Laser-driven particle mechanics

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Abstract

The dynamics of laser-driven charged particles is developed telegraphically from four different perspectives, all of them relativistic: Lagrangian, Hamiltonian, Hamilton-Jacobi, and action-angle. Their unifying idea is the principle of constructive interference. Its physical and mathematical appeal arises from its conceptual unit economy and simplicity.

Being driven by a travelling plane wave, the dynamics of a charged particle is that of an integrable exactly soluble dynamical system having four degrees of freedom. Each one is a uniform rotation whose frequency depends on the action of the that degree, and whose motion is related to the physical motion by the principle of constructive degree of freedom.

Contents

1 Motivational Overview

The motivating theme of these notes is the direct interaction of ultra-intense lasers with matter. Indeed, during the past decade a confluence of advances in laser science has opened the door to the study of laser-matter interaction as the new frontier of the 21st century. The extraordinarily high intensity (petawatt) of the laser pulses have pushed the relativistic, and hence nonlinear, nature of laser-matter interactions to the forefront of science. These processes are characterized by ultra-relativistic velocities and accelerations of such extreme violence that relativistic physics and mathematics is not merely optional but mandatory.

At present there is a considerable amount of highly visible experimental activity whose purpose is to grasp the properties and the nature of these extreme laser processes. However, a mathematically solid understanding in the form of principles, formulations, equations, etc., is sorely lacking. This lack extends from the laser-driven mechanics of particles, to that of plasmas, and onto that of fluids.

Our present focus is on the mechanics of particles. They are the fundamental building blocks of matter, and as such their interactions with ultra-intense laser radiation plays a fundamental role in physics. The nature of these interactions, which manifests itself through the mechanical trajectories of test particles, is controlled by the externally given laser-radiation field.

The set of all posible test particles, each with a given charge and mass, placed into such a field, and hence subject to well-determined measurable dynamical motions, forms a *dynamical system*. Thus a dynamical system is identified uniquely by the given laser radiation field.

There are a number of dynamical systems whose importance derives from laser fields readily implemented in the laboratory.

1. Particles moving in the e.m. field of a travelling wave.
2. Particles moving in the e.m. field of a standing wave.
3. Particles moving in the e.m. field of two counter-propagating waves, one having finite but small amplitude compared to the other,

4. Particles moving in the bichromatic e.m. field of two laser beams: one ultra-intense beam with a second weaker counter-propagating beam at a harmonic frequency.

5. Particles moving in the e.m. field of a plane wave beam of finite width.

6. Particles moving in the e.m. field of a weakly focussed travelling (or standing) wave Gaussian beam.

The first four, depicted in Figure 1, are charaterized by flat plane wave phase fronts infinite in extent. The last one has weakly curved phase fronts of finite extent. Even though Figures 1(a) and 1(c) are depict two archetypical laser fields, it is Figure 1(b) which is usually achieved in the laboratory. However, as shown in Figure 2, the laser field of two counter-propagating waves is equivalent to the superposition of a standing wave and a travelling wave,

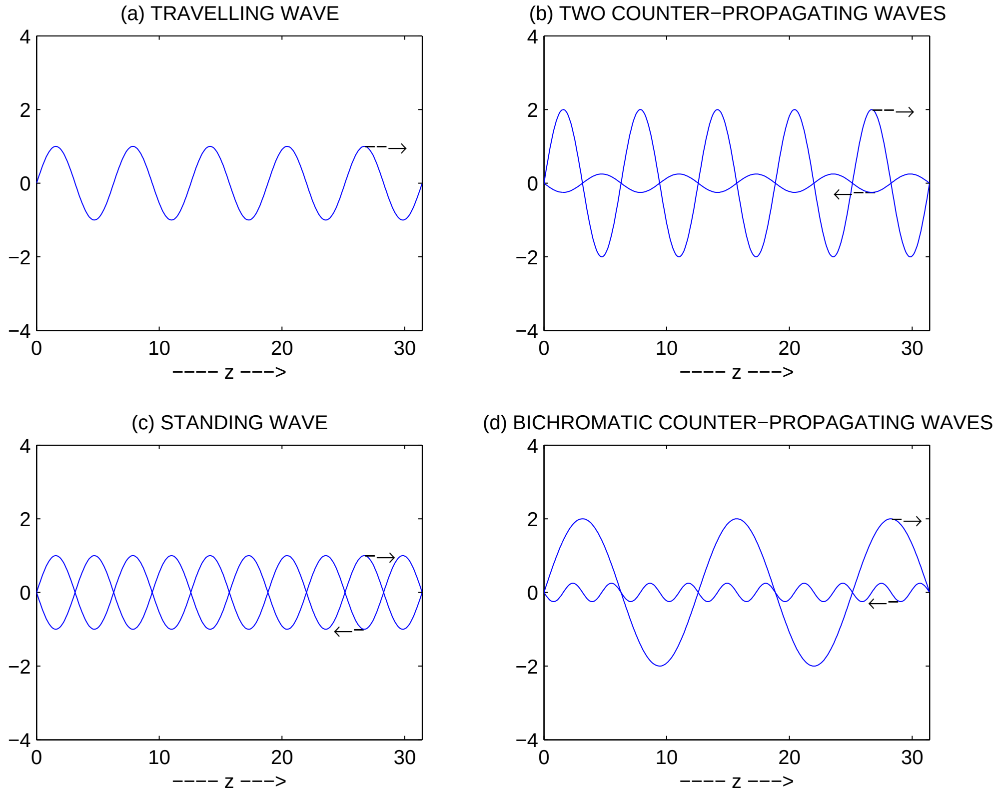

Figure 1: Amplitude graphs of four e.m. wave fields. Their potentials charterize four distinct laser-driven particle systems.

$$A \sin \omega(t+z) + B \sin \omega(t-z) = 2A \sin \omega t \cos \omega z + (B-A) \sin \omega(t-z) \ .$$

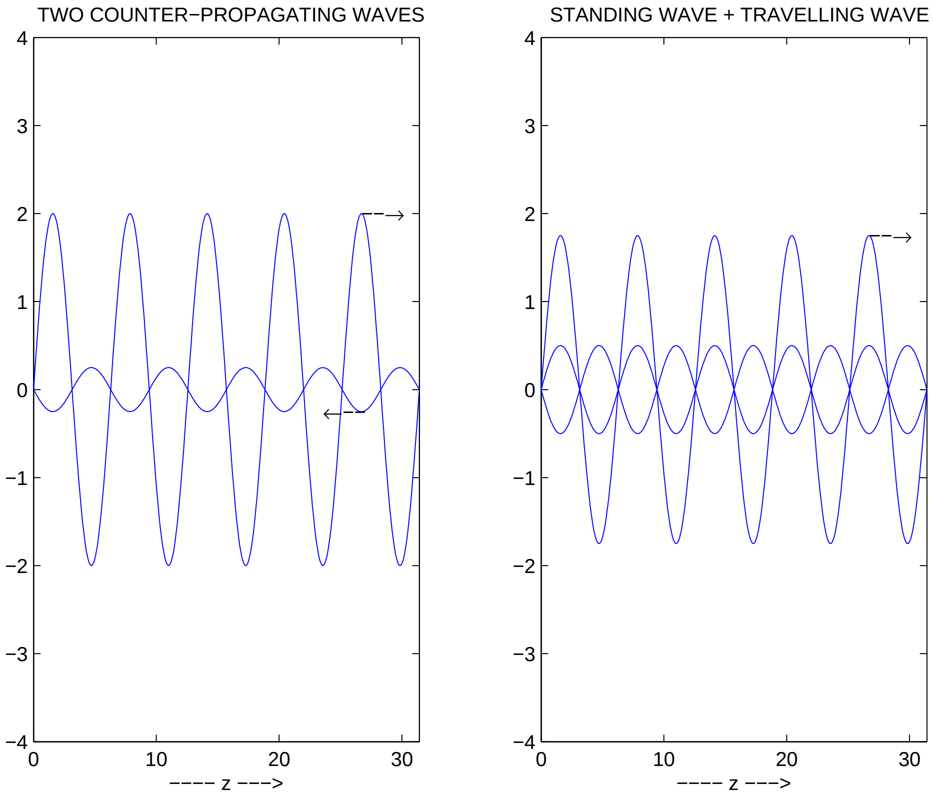

Figure 2: Two counter-propagating waves of equal frequency are equivalent to a superposition of a standing wave and a travelling wave.

