A BIG DIFFERENCE BETWEEN INTERPRETABILITY AND DEFINABILITY IN AN EXPANSION OF THE REAL FIELD

HARVEY FRIEDMAN AND CHRIS MILLER

Preliminary Report*. November 6, 1998

Given an expansion $\mathfrak{R}$ of the real line and $E \subseteq \mathbb{R}$, let $(\mathfrak{R}, E)^\#$ denote the expansion of $\mathfrak{R}$ by a predicate for each subset of each cartesian product E^k ($k \geq 1$). We say that E is **$\mathfrak{R}$-sparse** if $f(E^k)$ has no interior, for each $k \in \mathbb{N}$ and $f : \mathbb{R}^k \to \mathbb{R}$ definable in $\mathfrak{R}$.

(Throughout, "definable" means "definable *without* parameters".)

In this note, we consider the extent to which basic metric and topological properties of subsets of $\mathbb{R}$ definable in $(\mathfrak{R}, E)^\#$ are determined by the corresponding properties of subsets of $\mathbb{R}$ definable in $(\mathfrak{R}, E)$, when $\mathfrak{R}$ is an o-minimal expansion of $(\mathbb{R}, <, +, 0, 1)$ and E is $\mathfrak{R}$-sparse. The precise statement of the main result is a bit complicated, but we can state some special cases now:

Let $\mathfrak{R}$ be an o-minimal expansion of $(\mathbb{R}, <, +, 0, 1)$ and $E \subseteq \mathbb{R}$ be $\mathfrak{R}$-sparse. If every subset of $\mathbb{R}$ definable in $(\mathfrak{R}, E)$ has interior or is nowhere dense, then the same holds true of $(\mathfrak{R}, E)^\#$; and similarly with "nowhere dense" replaced by any of "null" (in the sense of Lebesgue), "countable", or "a finite union of discrete sets".

Example. Putting $2^{\mathbb{Z}} := \{ 2^k : k \in \mathbb{Z} \}$ and using results of van den Dries [D1], we obtain that every subset of $\mathbb{R}$ (parametrically) definable in $(\mathbb{R}, +, \cdot, 2^{\mathbb{Z}})^\#$ is the union of an open set and finitely many discrete sets. Hence, although this structure interprets *every* countable first-order structure, it does not define $\mathbb{Q}$, and hence not $\mathbb{Z}$. (This settles a question that arose in discussions between T. Scanlon and Miller while at MSRI in the spring of 1998.)

From now on, $\mathfrak{R}$ denotes an expansion of $(\mathbb{R}, <, +, 0, 1)$ in a language L extending $\{<, +, 0, 1\}$, and E is a fixed subset of $\mathbb{R}$. Enlarging L as necessary, we assume (for convenience only) that there is a symbol in L for each definable (in $\mathfrak{R}$) function $\mathbb{R}^k \to \mathbb{R}$ and $k \in \mathbb{N}$. (We regard $\mathbb{R}^0$ as the one-point space $\{0\}$, and a function $f : \mathbb{R}^0 \to \mathbb{R}$ as the corresponding constant $f(0)$.)

For $m, n \in \mathbb{N}$, $X \subseteq \mathbb{R}^{m+n}$ and $u \in \mathbb{R}^m$, X_u denotes the fiber

$$\{ x \in \mathbb{R}^n : (u, x) \in X \}.$$

---

*__NOTE.__ Work in progress. Comments welcome. Please regard as personal correspondence, not a preprint. Thanks. -CLM & HMF

Definition. Let $n \in \mathbb{N}$. An $(\mathfrak{R}, E)^\infty$**-set** (in $\mathbb{R}^n$) is a set of the form

$$\bigcup_{\alpha \in I} \bigcap_{u \in P_\alpha} X_u$$

where
(i) $X \subseteq \mathbb{R}^{m+n}$ is definable in $\mathfrak{R}$ (for some $m \in \mathbb{N}$);
(ii) I is an index set;
(iii) each $P_\alpha \subseteq E^m$.
An $(\mathfrak{R}, E)^\infty$-set in $\mathbb{R}^{n+1}$ is **special** if $X \subseteq \mathbb{R}^{m+n+1}$ is defined by an L-formula of one of the following forms:
(1) $\varphi(u, x)$
(2) $\varphi(u, x)$ & $f(u, x) = t$
(3) $\varphi(u, x)$ & $f(u, x) < t$
(4) $\varphi(u, x)$ & $t < g(u, x)$
(5) $\varphi(u, x)$ & $f(u, x) < t < g(u, x)$
where φ is an $(m+n)$-ary L-formula and f, g are $(m+n)$-ary function symbols.

