439.html
http://www.cs.nyu.edu/pipermail/fom/1998-November/002443.html
http://www.cs.nyu.edu/pipermail/fom/1999-March/002752.html
http://www.cs.nyu.edu/pipermail/fom/1999-May/003134.html
http://www.cs.nyu.edu/pipermail/fom/1999-July/003251.html
http://www.cs.nyu.edu/pipermail/fom/1999-July/003253.html
http://www.cs.nyu.edu/pipermail/fom/2006-March/010292.html
http://www.cs.nyu.edu/pipermail/fom/2006-March/010293.html
http://www.cs.nyu.edu/pipermail/fom/2006-March/010290.html
http://www.cs.nyu.edu/pipermail/fom/2006-March/010279.html
http://www.cs.nyu.edu/pipermail/fom/2006-March/010281.html
http://www.cs.nyu.edu/pipermail/fom/2006-April/010305.html

Suppose we have a Π^0_2 theorem $(\forall k)(\exists n)(A(k,n))$. We then get a recursive function F(k) = the least n such that A(k,n) holds. In the intended cases, we have F(0) < F(1) < ... .

We want to look at F(0),F(1),... and determine where there is a "qualitative jump" in size. I.e., a "phase transition". In the cases we focus on, after the first few terms - say about 16 or less - we simply get qualitatively indistinguishable very large integers.

There are a number of forms that such results can take. Some are more descriptive than quantitative and others are more quantitative than descriptive.

QUANTITATIVE APPROACH.

In the quantitative approach, we simply provide upper and lower bounds on some of the terms in F(0),F(1),... using a notation system for integers. We try to make exact calculations, if possible.

But what notation system to use for the integers?

In case the numbers involved are less than, say, 10^{100}, the usual base 10 notation is the clear choice.

But if the numbers involved are greater than, say, $10^{10\wedge 100}$, base 10 notation is generally of no use whatsoever since it cannot even be presented.

There are many special approaches that may be particularly illuminating. However, we are led to the following general approach. We identify a finite list of constants and basic functions of one or more variables on the nonnegative integers. We then use closed terms to name particular nonnegative integers, which are used for lower and upper bounds. Of course, we should strive to use small terms - or natural terms - for this purpose.

This leads to a very interesting computational complexity problem. Suppose we start with a finite list of constants and basic functions. We can ask, for example, whether two terms, using 0, are equal or not, or how they compare under <. What is the computational complexity of this problem? Is there a systematic theory here that involves the structure of the integer notation system being used?

Let us now focus on a very natural system. We use constants

0,1, addition, multiplication, and exponentiation to any positive base. We can choose to use terms in these basics with at most, say, 100 symbols, or even say, just 16 symbols, before wishing to use a "more powerful" system. The obvious motivation for moving to a "more powerful system" is because the number in question is larger than anything given by a term of, say, 100 or maybe 16 symbols.

The above system - or some more convenient variant - is a good system for investigating the finite Ramsey numbers, or numbers in Adjacent Ramsey Theory.

We think of the above system as an integer notation system.

I used the integer notation system based on the Ackermann hierarchy of functions, in my work on Long Finite Sequences.

The Ackermann hierarchy of functions is encapsulated in terms of a binary function A(n,m), which is the n-th Ackermann function at m. (There are many ways to present this hierarchy and the Ackermann function itself. They are all minor variants of each other). See

H. Friedman, Long Finite Sequences, Journal of Combinatorial Theory, Series A 95, 102-144 (2001).

There I gave the lower bound $A_{7198}(158386) < n(3)$.

In terms of the binary Ackermann function, this lower bound reads

$$A(7198,158386) < n(3).$$

I didn't give an upper bound there, but I later conjectured

$$n(3) < A(A(5,5),A(5,5)).$$

So base 10 notation and the binary Ackermann function can serve as a reasonable notation system for integers.

For a unified approach to integers too large to be bounded by a reasonable sized term in the above notation system for integers, we can use ϵ_0 with its usual system of fundamental sequences. At successor ordinals, we use the indefinite iteration as in the Ackermann hierarchy. At limits, we use the fundamental sequence. The Ackermann hierarchy appears at the first ω levels, and the Ackermann function essentially appears at the ω-th level.

This provides a notation system for integers, using 0 and the binary function corresponding to the above Hardy hierarchy (or Wainer hierarchy). We may want to sugar it with base 10 notation.

For integers too large to be bounded by a reasonable sized term in this notation system for integers, we would use the obvious extension of this for larger proof theoretic ordinals.

But there remains the question: what do we mean by a qualitative jump in size? What is a phase transition in this context?

We can, of course, let the estimates speak for themselves. However, we may demand a more principled answer. We offer the following Qualitative Approach.

QUALITATIVE APPROACH.

In the qualitative approach, we look to formal systems for the more principled answer. In particular, we associate an integer to every formal system.

Now, there will be some ad hoc features involved in the associated integer. However, we Conjecture that there is a great deal of robustness here.

We call this associated integer - defined below - the PROOF THEORETIC INTEGER OF T.

We assume that T is

#) a formal system in a finite relational type in many sorted predicate calculus with equality, containing a sort for natural numbers, with 0,S,+,•,exp,<.