Thus a particle moving in the e.m. field of Figure 1(b) is a combination of the motions in the two archetypical laser fields. Consequently, a mathematical formulation of this combined motion demands a mathematical formulation of the laser-driven particle mechanics for each of these two fields.

2 Lagrangian and Hamiltonian Formulation of Mechanics.

The action integral for a particle of mass m and charge q is

$$\begin{aligned} I &= \int_{\tau_1}^{\tau_2} \left\{ \frac{m}{2}\, \eta_{\alpha\beta} \frac{dx^\alpha}{d\tau}\, \frac{dx^\beta}{d\tau} + qA_\alpha(x^\gamma) \frac{dx^\alpha}{d\tau} \right\} d\tau \\ &\equiv \int_{\tau_1}^{\tau_2} L(x, \dot{x}) d\tau \end{aligned} \tag{1}$$

where $[\eta_{\alpha\beta}] = diag(-1, 1, 1, 1)$ and A_α is the given electromagnetic vector potential. The integral I is an extremum for those worldlines between $x^\alpha(\tau_1)$ and $x^\beta(\tau_2)$ which satisfy the Euler-Lagrange equation

$$\frac{d}{d\tau}\, \frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0,$$

namely,

$$m\eta_{\alpha\beta}\, \frac{d^2x^\beta}{d\tau^2} = q\, \frac{\partial A_\beta}{\partial x^\alpha}\, \frac{dx^\beta}{d\tau} - q\, \frac{\partial A_\alpha}{\partial x^\beta}\, \frac{dx^\beta}{d\tau}$$

or

$$\frac{d^2x^\beta}{d\tau^2} = \frac{q}{m}\, F^\beta_{\ \alpha}\, \frac{dx^\alpha}{d\tau} \quad \text{with } F^\beta_{\ \alpha} = \eta^{\beta\gamma} \left(\frac{\partial A_\alpha}{\partial x^\gamma} - \frac{\partial A_\gamma}{\partial x^\alpha} \right),$$

which are the Lorentz equations of motion for the charged particle.

A physically and mathematically more advantageous set of equations is based on the introduction of the momentum variables

$$p_\alpha \equiv \frac{\partial L}{\partial \dot{x}^\alpha} = \eta_{\alpha\beta}\, \frac{dx^\beta}{d\tau} m + qA_\alpha$$

and the superhamiltonian

$$\begin{aligned} \mathcal{H} &= \dot{x}^\alpha\, \frac{\partial L}{\partial \dot{x}^\alpha} - L \\ &= \frac{1}{2m}\, \eta^{\alpha\beta}(p_\alpha - qA_\alpha)(p_\beta - qA_\beta)\ . \end{aligned} \tag{2}$$

In terms of these one has the two sets of Hamiltonian equations of motion

$$\frac{dx^\alpha}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_\alpha} = \frac{1}{m}\, \eta^{\alpha\beta}(p_\beta - qA_\beta) \tag{3}$$

and

$$\frac{dp_\gamma}{d\tau} = -\frac{\partial \mathcal{H}}{\partial x^\gamma} = \frac{q}{m}\, \eta^{\alpha\beta}(p_\alpha - qA_\alpha)\, \frac{\partial A_\beta}{\partial x^\gamma}. \tag{4}$$

One can readily check that this system of first order equations implies the Euler-Lagrange equations.

3 Advantage of the Hamiltonian Formulation.

One of the chief virtues of the Lagrangian equations of motion is that they remain invariant under an arbitrary point transformation

$$\{x^\alpha\} \rightsquigarrow \{Q^\beta = Q^\beta(x)\}$$

Hamilton's equations of motion not only share this virtue but they take it to a higher level: they are invariant under certain more general transformations

$$\{x^\alpha, p_\gamma\} \rightsquigarrow \{Q^\beta = Q^\beta(x,p), P_\beta = P_\beta(x,p\}, \tag{5}$$

which is to say,

$$\frac{dQ^\beta}{d\tau} = \frac{\partial H}{\partial P_\beta} \quad \text{and} \quad \frac{dP_\delta}{d\tau} = -\frac{\partial H}{\partial Q^\delta}\ . \tag{6}$$

Here H is the transformed superhamiltonian obtained from $\mathcal{H}$ with the help of Eq.(5):

$$H(Q,P) = \mathcal{H}\left(x(Q,P), p(Q,P)\right)\ .$$

Termed *canonical*, such transformations have the distinguishing property that they leave invariant the representation of the antisymmetric tensor

$$dx^\alpha \wedge dp_\alpha = dQ^\beta \wedge dP_\beta .$$

A sufficient condition for a transformation, Eq.(5), to be canonical is that there exist a scalar function of $\{x^\alpha\}$ and $\{P_\beta\}$,

$$S = S(x,P).$$

Indeed, letting

$$\frac{\partial S(x,P)}{\partial x^\alpha} \equiv\ p_\alpha \ \text{ and } \frac{\partial S(x,P)}{\partial P_\beta} \equiv Q^\beta \tag{7}$$

one finds that

$$\begin{aligned}
dx^\alpha \wedge dp_\alpha &= dx^\alpha \wedge \frac{\partial}{\partial x^\beta}\left(\frac{\partial S}{\partial x^\alpha}\right) dx^\beta + dx^\alpha \wedge \frac{\partial}{\partial P_\beta}\left(\frac{\partial S}{\partial x^\alpha}\right) dP_\beta \\
&= dx^\alpha \wedge dx^\beta \frac{\partial^2 S}{\partial x^\beta \partial x^\alpha} + dx^\alpha \wedge \frac{\partial}{\partial x^\alpha}\left(\frac{\partial S}{\partial P_\beta}\right) dP_\beta \\
&= \text{zero }\ + \frac{\partial}{\partial x^\alpha}\left(\frac{\partial S}{\partial P_\beta}\right) dx^\alpha \wedge dP_\beta \\
&= \frac{\partial}{\partial P_\alpha}\left(\frac{\partial S}{\partial P_\beta}\right) dP_\alpha \wedge dP_\beta + \frac{\partial}{\partial x^\alpha}\left(\frac{\partial S}{\partial P_\beta}\right) dx^\alpha \wedge dP_\beta \\
&= d\left(\frac{\partial S}{\partial P_\beta}\right) \wedge dP_\beta \\
&= dQ^\beta \wedge dP_\beta
\end{aligned} \tag{8}$$

Problem 1: Prove that the representation invariance of Eq.(8) implies the local existence of a scalar $S(x,P)$ whose gradients yield Eq.(7).

Thus the existence of a scalar S is both a necessary and a sufficient condition for the invariance expressed by Eq.(8). It also is a sufficient condition for the invariance of Hamilton's equations of motion.

Problem 2: Show that a transformation such as the one given by Eq.(7) transforms the given euations of motion, Eqs.(3)-(4), into the same form, and given by Eq.(6).

Discussion: Taking advantage of the chain rule, let

$$\dot{\mathbf{g}}^\tau_{xp} \equiv \frac{dx^\alpha}{d\tau}\frac{\partial}{\partial x^\alpha}+\frac{dp_\beta}{d\tau}\frac{\partial}{\partial p^\beta}=\frac{dQ^\alpha}{d\tau}\frac{\partial}{\partial Q^\alpha}+\frac{dP_\beta}{d\tau}\frac{\partial}{\partial P_\beta}\equiv \dot{\mathbf{g}}^\tau_{QP}$$

be a vector tangent to the phasespace trajectory $\mathbf{g}^\tau$ relative to the given $(x^\alpha;p_\beta)$ and the new $(Q^\alpha;P_\beta)$ coordinates respectively.

a) Show that

$$\langle dx^\alpha \wedge dp_\alpha | \dot{\mathbf{g}}^\tau_{xp}\rangle = dp_\alpha \frac{dx^\alpha}{d\tau} - dx^\beta \frac{dp_\beta}{d\tau} \tag{9}$$

and

$$\langle dQ^\alpha \wedge dP_\alpha | \dot{\mathbf{g}}^\tau_{QP}\rangle = dP_\alpha \frac{dQ^\alpha}{d\tau} - dQ^\beta \frac{dP_\beta}{d\tau} \tag{10}$$

b) Point out why

$$\langle dx^\alpha \wedge dp_\alpha | \dot{\mathbf{g}}^\tau_{xp}\rangle = \langle dQ^\alpha \wedge dP_\alpha | \dot{\mathbf{g}}^\tau_{QP}\rangle\ .$$

c) Show that

$$\langle dx^\alpha \wedge dp_\alpha | \dot{\mathbf{g}}^\tau_{xp}\rangle = d\mathcal{H}(x^\alpha,p_\beta)$$

implies Hamilton's equations of motion, Eq.(3)-(4).

d) Show that the introduction of the new coordinates $\{Q^\alpha,P_\alpha\}$ into $d\mathcal{H}$,

$$d\mathcal{H}(x^\alpha(Q^\gamma,P_\delta),p_\beta(Q^\gamma,P_\delta)) \equiv dH(Q^\gamma,P_\delta)$$

yields Eq.(6), Hamilton's equations relative to the new coordinates $\{Q^\gamma,P_\delta\}$.