We'll refer to the special sets corresponding to the above five forms as being of types (1) through (5) respectively.
Let $(\mathfrak{R}, E)^\infty$ denote the expansion of $\mathfrak{R}$ obtained by adding a predicate for each $(\mathfrak{R}, E)^\infty$-set in each $\mathbb{R}^n$, $n \geq 1$. (Note that $(\mathfrak{R}, E)^\#$ is reduct of $(\mathfrak{R}, E)^\infty$.)

We can now state our main results.

Proposition 1. *Suppose that $\mathfrak{R}$ is o-minimal. Then for every $n \in \mathbb{N}$, every subset of $\mathbb{R}^{n+1}$ definable in $(\mathfrak{R}, E)^\infty$ is a finite union of special sets.*

(We delay the proof.)

Proposition 2. *Let $A \subseteq \mathbb{R}$ be special. Suppose that A has no interior. Then there exist $m \in \mathbb{N}$ and $f : \mathbb{R}^m \to \mathbb{R}$, definable in $\mathfrak{R}$, such that $A \subseteq \mathrm{cl}(f(E^m))$.*

Proof. If $\emptyset \neq A \subseteq \mathbb{R}$ is special and has no interior, then A is of type (2) or of type (5). If the former, then $A = f(P)$ for some $m \in \mathbb{N}$, $P \subseteq E^m$, and definable $f : \mathbb{R}^m \to \mathbb{R}$. If the latter, then

$$A = \bigcup_{\alpha \in I} \bigcap_{u \in P_\alpha} \{\, t \in \mathbb{R} : \varphi(u) \ \& \ f(u) < t < g(u) \,\}$$

where φ is an m-ary L-formula and $f, g : \mathbb{R}^m \to \mathbb{R}$ are definable. Since A has no interior, each set $\bigcap_{u \in P_\alpha} \{\, t \in \mathbb{R} : \varphi(u) \ \& \ f(u) < t < g(u) \,\}$ has no interior. Hence,

$$A \subseteq \mathbb{R} \cap \{\, \sup f(P_\alpha) : \alpha \in I \,\} \subseteq \mathrm{cl}(f(E^m)). \quad \square$$

Theorem. *Suppose that $\mathfrak{R}$ is o-minimal. Let $E \subseteq \mathbb{R}$ be such that $\mathrm{cl}(f(E^k))$ is nowhere dense, for each $k \in \mathbb{N}$ and definable $f : \mathbb{R}^k \to \mathbb{R}$. Then every subset of $\mathbb{R}$ definable in $(\mathfrak{R}, E)^\infty$ has interior or is nowhere dense.*

Corollary. *Suppose that $\mathfrak{R}$ is o-minimal and E is $\mathfrak{R}$-sparse. Then the following are equivalent:*

(1) *Every subset of $\mathbb{R}$ definable in $(\mathfrak{R}, E)^\infty$ has interior or is nowhere dense.*
(2) *Every subset of $\mathbb{R}$ definable in $(\mathfrak{R}, E)$ has interior or is nowhere dense.*
(3) *For all $k \in \mathbb{N}$ and functions $f : \mathbb{R}^k \to \mathbb{R}$ definable in $\mathfrak{R}$, $\mathrm{cl}(f(E^k))$ is nowhere dense.*

The Theorem and Corollary hold if we uniformly replace "nowhere dense" by any of "countable", "a finite union of discrete sets" or "null" (that is, of Lebesgue measure 0). Of course, when dealing with the nowhere-dense case, we need only consider the images $f(E^k)$, rather than their closures.

Obviously, if E has cardinality less than the continuum, then E is $\mathfrak{R}$-sparse.