The Δ_0 formulas are defined as usual, using bounded quantifiers. The Sigma01 formulas are obtained from the Δ_0 formulas by putting zero or more existential quantifiers in front of Δ_0 formulas.

The proof theoretic integer of T is the least integer n such that every Σ^0_1 sentence that has a proof in T with at most 10,000 symbols, has witnesses less than n.

Of course, this definition needs some exact spelling out - e.g., what exact proof system is to be used, and what exactly counts as a symbol (what about parentheses), etcetera?

However, it is expected that there is a lot of robustness. Of course, not robustness in the form of the exact number being unchanged. But robustness in a more subtle sense.

In particular, we make the following robustness conjecture.

ROBUSTNESS CONJECTURE. Let S,T be two naturally occurring formal systems obeying #), which prove EFA (exponential function arithmetic). Suppose S proves the 1-consistency of T. Then the proof theoretic integer of S is at least a double exponential of the proof theoretic integer of T.

QUESTION: Can we use a significantly smaller number than 10,000 in the definition of the proof theoretic integer, and still have the robustness conjecture?

I used 10,000 because I want to accommodate some technically neat but entirely crude Hilbert style syst em, without any sugar.

Because this Conjecture has "naturally occurring", it takes on an experimental character. We Conjecture that there is a form of the Conjecture that can be proved, where we assume instead that the complexity of S,T is low, and the size of the proof of 1-consistency of T in S is also low.

We propose using proof theoretic integers of basic formal systems (such as EFA, 1 quantifier induction, two quantifier induction, PA, ACA_0, ACA, ATR_0, ATR, Π11-CA0).

AN IMPORTANT EXAMPLE.

A good source of examples is in the area surrounding Kruskal's theorem (starting with k = 0). We will not allow an empty tree. Here is my original finite form of Kruskal's theorem.

THEOREM. For all k ≥ 0 there exists n ≥ 0 such that the following holds. For all structured finite trees

$T_1,...,T_n$, where each T_i has at most i+k vertices, there exists i < j such that T_i is inf and structure preserving embeddable into T_j.

For each k ≥ 0, let n be least such that the Theorem holds.

F(0) = 2.
F(1) = 3.
F(2) = 6.
F(3) is greater than 58.
f(4) is greater than the proof theoretic integer of 1 quantifier induction.
F(5) is greater than the proof theoretic integer of PA.
F(6) is greater than the proof theoretic integer of ATR (or even ATR + Σ^1_1-DC).

We now sketch the ideas for these lower bounds. The first three are easy.

We use the following notation for structured trees. Th etree consisting only of its root is x. $(t_1,...,t_k)$, k ≥ 1, is the joining together of trees $t_1,...,t_k$ by a root.

For F(3), start with a tree with 4 vertices and no splitting. This is of height 4, and written (((x))). This is followed by (x,x,x,x), which is of height 2.

For trees of height at most 3, it is very convenient to use the following special notation. $a_1,...,a_k$, where k ≥ 1 and $a_1,...,a_k \geq 0$, is the tree where the root has k sons, and the i-th son has a_i sons.

We use the following construction.

(((x)))
(x,x,x,x) = 0,0,0,0
2,0,0
1,2,0
1,1,2
1,0,4

0,6,0
...
0,0,12
14,0
13,2
...
0,28
30
...
0

The length of this sequence of trees is 58. So F(3) is greater than 58.

For F(4), we use the following construction.

((((x))))
(((x,x,x)))
(((x,x),x,x))
(((x,x),x),(x))
(((x,x)),(x),(x))
...
(((x,x)),x,x,x,x,x,x)
...
(((x)),x,x,x,x,x,x,x,x,x,x)

with 10 consecutive x's. Note that we have the tree 0,0,0,0,0,0,0,0,0,0 conveniently present here. We continue by operating only on 0,0,0,0,0,0,0,0,0,0, arriving at

(((x)))

which has ordinal ω^ω. Since we are at some gigantic Ackermann level stage, we have F(4) > the proof theoretic integer of single quantifier induction.

For F(5), we use the following construction.

((((x))))
(((x,x,x,x)))
(((x,x,x),x,x))
(((x,x,x),x),(x))

(((x,x,x)),(x,x,x))
(((x,x,x)),(x,x),(x))
(((x,x,x)),(x),(x),(x))
...
(((x,x,x)),x,x,x,x,x,x,x,x,x)

Note that we have the tree 0,0,0,0,0,0,0,0,0 conveniently present here. We continue by operating only on 0,0,0,0,0,0,0,0,0, arriving at

(((x,x,x)))

after a gigantic stage in the construction, of Ackermann level size. Because of this 3 splitting, we are now free to work within the structured binary trees, starting with one of Ackermann level size. We proceed by descent, confirming the estimate F(4) > the proof theoretic integer of PA.

For F(6), we use the following construction.

(((((x)))))
(((x,x,x,x,x)))
...
(((x,x,x,x)))

as in the construction for F(5), after a gigantic stage in the construction, of Ackermann level size. Because of this 4 splitting, we are now free to work within the structured ternary trees, starting with one of Ackermann level size. We proceed by descent, confirming the estimate F(5) > the proof theoretic integer of ATR + Σ^1_1-DC, and of course, more.

We will not get into the upper bounds at this time.

However, appropriate upper bound results are needed to establish that we do indeed have the above suggested phase transitions.

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