4 The Dynamical Phase

The most important scalar function for a dynamical system is its dynamical phase. This phase is the (value of) the integral, Eq.(1), evaluated for a worldline $x^\alpha(\tau)$ which starts at $x^\alpha(\tau_1)$, which satisfies the Lagrange equations of motion,

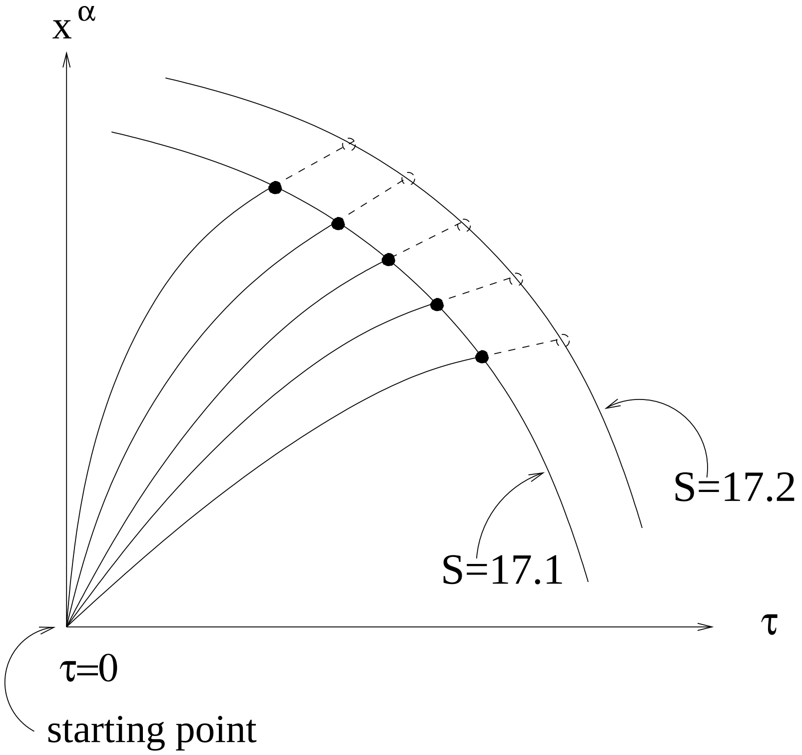

Figure 3: Five different world lines having the same starting point and terminating at points where the dynamical phase along the world lines has reached the value $S = 17.1$. The locus of such termination points of such world lines with common starting point $\tau = 0$ forms the isogram of a scalar function $S(x^\alpha, \tau)$, the dynamical phase ("Hamilton-Jacobi function") of the Hamiltonian system.

and which therefore extremizes this integral. Let us designate this extremal value by

$$S = \int_{\tau_1}^{\tau} L(x, \dot{x}) d\tau \ .$$

This integral is a function of the worldline's termination point which we take to be $x^\alpha(\tau)$. If there is only one worldlines between $x^\alpha(\tau_1)$ and $x^\alpha(\tau)$, as is usually the case, the S is a single valued function of the termination point. This is depicted in Figure 3. (If there were several such worldlines, then S would be multivalued.) Thus S is a function of the location $\{x^0(\tau), x^1(\tau),\, x^2(\tau),\, x^3(\tau)\}$ of that termination point. It also is a function of the parameter τ which one uses to parametrize the worldline. Thus one has

$$\int_{\tau_1}^{\tau} L(x,\dot{x})d\tau = S(x^0,x^1,x^2,x^3,\tau) \ .$$

5 The Hamilton-Jacobi Equation for a Relativistic Particle.

Being defined in terms of the action integral, the dynamical phase satisfies a differential equation which one obtains by a simple argument:

Let $x^\alpha(\tau)$ and $\overline{x}^\alpha(\tau) = x^\alpha(\tau) + h^\alpha(\tau)$ be two worldlines having the same starting point

$$x^\alpha(\tau_1) = \overline{x}^\alpha(\tau_1),$$

both satisfying Lagrange's equation of motion, but having slightly different termination points

$$x^\alpha(\tau) \text{ and } \overline{x}^\alpha(\tau+\delta\tau) = x^\alpha(\tau) + h^\alpha(\tau) + \dot{x}^\alpha\delta\tau + \cdots$$

as in Figure 4. Then the (principal linear part of the) difference in the value of the dynamical phase at these termination points is

$$\begin{aligned}\delta S &= \int_{\tau_1}^{\tau+\delta\tau} L(\overline{x}^\alpha,\dot{\overline{x}}^\alpha)d\tau - \int_{\tau_1}^{\tau} L(x^\alpha,\dot{x}^\alpha)d\tau \\ &= \int_{\tau_1}^{\tau}\sum_\alpha\left(\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\alpha}\right)h^\alpha(\tau)d\tau + \sum_\alpha \frac{\partial L}{\partial \dot{x}^\alpha}\ h^\alpha\Big|_{\tau_1}^{\tau+\delta\tau} + L\delta\tau\Big|_{\tau_1}^{\tau+\delta\tau} \ .\end{aligned}$$

The fact that $x^\alpha(\tau)$ satisfies Lagrange's equation of motion implies that the integral vanishes. Recalling the definition of δx^α, or looking at Figure 4, one sees that at the two termination points one has

$$h^\alpha = \delta x^\alpha - \dot{x}\delta\tau.$$

Consequently, the principal linear part of the difference between the two S values at the termination point is

$$\delta S = \sum_\alpha p_\alpha \delta x^\alpha - \mathcal{H}\delta\tau. \tag{11}$$

Here

$$p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha}\left(= \eta_{\alpha\beta}\dot{x}^\beta + qA_\alpha\right)$$

are the momentum components and

$$\mathcal{H} \equiv \frac{\partial L}{\partial \dot{x}^\alpha}\ \dot{x}^\alpha - L = \frac{\eta^{\alpha\beta}}{2m}(p_\alpha - qA_\alpha)(p_\beta - qA_\beta)$$

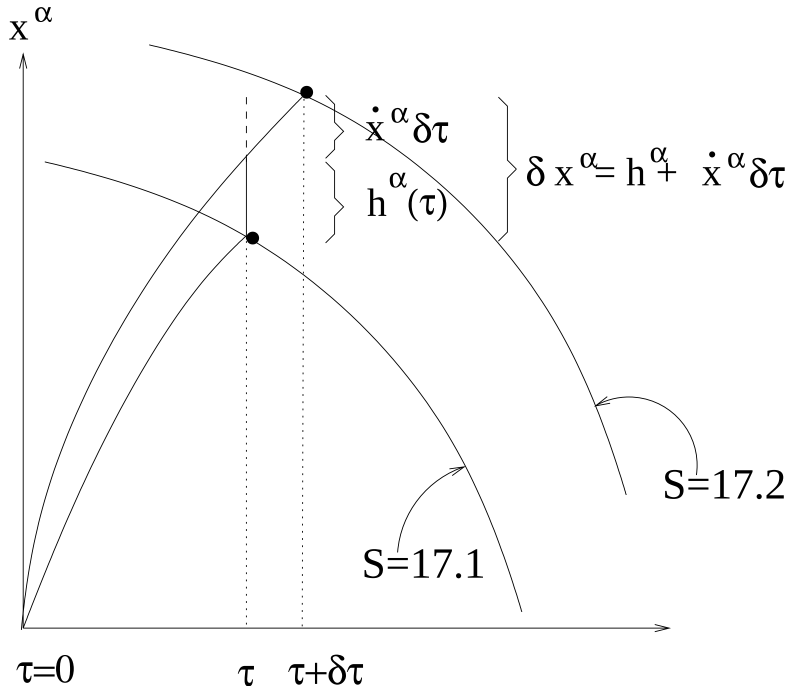

Figure 4: Differential of the action function S as a function of variations $\delta\tau$ and δx^α in the endpoint of an extremal world line.

is the superhamiltonian of the charged particle at the termination point of its worldline. Equation (11) is the expression for the differential of S. One has

$$\frac{\partial S}{\partial x^\alpha} = p_\alpha$$
$$\frac{\partial S}{\partial \tau} = -\mathcal{H}(x^\alpha, p_\gamma)$$

Thus the differential equation for the dynamical phase function S is

$$\mathcal{H}\left(x^\alpha, \frac{\partial S}{\partial x^\gamma}\right) + \frac{\partial S}{\partial \tau} = 0, \tag{12}$$

or explicitly

$$\frac{\eta^{\alpha\beta}}{2m}\left(\frac{\partial S}{\partial x^\alpha} - qA_\alpha\right)\left(\frac{\partial S}{\partial x^\beta} - qA_\beta\right) + \frac{\partial S}{\partial \tau} = 0.$$

This is the Hamilton-Jacobi for a charged particle in an electromagnetic vector potential $A_\alpha(x)$.