If the substructure of $\mathfrak{R}$ generated by E is proper, then E is $\mathfrak{R}$-sparse, since $\mathfrak{R}$ expands $(\mathbb{R}, +)$; in particular, it's possible for E to be $\mathfrak{R}$-sparse and yet have the cardinality of the continuum.

If $\mathfrak{R}$ is o-minimal and the substructure of $\mathfrak{R}$ generated by E is proper, then E is $\mathfrak{R}'$-sparse, where $\mathfrak{R}'$ is the expansion of $\mathfrak{R}$ by constants for all real numbers; see Lemma 4.1 of [D2].

If L is countable, $\mathfrak{R}$ is o-minimal and E is a countable union of compact sets, each having Hausdorff dimension 0, then the substructure of $\mathfrak{R}$ generated by E is proper. (This is proved in Section 4.1 of [MS] for the case that $\mathfrak{R}$ expands the field, but it's also true when working just over the additive group.) Hence, E is $\mathfrak{R}$-sparse. (See also the preceding paragraph).

If E is $\mathfrak{R}$-sparse and measurable, then E is null (since otherwise the difference set $\{ a \Leftrightarrow b : a, b \in E \}$ has interior). Similarly, if E is $\mathfrak{R}$-sparse and has the property of Baire, then E is meager.

One easily modifies all of the above results to take into account definability *with* parameters.

We are still investigating consequences for the definable sets in arbitary $\mathbb{R}^n$.

We now begin to work toward the proof of Proposition 1.

We start with four lemmas, whose proofs we leave to the reader. The first three involve nothing but (somewhat tedious, in spots) elementary set theory. (Recall that we allow *arbitrary* index sets I in the definition of the $(\mathfrak{R}, E)^\infty$-sets.) The fourth is an easy observation about sets of real numbers.

Lemma 1. *Let $n \in \mathbb{N}$. A finite intersection of $(\mathfrak{R}, E)^\infty$-sets in $\mathbb{R}^n$ is an $(\mathfrak{R}, E)^\infty$-set. A finite intersection of special sets in $\mathbb{R}^{n+1}$ is special.*

(Note that the collection of all subsets of $\mathbb{R}^{m+n+1}$ defined by L-formulas of the forms (1) through (5) in the definition of "simple set" is closed under taking finite intersections.)

Lemma 2. *Let $m, n \in \mathbb{N}$, I be a set, and $P_\alpha \subseteq E^m$ for $\alpha \in I$. Let $A \subseteq \mathbb{R}^{m+n+1}$ and put*

$$B := \bigcup_{\alpha \in I} \bigcap_{u \in P_\alpha} A_u .$$

(1) *If A is special, then B is special.*
(2) *If A is a finite union of special sets, then B is a finite union of special sets.*

(Use (1) and Lemma 1 to get (2).)

Lemma 3. *Let $n \in \mathbb{N}$. The complement of an $(\mathfrak{R}, E)^\infty$-set in $\mathbb{R}^n$ is an $(\mathfrak{R}, E)^\infty$-set. The complement of a finite union of special sets in $\mathbb{R}^{n+1}$ is a finite union of special sets.*

(Use Lemmas 1 and 2(2) for the second part.)

Lemma 4. *Let $\emptyset \neq S \subseteq \mathbb{R}$. Then:*

(1) $\sup S < +\infty$ *if and only if there exists $s \in S$ such that $s' < s + 1$ for all $s' \in S$;*
(2) $\inf S > \Leftrightarrow\infty$ *if and only if there exists $s \in S$ such that $s < s' + 1$ for all $s' \in S$;*
(3) $\inf S > 0$ *if and only if there exists $s \in S$ such that $0 < s < 2s'$ for all $s' \in S$.*

For suggestiveness (as well as economy of space) we adopt the following notation for the proof of the next lemma: For any set Y, we write "$\bigvee_{y \in Y}$" instead of "$\exists y \in Y$", and "$\bigwedge_{y \in Y}$" instead of "$\forall y \in Y$".