6 Solution to the Hamilton-Jacobi Equation

As an example, consider the problem of solving the H-J equation for a charged particle in the vector potential

$$\{A_0, A_1, A_2, A_3\} = \{0, A_x(t-z), A_y(t-z), 0\}. \tag{13}$$

It expresses a generic plane wave traveling towards positive z. The H-J equation is

$$\frac{1}{2m}\left\{-\left(\frac{\partial S}{\partial t}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 + \right.$$
$$\left.\left(\frac{\partial S}{\partial x} - qA_x(t-z)\right)^2 + \left(\frac{\partial S}{\partial y} - qA_y(t-z)\right)^2\right\} + \frac{\partial S}{\partial \tau} = 0\ .$$

To make this equation physically more transparent and mathematically more manageable one introduces the retarded and advanced time coordinates

$$t - z \equiv u$$

and

$$t + z \equiv v$$

In terms of these the H-J equation becomes

$$\frac{1}{2m}\left\{-4\,\frac{\partial S}{\partial u}\,\frac{\partial S}{\partial v} + \left(\frac{\partial S}{\partial x} - qA_x(u)\right)^2 + \left(\frac{\partial S}{\partial y} - qA_y(u)\right)^2\right\} + \frac{\partial S}{\partial \tau} = 0 \tag{14}$$

This equation is readily solved by the method of separation of variables according to which the solution,

$$S = \int^u p_u(u)du + \int^v p_v(v)dv + \int^x p_x(x)dx + \int^y p_y(y)dy + \int^\tau p_\tau(\tau)d\tau\ ,$$

is a sum of antiderivatives, each one depending only on its respective integration variable. With this stipulation one finds that the components of the gradient of S are

$$\frac{\partial S}{\partial u} \equiv p_u = \frac{1}{4P_v}\left[(P_x - qA_x(u))^2 + (P_y - qA_y(u))^2\right] + 2mP_\tau \tag{15}$$

$$\frac{\partial S}{\partial v} \equiv p_v = P_v \tag{16}$$

$$\frac{\partial S}{\partial x} \equiv p_x = P_x \tag{17}$$

$$\frac{\partial S}{\partial y} \equiv p_y = P_y \tag{18}$$

$$\frac{\partial S}{\partial \tau} \equiv p_\tau = P_\tau \tag{19}$$

where P_v, P_x, P_y, as well as P_τ, are the constants of separation[1]. Consequently, the Hamilton-Jacobi function ("Hamilton's principal function", "Schroedinger phase", dynamical phase)

$$S = S(u, v, x, y; P_\tau, P_v, P_x, P_y; \tau) \tag{20}$$

has the form

$$\begin{aligned} S =& \frac{1}{4P_v}\int_{u_0}^{u}\left[(P_x - qA_x)^2 + (P_y - qA_y)^2 + 2mP_\tau\right] du + vP_v \\ &+ xP_x + yP_y + \tau P_\tau \end{aligned} \tag{21}$$

or more generally

$$S' = S - \beta(P_\tau, P_v, P_x, P_y) \ .$$

Both S and S' are solutions to the H-J equation, but S' has an additive constant $\beta(P_\tau, P_v, P_x, P_y)$ which is a function of the four separation constants. Although the difference between S and S' is trivial from the perspective of solving the H-J equation, the opposite is true from the viewpoint of physics and mathematics. Indeed, suppose one let's

$$\beta = S(u_0, v_0, x_0, y_0; P_\tau, P_v, P_x, P_y; \tau_0) \ .$$

Then one obtains

$$\begin{aligned} S' =& \frac{1}{4P_v}\int_{u_0}^{u}\left[(P_x - qA_x)^2 + (P_y - qA_y)^2 + 2mP_\tau\right] du + (v - v_0)P_v \\ &+ (x - x_0)P_x + (y - y_0)P_y + (\tau - \tau_0)P_\tau \ . \end{aligned} \tag{22}$$

The usefulness of this H-J function S' is that it generates new phase space coordinates relative to which the solutions of Hamilton's Eqs.(3)-(4) are straight lines. They are depicted in Figure 8 on page 18. In other words, solving the H-J equation for S' is tantamount to solving Hamilton's equations for all possible initial conditions.

7 Dynamical Phase as Physical

However, before validating this claim, it is worthwhile to remind oneself that the phase function S' is not a floating abstraction, i.e. disconnected from reality. It is a sum of five parts,

$$S' = W_0(u) + W_1(v) + W_2(x) + W_3(y) + (\tau - \tau_0)P_\tau \ .$$

The first four are the contributions to the phase which depend on (u, v, x, y) and thus express the dependence on the arrangement of laboratory meter sticks and clocks used to establish spacetime events.

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[1] $\mathcal{H} = -P_\tau$ is the value of the conserved superhamiltonian $\mathcal{H}$ which for the particle under consideration is $\mathcal{H} = -\frac{m}{2}$

The last part, which depends on the proper time τ (the particle's "wrist watch" time), is the contribution which depends on the intrinsic properties of the particle. Its rest mass, or more precisely its quantum mechanical Compton frequency ($mc^2/\hbar$), determines the rate at which it contributes to the dynamical phase. This phase, in turn, gives rise to interference effects measurable in the laboratory. It is, as it were, that the particle carries its own intrinsic clock.

8 The Principle of Constructive Interference

The principle that gives rise to the above-mentioned new phase space coordinates is *the principle of constructive interference* according to which the spacetime trajectory of a particle is the locus of events where the semi-classical wavefunction

$$\psi \sim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(P_\tau, P_v, P_x, P_y)\mathcal{A}(x^\alpha)e^{iS'/\hbar}dP_\tau dP_v dP_x dP_y$$

has maximum modulus.

The mathematical formulation of this principle is based on the evaluation of the superposition expressed by this integral whenever the exponential phase factor is a function varying rapidly compared to the slowly varying amplitude $\mathcal{A}$. Based on a Gaussian weight factor f, the superposition integral is a Gaussian also. Its maximum is located at those events which satisfy

$$\left.\begin{aligned} \frac{\partial S'}{\partial P_\tau} &= 0 \\ \frac{\partial S'}{\partial P_v} &= 0 \\ \frac{\partial S'}{\partial P_x} &= 0 \\ \frac{\partial S'}{\partial P_y} &= 0 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \frac{\partial S}{\partial P_\tau} &= Q^\tau \\ \frac{\partial S}{\partial P_v} &= Q^v \\ \frac{\partial S}{\partial P_x} &= Q^x \\ \frac{\partial S}{\partial P_y} &= Q^y \end{aligned}\right. \tag{23}$$

The left column are the conditions for constructive interference. They comprise in Minkowski spacetime the particle's world line

$$\{x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)\} = \{u, v, x, y\}$$

as obtained from

$$\frac{m}{2P_v}(u-u_0)+\tau \equiv Q^{\tau} = \tau_0 \tag{24}$$

$$\begin{aligned} v - \frac{1}{4P_v^2}\int_{u_0}^{u} \left[(P_x - qA_x(u))^2 + (P_y - qA_y(u))^2 \ +2mP_\tau\right] du & \\ \equiv Q^{v} = v_0 & \end{aligned} \tag{25}$$

$$x + \frac{1}{2P_v}\int_{u_0}^{u} (P_x - qA_x(u))\, du \equiv Q^{x} = x_0 \tag{26}$$

$$y + \frac{1}{2P_v}\int_{u_0}^{u} (P_y - qA_y(u))\, du \equiv Q^{y} = y_0 \ . \tag{27}$$

The capitalized P's and Q's refer to the initial value data of the world line at $\tau = \tau_0$.

The pictorial representation of the principle of constructive interference consists of the intersection of the isograms of two slightly different solutions to the H-J equation. The points of intersection are where constructive interference takes place. The particle worldline is understood to pass through these successive points. Figure 5 and 6 illustrate this process for free charge and and for a charge driven by the e.m. field of a plane wave. Figure 7 illustrates it for an e.m. pulse with a finite number of oscillations. Note that, once suitably averaged, its spacetime region acts a refractive medium for the particle world line.

> *Problem 3*: a) For Figure 7 formulate what in Euclidean space corresponds to Snell's law.
>
> b) Can one identify a refractive index for the laser pulse history? If so, what is it?

9 Laser-driven Particle Mechanics via Celestial Mechanics

Compared to laser-driven particle mechanics, celestial mechanics is a highly developed science. Progress in the former could be speeded up considerably if the questions that have been asked, the mathematical techniques that have been developed, and the principles that have been formulated could be extended from the latter to the former.

It is a well-known fact that the Kepler problem of, say, a planet orbiting around a star ("two-body problem") plays a role of key importance in celestial mechanics. This importance extends to the problem of a charge in the e.m. field of a travelling plane wave in laser-driven particle mechanics in the following three respects:

- Both problems are exactly soluble, i.e. are integrable dynamical systems.

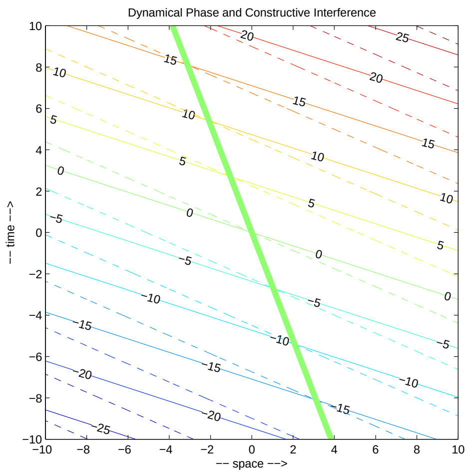

Figure 5: Constructive interference between two sets of wave front histories, the solid isograms of $S(t, z; P_v)$ and dashed isograms of $S(t, z; P_v + \Delta P_v)$. These two sets of wave front histories are the familiar relativistic De Broglie matter waves. The particle is understood to be located at that event where an isogram of one intersects with an equal-value isogram of the other. The fact that the heavy world line and the intersecting isograms are straight is a reflection of the fact that the charged particle is free: there is no e.m. field.

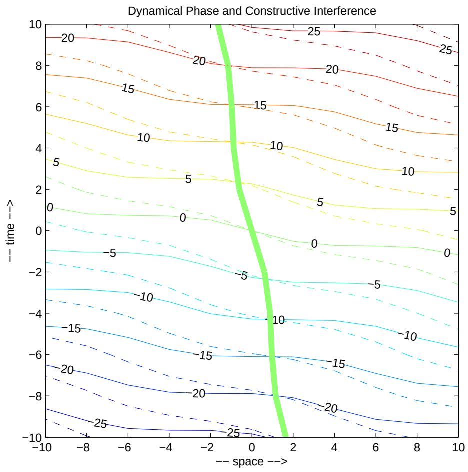

Figure 6: Same as Figure 5 except that the charged particle is driven by the periodic e.m. field of a travelling wave. In such a circumstance the oscillating e.m. field distorts the De Broglie wave front histories so that constructive interference results in an oscillating particle trajectory.

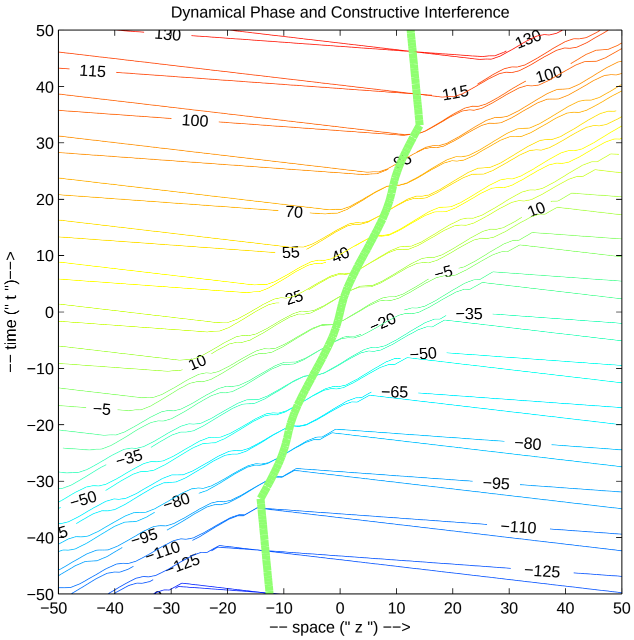

Figure 7: Relativistic De Broglie wave front histories (solid and dashed isograms) distorted by a three cycle laser pulse. Its spacetime history occupies the diagonal 45^o swath. The constructive interference expresses the circumstance where the particle with slight negative z-velocity passes through the +z travelling pulse, gets jiggled three times, before it emerges with its original velocity from the back of the pulse. The net effect of this process is that the particle gets shifted by an amount proportional to the duration of the pulse. If one ignores the oscillatory effect on the De Broglie waves by averaging over each of their histories, then the spacetime history of the laser pulse acts as a refractive medium for the averaged particle world line.

- Both problems have their repective laws: (i) Kepler's second law (equal areas swept out in equal times), and (ii) equal phases (educed from Eqs.(31)-(32) elapse in equal proper times. The corresponding conserved quantities are (i) the angular momentum and (ii) the momentum $P_v = P_t + P_z$, as one can see from Eq.(24).
- By having the planet orbit around a double star system Kepler's problem becomes an in-general nonintegrable dynamical system.

  !!!!! parallelism to be completed

10 Phase Flow: Its Trajectories and Transformations

Together with the spacetime gradient of S, Eqs.(15)-(19), the constructive interference conditions, Eq.(23), also yields a moving point in phase space. The starting point is $\{Q^\alpha; P_\beta\}$ when $\tau = \tau_0$. When $\tau = \tau$ the point has reached $\{x^\alpha; p_\beta\}$. This moving point forms a trajectory in phase space,

$$\{x^\alpha(\tau); p_\beta(\tau) : \alpha, \beta = 0, 1, 2, 3\} = \{u, v, x, y; p_u, p_v, p_x, p_y\}.$$

It is a solution to Hamilton's equations of motion, Eqs.(3)-(4).

> *Problem 4*: Show that (a) the conditions for constructive interference, Eq.(23), together with (b) the fact that the dynamical phase $S(x^\alpha; P_\beta)$ satisfies the H-J equation, Eq.(12), imply that each phase trajectory satisfies the Hamiltonian equations of motion, Eqs.(3) and (4).
>
> *Hint:* Differentiate the H-J equation
>
> $$\mathcal{H}\left(x^\alpha(\tau), \frac{\partial S(x^\gamma(\tau), P_\beta; \tau)}{\partial x^\alpha}\right) + \frac{\partial S(x^\gamma(\tau), P_\beta; \tau)}{\partial \tau} = 0$$
>
> with respect to each of the constants of motion P_β. Differentiate the four equations for constructive interference
>
> $$\frac{\partial S(x^\gamma(\tau), P_\beta; \tau)}{\partial P_\beta} = 0$$
>
> with respect to the world line parameter τ. Compare the results. Next differentiate the H-J equation with respect to each of the coordinates x^α.