Lemma 5. *For each $n \in \mathbb{N}$, the projection on the first n variables of a special set in $\mathbb{R}^{n+1}$ is an $(\mathfrak{R}, E)^\infty$-set in $\mathbb{R}^n$.*

Proof. Let $n \in \mathbb{N}$ and $A \subseteq \mathbb{R}^{n+1}$ be special.

If A is of type (1), then $A = B \times \mathbb{R}$ for some $(\mathfrak{R}, E)^\infty$-set $B \subseteq \mathbb{R}^n$.

Let $x \in \mathbb{R}^n$.

Suppose A is type (2). Then

$$\bigvee_{t \in \mathbb{R}} \bigvee_{\alpha \in I} \bigwedge_{u \in P_\alpha} \left[\varphi(u, x) \;\;\&\;\; f(u, x) = t\right]$$

$$\Leftrightarrow$$

$$\bigvee_{\alpha \in I} \bigvee_{v \in P_\alpha} \bigwedge_{u \in P_\alpha} \left[\varphi(u, x) \;\;\&\;\; f(u, x) = f(v, x)\right].$$

Suppose A is of type (3). Let $\alpha \in I$. By Lemma 4(1),

$$\sup f(P_\alpha, x) < +\infty \Leftrightarrow \bigvee_{v \in P_\alpha} \bigwedge_{u \in P_\alpha} \left[f(u, x) < f(v, x) + 1\right].$$

Hence,

$$\bigvee_{t\in\mathbb{R}}\bigvee_{\alpha\in I}\bigwedge_{u\in P_\alpha}\big[\varphi(u,x)\ \ \&\ \ f(u,x)<t\big]$$

$$\Leftrightarrow$$

$$\bigvee_{\alpha\in I}\bigvee_{v\in P_\alpha}\bigwedge_{u\in P_\alpha}\big[\varphi(u,x)\ \ \&\ \ f(u,x)<f(v,x)+1\big]\,.$$

The case that A is of type (4) is similar, using Lemma 4(2).

Suppose that A is of type (5). Let $\alpha\in I$. Now,

$$\bigvee_{t\in\mathbb{R}}\bigwedge_{u\in P_\alpha}\big[f(u,x)<t<g(u,x)\big]$$

if and only if

$$\bigwedge_{u\in P_\alpha}\bigwedge_{v\in P_\alpha} f(u,x)<g(v,x)$$

and at least one of the following hold: $\max f(P_\alpha,x)$ does not exist; $\min g(P_\alpha,x)$ does not exist; or $\inf\{\,g(v,x)\Leftrightarrow f(u,x):u,v\in P_\alpha\,\}>0$. Note that

$$\max f(P_\alpha,x)\text{ does not exist }\ \Leftrightarrow\bigwedge_{u\in P_\alpha}\bigvee_{v\in P_\alpha}\big[f(u,x)<f(v,x)\big]$$

and

$$\min g(P_\alpha,x)\text{ does not exist}\Leftrightarrow\bigwedge_{u\in P_\alpha}\bigvee_{v\in P_\alpha}\big[g(v,x)<g(u,x)\big]\,.$$

Use Lemmas 1, 3 and 4(3) to finish. □

Proof of Proposition 1. Suppose that $\mathfrak{R}$ is o-minimal. By Lemmas 3 and 5, it suffices to show that for each $n\in\mathbb{N}$, every $(\mathfrak{R},E)^\infty$-set in $\mathbb{R}^{n+1}$ is a finite union of special sets, which is immediate from cell decomposition and Lemma 2(2). □

References

[D1] L. van den Dries, *The field of reals with a predicate for the powers of two*, Manuscripta Math. **54** (1985), 187–195.

[D2] ———, *Dense pairs of o-minimal structures*, Fund. Math. **157** (1998), 61–78.

[MS] C. Miller and P. Speissegger, *Expansions of the real line by open sets: o-minimality and open cores* (preprint).

Department of Mathematics, The Ohio State University, 231 W. 18th Avenue, Columbus, Ohio 43210

E-mail address: miller@math.ohio-state.edu

Department of Mathematics, The Ohio State University, 231 W. 18th Avenue, Columbus, Ohio 43210

E-mail address: friedman@math.ohio-state.edu

URL: www.math.ohio-state.edu/~friedman/