Consider the set of phase space trajectories obtained from the dynamical phase S. For fixed τ they yield the map

$$\boxed{\begin{array}{ccl} \mathbb{R}^8 & \xrightarrow{\mathbf{g}^\tau} & \mathbb{R}^8 \\ \{Q^\alpha; P_\beta\} & \rightsquigarrow & \mathbf{g}^\tau(Q^\alpha; P_\beta) \equiv \{x^\alpha(\tau); p_\beta(\tau)\}\ , \end{array}} \tag{28}$$

where, according to Eqs.(16)-(18) and (24)-(27),

$$\{x^\alpha, p_\alpha\} = (u, v, x, y; p_u, p_v, p_x, p_y).$$

The relation $\mathbf{g}$ is called the *phase flow* generated by the superhamiltonian $\mathcal{H}$. Physically this flow expresses the evolution of a collisionless ensemble of charged particles each launched with its own (Q^α, P_β) in the field of a laser. Mathematically this flow combines two concepts into one: letting τ vary while keeping $\{Q^\alpha; P_\beta\}$ fixed yields a phase space trajectory; letting $\{Q^\alpha; P_\beta\}$ vary while keeping τ fixed yields a (canonical) transformation, Eq.(28). Combining the two, one obtains $\mathbf{g}^\tau$, a τ-parametrized family of transformations which expresses the nonintersecting trajectories of a set of moving points, each one labelled by eight coordinates.

The geometrical representation of these transformations is depicted in Figure 8: Points of the intersection of the straight trajectories with a $(Q^\alpha; P_\beta)$-plane in (b) get mapped into the intersection of the curved trajectories with a $(q^\alpha; p_\beta)$-plane in (a).

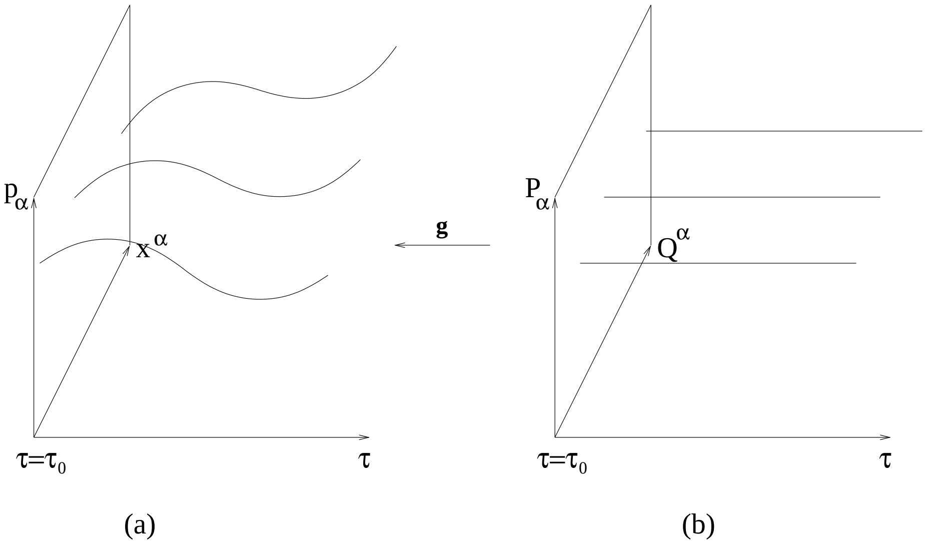

Figure 8: Three trajectories in the extended phase space relative to (a) the given physical coordinates $\{x^\alpha, p_\beta, \tau - \tau_0\} \equiv (u, v, x, y; p_u, p_v, p_x, p_y, \tau - \tau_0)$ and (b) the new coordinates $\{Q^\alpha, P_\beta, \tau - \tau_0\} \equiv (Q^\tau, Q^v, Q^x, Q^y; P_\tau, P_v, P_x, P_y, \tau - \tau_0)$ relative to which the trajectories are straight lines. The mapping $\mathbf{g}$ is the phase flow map discussed in the text.

Consider the transformation corresponding to $\tau = \tau_1$,

$$\mathbf{g}^{\tau_1}(Q^\alpha; P_\beta) = \{Q_1^\alpha; P_{1\beta}\} \ . \tag{29}$$

Then

$$\mathbf{g}^{\tau}(Q_1^\alpha; P_{1\beta}) = \mathbf{g}^{\tau} \circ \mathbf{g}^{\tau_1}(Q^\alpha; P_\beta) \tag{30}$$

is the composite of $\mathbf{g}^\tau$ and $\mathbf{g}^{\tau_1}$. But both $\mathbf{g}^\tau \circ \mathbf{g}^{\tau_1}(Q^\alpha; P_\beta)$ and $\mathbf{g}^{\tau+\tau_1}(Q^\alpha; P_\beta)$ are solutions to the same (time-independent!) Hamilton's equations of motion with the same starting point. Consequently,

$$\mathbf{g}^{\tau} \circ \mathbf{g}^{\tau_1}(Q^\alpha; P_\beta)) = \mathbf{g}^{\tau+\tau_1}(Q^\alpha; P_\beta)$$

One sees that $\mathbf{g}^{\tau=0}$ is the identity map, and $\mathbf{g}^{-\tau} = (\mathbf{g}^\tau)^{-1}$ is the inverse map. Consequently, Eq.(28) is a τ-parametrized group of phase space coordinate transformations, the phase flow of $\mathcal{H}$. Being generated by the dynamical phase S, they are canonical transformations whose defining property is Eq.(8).

These transformations have a striking simplifying effect on the phase space trajectories. Suppose one considers the *extended* phase space,

$$\mathbb{R}^9 = \mathbb{R}^8 \times \mathbb{R} \quad \text{where } \mathbb{R} = \tau\text{-axis}$$

It is (8+1)-dimensional and it is obtained from $\mathbb{R}^8$ by adding the τ-line as an extra coordinate axis. In this space the phase space trajectories are represented by curved non-intersecting lines as in Figure 8(a). The τ-parametrized canonical transformation constitute a map from a $(x^\alpha; p_\beta, \tau)$-coordinatized copy of the extended phase space $\mathbb{R}^9$ to a $(Q^\alpha; P_\beta, \tau)$-coordinatized copy of the same extended phase space $\mathbb{R}^9$. The benefit of the latter is that relative to it the phase space trajectories are straightened out as in Figure 8(b).

However, as intimated by Eq.(23), the same constructive interference conditions also imply a τ-dependent canonical coordinate transformation,

$$\begin{aligned} \{Q^\alpha, P_\alpha\} \equiv (Q^\tau, Q^v, Q^x, Q^y; P_\tau, P_v, P_x, , P_y) & \overset{\mathbf{g}^\tau}{\rightsquigarrow} \\ & \{x^\alpha, p_\alpha\} \equiv (u, v, x, y; p_u, p_v, p_x, y, p_y) \end{aligned}$$

This transformation, which is given (implicitly) by Eqs.(19)-(18) and (25)-(27), is generated by the Hamilton-Jacobi function, Eq.(20). This transformation straightens out the phase space trajectories (as depicted in Figure 8) of Hamilton's equations of motion (3) and (4).

> *Problem 5*: Write down and solve Hamilton's equations of motion, (3)-(4), and verify that the solution coincides with the constuctive interference condition, Eqs.(24)-(27)

11 Action-Angle Representation

In mechanics, periodicity, one of the most pervasive aspects of nature, leads to the introduction of action-angle variables as phase space coordinates for a dynamical system. One of the chief virtues of these variables is that they decompose the given dynamical system into its fundamental components, a set of noninteracting subsystems ("degrees of freedom") each having its own frequency. This is why some mathematicians[2], call these coordinates "normal coordinates", in analogy with the normal modes of a linear vibrating system.

Suppose the charged particle moves in the periodic electromagnetic field of a plane wave

$$A_x(u) = -\frac{E_x}{\omega_1} \sin \omega_1 u \tag{31}$$

$$A_y(u) = -\frac{E_y}{\omega_1} \sin(\omega_1 u + \delta) \ , \tag{32}$$

---

[2] J. Moser, *Stable and Random Motions in Dynamical Systems* (Princeton University Press, Princeton, N.J., 1973), pages 41, 43

which propagates along the positive z-direction with frequency be ω_1. Then the vector potential, Eq.(13), satisfies

$$A_\mu(u + 2\pi/\omega_1) = A_\mu(u) \ , \tag{33}$$

and so does the superhamiltonian, Eq.(2).

11.1 Multiple Periodicity expressed in Terms of a Torus

Such periodicity divides the infinite u-domain of the dynamical system into countably many equivalent subdomains of size $\Delta u = 2\pi/\omega_1$. Taking into account that the ultimate goal is to uncover the statistical properties of the motion and to make a perturbational analysis of the system, one desires to compare the motion in successive subdomains. This is achieved by viewing them as a single equivalence class, a circle whose perimeter is $\Delta u = 2\pi/\omega_1$. The traversal of successive subdomains corresponds to rotations by 2π on this circle.

Analogous statements hold for the other three coordinates v, x, and y. They are cyclic, which expresses the fact that the dynamical system is invariant under translation into these three directions. This implies invariance under discrete but arbitrary translations, say, $\Delta v = 2\pi/\omega_2$, $\Delta x = L_x$, and $\Delta y = L_y$. These magnitudes are to be justified later in the presence of perturbations.

For the same reason and in the same manner as was done with the u-domain, one divides each of the v, x, and y domains into three equivalence classes which are three circles with respective perimeters $\Delta v = 2\pi/\omega_2$, $\Delta x = L_x$, and $\Delta y = L_y$. The compactification of the four rectilinear spacetime coordinates into four circles means that Minkowski spacetime $\mathbb{R}^4$ has been compactified (for the purpose of dynamical sytems analysis) into a four-dimensional torus,

$$\mathbb{T}^4 = S^1 \times S^1 \times S^1 \times S^1 \ . \tag{34}$$

It is a topological expression of the four-fold periodicity of the dynamical system. The spacetime trajectory of a particle gets represented as a winding trajectory on this torus.

11.2 Action Variables

The four periods give rise to the four action variables

$$I_u \equiv \int_0^{2\pi/\omega_1} p_u du \tag{35}$$

$$I_v \equiv \int_0^{2\pi/\omega_2} p_v dv \tag{36}$$

$$I_x \equiv \int_0^{L_x} p_x dx \tag{37}$$

$$I_y \equiv \int_0^{L_y} p_y dy \ . \tag{38}$$

The evaluation of these integrals is done with the help of Eqs.(15)-(19) and (31)-(32). The result is the transformation

$$\mathbf{P} \equiv (P_v, P_x, P_y, P_\tau) \rightsquigarrow \begin{cases} I_u = & \dfrac{1}{4P_v}\dfrac{2\pi}{\omega_1}\left\{(E_x^2+E_y^2)\dfrac{q^2}{2\omega_1^2} + P_x^2 + P_y^2 + 2mP_\tau\right\} \\ I_v = & P_v\dfrac{2\pi}{\omega_2} \\ I_x = & P_x L_x \\ I_y = & P_y L_y \end{cases} \tag{39}$$

and its inverse

$$\mathbf{I} \equiv (I_u, I_v, I_x, I_y) \rightsquigarrow \begin{cases} -P_\tau = & \dfrac{1}{2m}\left\{\dfrac{I_x^2}{L_x^2} + \dfrac{I_y^2}{L_y^2} - 4I_v I_u \dfrac{\omega_1}{2\pi}\dfrac{\omega_2}{2\pi} + \dfrac{q^2}{2\omega_1^2}\left(E_x^2+E_y^2\right)\right\} \\ \equiv & H(\mathbf{I}) \\ P_v = & I_v\dfrac{\omega_2}{2\pi} \\ P_x = & I_x/L_x \\ P_y = & I_y/L_y \end{cases} \tag{40}$$

The effect of this transformation is that it decomposes – with mathematical precision – the dynamical system into its fundamental physical components, each having its own frequency.

Indeed, introducing the action variables into the dynamical phase, Eq.(22), one finds that, with the help of Eqs.(31) and (32), its form (a.k.a. "Hamilton's *principal* function") is

$$S'(x;\mathbf{P}(\mathbf{I});\tau) \equiv \tilde{S}'(x;\mathbf{I};\tau) = \overbrace{\underbrace{W(x^\alpha,\mathbf{I})}}^{W_0(u;\mathbf{I}) + W_1(v;\mathbf{I}) + W_2(x;\mathbf{I}) + W_3(y;\mathbf{I})} - W(x_0^\alpha,\mathbf{I}) - (\tau-\tau_0)H(\mathbf{I}) \tag{41}$$

where the four contributions to Hamilton's *characteristic* function

$$W(x^\alpha;\mathbf{I}) = \int_0^u p_u du + \int_0^v p_v dv + \int_0^x p_x dx + \int_0^y p_y dy \tag{42}$$

are

$$\begin{aligned} W_0(u,\mathbf{I}) &= u\frac{\omega_1}{2\pi}I_u \\ &+ \frac{1}{4I_v}\frac{2\pi}{\omega_2}\int_0^u 2m\,du\left\{\frac{I_x}{L_x}\eta_x\sin\omega_1 u+\frac{I_y}{L_y}\eta_y\sin(\omega_1 u+\delta)\right. \\ &\qquad\left. -\frac{\eta_x^2}{4}m\cos 2\omega_1 u-\frac{\eta_y^2}{4}m\cos(2\omega_1 u+2\delta)\right\} \qquad (43) \\ W_1(v,\mathbf{I}) &= v\frac{\omega_2}{2\pi}I_v \qquad (44) \\ W_2(x,\mathbf{I}) &= \frac{x}{L_x}I_x \qquad (45) \\ W_3(y,\mathbf{I}) &= \frac{y}{L_y}I_y \qquad (46) \end{aligned}$$

with

$$\eta_x\equiv\frac{qE_x}{m\omega_1}\quad\text{and}\quad\eta_y\equiv\frac{qE_y}{m\omega_1} \qquad (47)$$

as the dimensionless relativistic impulse factors, which express the interaction between the laser and the charge, while

$$H(\mathbf{I})=\frac{1}{2m}\left\{\frac{I_x^2}{L_x^2}+\frac{I_y^2}{L_y^2}-4I_vI_u\frac{\omega_1}{2\pi}\frac{\omega_2}{2\pi}+\frac{m^2}{2}\left(\eta_x^2+\eta_y^2\right)\right\}$$

is the conserved superhamiltonian, $-P_\tau$ in Eq.(40), expressed in terms of the four action variables.

The introduction of these variables into the dynamical phase, and their application to the principle of constructive interference,

$$\frac{\partial\tilde{S}'(x;\mathbf{I};\tau)}{\partial I_\beta}=0 \qquad \beta=u,v,x,y$$

again leads to a straightening out of the phase space trajectories, just like Eq.(23) led to Figure 8b. The only difference is that now the straight lines are tilted relative to the coordinate plane $\tau=\tau_0$. This means that the straight lines have nonzero projections onto this plane. These projected paths are

$$\frac{\partial W(x^\alpha;I)}{\partial I_\beta}=\frac{\partial W(x_0^\alpha;I)}{\partial I_\beta}+(\tau-\tau_0)\frac{\partial H(I)}{\partial I_\beta} \qquad \beta=u,v,x,y\ .$$

They lead to a number of conclusions.

> *Problem6*: a) Point out why these four equations express the same space-time trajectory as Eqs.(23).

b) show that their explicit form is

$$\begin{aligned}
(u-u_0)\frac{\omega_1}{2\pi} &= -(\tau-\tau_0)\frac{2}{m}\frac{\omega_1}{2\pi}\frac{\omega_2}{2\pi}I_v \\
(v-v_0)\frac{\omega_2}{2\pi} &+ \frac{1}{4I_v^2}\frac{2\pi}{\omega_2}\frac{2m}{\omega_1}\left[\frac{\eta_x}{L_x}I_x\cos\omega_1 u + \frac{\eta_y}{L_y}I_y\cos(\omega_1 u+\delta)\right. \\
&\qquad\qquad \left. + \frac{m}{8}\left(\eta_x^2\sin 2\omega_1 u + \eta_y^2\sin(2\omega_1 u+\delta)\right)\right]_{u_0}^{u} \\
&= -(\tau-\tau_0)\frac{2}{m}\frac{\omega_1}{2\pi}\frac{\omega_2}{2\pi}I_u \\
\frac{x-x_0}{L_x} &- \frac{1}{4I_v}\frac{2\pi}{\omega_2}\frac{2m}{\omega_1}\;\frac{\eta_x}{L_x}\cos\omega_1 u\Big|_{u_0}^{u} = (\tau-\tau_0)\frac{I_x}{mL_x^2} \\
\frac{y-x_0}{L_y} &- \frac{1}{4I_v}\frac{2\pi}{\omega_2}\frac{2m}{\omega_1}\;\frac{\eta_y}{L_y}\cos(\omega_1 u+\delta)\Big|_{u_0}^{u} = (\tau-\tau_0)\frac{I_y}{mL_y^2}
\end{aligned}$$

11.2.1 Linear Representation

1. For fixed $\mathbf{I}=(I_u,I_v,I_x,I_y)$ these paths have the linear representation

$$\phi^u = \phi_0^u + (\tau-\tau_0)\frac{\partial H(\mathbf{I})}{\partial I_u} \tag{48}$$

$$\phi^v = \phi_0^v + (\tau-\tau_0)\frac{\partial H(\mathbf{I})}{\partial I_v} \tag{49}$$

$$\phi^x = \phi_0^x + (\tau-\tau_0)\frac{\partial H(\mathbf{I})}{\partial I_x} \tag{50}$$

$$\phi^y = \phi_0^y + (\tau-\tau_0)\frac{\partial H(\mathbf{I})}{\partial I_y} \tag{51}$$

It is related to the spacetime representation relative to the physically given coordinates $x^\alpha=(u,v,x,y)$ by the transformation

$$\left.\begin{aligned}
\phi^u &= \frac{\partial W(u,v,x,y;\mathbf{I})}{\partial I_u} \\
\phi^v &= \frac{\partial W(u,v,x,y;\mathbf{I})}{\partial I_v} \\
\phi^x &= \frac{\partial W(u,v,x,y;\mathbf{I})}{\partial I_x} \\
\phi^y &= \frac{\partial W(u,v,x,y;\mathbf{I})}{\partial I_y}\ .
\end{aligned}\right\} \tag{52}$$

This transformation has changed a nonlinear representation into an equivalent representation which is linear.

11.2.2 Canonical Transformation

2. Recalling that the four-momentum of the particle is

$$p_\alpha = \frac{\partial W(u,v,x,y;\mathbf{I})}{\partial x^\alpha} \qquad \alpha = u,v,x,y\ , \tag{53}$$

one has the fact that these two sets of equations, (52) and (53), define implicitly the phase space transformation

$$(u, v, x, y, ; p_u, p_v, p_x, p_y) \longrightarrow (\phi^u, \phi^v, \phi^x, \phi^y; I_u, I_v, I_x, I_y)$$

Being generated from the scalar function $W(x^\alpha; \mathbf{I})$, this transformationis canonical, i.e. it leaves invariant the representation of the antisymmetric tensor, Eq.(8):

$$dx^\alpha \wedge dp_\alpha = d\phi^\beta \wedge dI_\beta \ .$$

11.2.3 Rotational Periodicity

3. Each of the four coordinates, Eq.(52), is the angular coordinate on one of the circles which makes up the torus, Eq.(34). Furthermore, the periodicity of each is measured in units of revolutions, one per period:

$$\left.\begin{array}{ll} \Delta\phi^u & = 1 \\ \Delta\phi^v & = 1 \\ \Delta\phi^x & = 1 \\ \Delta\phi^y & = 1 \end{array}\right\} \tag{54}$$

The existence and the magnitude of four periods expressed by these equations is validated by the ensuing calculation. One takes advantage of the four-fold periodicity of the spacetime environment of the particle's world line. This periodicity is characterized by the four periods

$$\left.\begin{array}{ll} \Delta u & = \frac{2\pi}{\omega_1} \\ \Delta v & = \frac{2\pi}{\omega_2} \\ \Delta x & = L_x \\ \Delta y & = L_y \end{array}\right\} \ . \tag{55}$$

The transformation, Eq.(52), maps the given spacetime periods, Eq.(55), into the periods, Eq.(54), of the torus. For example, using Eqs.(52) and (42) one has

$$\begin{aligned} \Delta\phi^u &= \frac{\partial W(u + 2\pi/\omega_1, v, x, y; \mathbf{I})}{\partial I_u} - \frac{\partial W(u, v, x, y; \mathbf{I})}{\partial I_u} \\ &= \frac{\partial}{\partial I_u}\left\{\int_u^{u+2\pi/\omega_1} p_u du + 0 + 0 + 0\right\} \\ &= \frac{\partial I_u}{\partial I_u} = 1 \end{aligned}$$

Analogous computations validate the remaining periodicities in Eq.(54).

For fixed action $\mathbf{I} = (I_u, I_v, I_x, I_y)$ the periodic angle coordinates $\{\phi^u, \phi^v, \phi^x, \phi^y, \}$ span a four-dimensional torus, Eq.(34), while a given spacetime trajectory, Eqs.(48)-(51), is represented by a straight line winding on it. The relation between these toroidal coordinates and the physically given ones is

$$
\begin{aligned}
\phi^u &= (u-u_0)\frac{\omega_1}{2\pi} \\
\phi^v &= (v-v_0)\frac{\omega_2}{2\pi} + \frac{1}{4I_v^2}\frac{2\pi}{\omega_2}\frac{2m}{\omega_1}\left[\frac{\eta_x}{L_x}I_x\cos\omega_1 u + \frac{\eta_y}{L_y}I_y\cos(\omega_1 u+\delta)\right. \\
&\qquad\left. + \frac{m}{8}\left(\eta_x^2\sin 2\omega_1 u + \eta_y^2\sin(2\omega_1 u+\delta)\right)\right]_{u_0}^{u} \\
\phi^x &= \frac{x-x_0}{L_x} - \frac{1}{4I_v}\frac{2\pi}{\omega_2}\frac{2m}{\omega_1}\;\frac{\eta_x}{L_x}\cos\omega_1 u\Big|_{u_0}^{u} \\
\phi^y &= \frac{y-y_0}{L_y} - \frac{1}{4I_v}\frac{2\pi}{\omega_2}\frac{2m}{\omega_1}\;\frac{\eta_y}{L_y}\cos(\omega_1 u+\delta)\Big|_{u_0}^{u}\quad .
\end{aligned}
$$

In the framework of this toroidal picture the physical coordinates are curvilinear, but the geometrical coordinates are rectilinear. Because of their geometrical simplicity, the action-angle variables are sometimes called *normal coordinates.*

11.2.4 Frequency of Rotation

4. The evolution of a particle in the periodic electromagnetic field of a travelling plane wave is mathematically equivalent to four rotation processes. Together these processes, Eq.(48)-(51), describe a point moving uniformly on a four dimensional torus. The four radii of the torus are proportional to the action of four degrees of freedom and their respective frequencies are

$$\nu_u \equiv \frac{d\phi^u}{d\tau} = \frac{\partial H(\mathbf{I})}{\partial I_u} = \left(-2\frac{\omega_2 I_v}{2\pi m}\right)\frac{\omega_1}{2\pi} \tag{56}$$

$$\nu_v \equiv \frac{d\phi^v}{d\tau} = \frac{\partial H(\mathbf{I})}{\partial I_v} = \left(-2\frac{\omega_1 I_u}{2\pi m}\right)\frac{\omega_2}{2\pi} \tag{57}$$

$$\nu_x \equiv \frac{d\phi^x}{d\tau} = \frac{\partial H(\mathbf{I})}{\partial I_x} = \left(\frac{I_x}{mL_x}\right)\frac{1}{L_x} \tag{58}$$

$$\nu_y \equiv \frac{d\phi^y}{d\tau} = \frac{\partial H(\mathbf{I})}{\partial I_y} = \left(\frac{I_y}{mL_y}\right)\frac{1}{L_y}\quad . \tag{59}$$

These equations will be recognized as the first half of Hamilton's equations relative to action-angle variables. The second half are simply

$$\frac{dI_u}{d\tau} = -\frac{\partial H((\mathbf{I})}{\partial\phi^u}\ (=0) \tag{60}$$

$$\frac{dI_v}{d\tau} = -\frac{\partial H((\mathbf{I})}{\partial\phi^v}\ (=0) \tag{61}$$

$$\frac{dI_x}{d\tau} = -\frac{\partial H((\mathbf{I})}{\partial\phi^x}\ (=0) \tag{62}$$

$$\frac{dI_y}{d\tau} = -\frac{\partial H((\mathbf{I})}{\partial\phi^y}\ (=0)\ , \tag{63}$$

and they express what one knew all along, namely that the action variables are constant along the particle path.

12 Conclusion

The key property of an integrable periodic dynamical system is that it decomposes into into degrees of freedom each one of which

- rotates uniformly,
- has a characteristic angular frequency which depends on the action of this degree of freedom, and
- executes a motion which is related to the physical motion by the *principle of constructive interference.*