SNELL'S LAW AND REFRACTION OF ELECTRON WORLD LINES BY INTENSE LASER FIELDS

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(Dated: November 8, 2008)

The dynamics of an electron driven by arbitrary plane wave laser radiation is formulated as a refraction process based on an exact index of refraction. This reformulation leads to the dynamics of the electron as being governed by (i) the Lorentzian version of what in Euclidean space is Snell's law, (ii) the spacetime version of Fermat's principle of least time, (iii) a Lorentzian stability criterion for the circumstance which in Euclidean space is the propagation of rays passing through a periodic wave guide of lenses which all have the same focal length. That criterion demands that the laser intensity satisfy $I_{rms} < .56 \cdot (10,000\text{Å}/\lambda)^2 \times 10^{18}\text{watts/cm}^2$.

PACS numbers:

When a particle of charge q and mass m is placed into a laser beam whose radiation frequency $\omega/2\pi$ has electric field amplitude E_0, then the particle executes oscillatory motion. The magnitude of the this effect is expressed by the dimensionless *impulse factor*

$$\frac{qE_0}{mc\omega} \equiv \eta \ .$$

In light of (a) the simultaneous presence of an oscillating magnetic field, and (b) the possibility of the motion being relativistic, it is not surprising that the resulting complexity in the actual motion of the particle implies a corresponding complexity of the mathematical description.

However, it has turned out that, hidden behind this complexity, there often exists a readily identifyable simplicity, which physicists have expressed in terms of what is known as the "ponderomotive potential" and its gradient, the "ponderomotive force". These quantities arise from the fact that quite often the full motion of a particle is characterized by two time scales. One characterizes the rapid quivering/oscillatory (fine-grained) aspect of the full motion. The other characterizes a slower "guiding center" (coarse-grained) motion around which the particle executes its fast quivering oscillations[1–5].

The ponderomotive force/potential, a slowly varying function of space and time, determines the slow motion dynamics of the particle. It is the result of performing a one-cycle average over the complex motion of the particle. There are a number of ways[1–4] of doing this, but their common drawback is that one not only loses potentially useful information about the particle's motion but, more importantly, misrepresents it, when the laser radiation becomes so intense ($I_{aver} \cdot (\lambda/10,000\text{Å})^2 \geq 1.37 \times 10^{18}$ watts/cm^2 $\Leftrightarrow$ $\eta \geq 1$) that the ponderomotive force varies as rapidly as the one due to the "rapid" quiverings/oscillations. Under such a circumstance, which includes a charge in a standing plane wave[1, 6, 7], an assumed decomposition into oscillatory plus averaged motion, does not apply.(See, however, the second footnote below). For one thing, an *ab initio* averaging hides the possibility of resonance[8] where the frequency of the oscillatory motion is a multiple of that of the averaged motion.

Furthermore, femtosecond pulsed laser radiation makes any averaging scheme meaningless. There is not enough time to establish an average which changes slowly. Neither does averaging apply to charges located in, and hence scattered by, the transient overlap region of two counter propagating few cycle pulses.

There is a superior way of understanding the dynamics of an electron driven by arbitrary plane wave laser radiation. We shall introduce an index of refraction which is exact for arbitrarily relativistic motion and/or arbitrarily short laser pulses and then reformulate the dynamics as a Lorentzian refraction process, as compared to its well known Euclidean cousin. Taking a cue from the principles of geometrical optics, this reformulation leads to (i) the Lorentzian version of what in Euclidean space is Snell's law, (ii) the spacetime version of Fermat's principle of least time, (iii) a Lorentzian stability criterion for the circumstance which in Euclidean space is the propagation of rays passing through a periodic wave guide of lenses which all have the same focal length.

Dynamics of a Charge in a Generic Plane Wave Field – Consider an electron born at a generic location of the electromagnetic plane wave field of a laser directed along the z-direction. The four components of the vector potential are

$$\{A_0, A_1, A_2, A_3\} = \{0, 0, A_y(t, z), A_x(t, z)\} \ . \quad (1)$$

The dynamics of the particle is governed by the

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four Lorentz (force law) equations of motion[9, 10] $md^2x^\alpha/d\tau^2 = qF^\alpha_{\ \beta}dx^\beta/d\tau$ ($\alpha = 0,1,2,3$) for an electron of mass m and charge q in the e.m. field $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. For a generic plane wave along the z-direction, these equations imply

$$m\frac{d^2t}{d\tau^2} = +\frac{\partial}{\partial t}\Phi(t,z) \tag{2}$$
$$m\frac{d^2z}{d\tau^2} = -\frac{\partial}{\partial z}\Phi(t,z) \tag{3}$$
$$m\frac{dy}{d\tau} = [p_y - qA_y(t,z)] \tag{4}$$
$$m\frac{dx}{d\tau} = [p_x - qA_x(t,z)]\ ,\ . \tag{5}$$

They are invariant under gauge changes having plane wave symmetry. Here time is measured in light travelling distance, (p_x, p_y) is the particle's conserved ("canonical") transverse momentum, and

$$\Phi(t,z) = \frac{m}{2}\left\{\left(\frac{p_x}{m} - \frac{q}{m}A_x(t,z)\right)^2 + \left(\frac{p_y}{m} - \frac{q}{m}A_y(t,z)\right)^2\right\}\ , \tag{6}$$

is the scalar potential which characterizes the longitudinal dynamics in the (t,z)-plane.

The dynamics of the laser-accelerated charge is controlled entirely by Eqs.(2) and (3). We therefore refer to them as the *master system of equations.* By contrast, Eqs. (4) and (5) are merely *slave equations.* In relation to the longitudinal degrees of freedom these equations are dynamically passive. They have no effect on the particle dynamics in the (t,z)-plane. Instead, they identify a physical measurable property, the transverse x and the y-velocity components.

The master system of equations implies that

$$\frac{1}{2}m\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2}m\left(\frac{dz}{d\tau}\right)^2 - \Phi(t,z) \equiv \mathcal{H} = \frac{m}{2} \tag{7}$$

is an integral of motion. The integration constant $\frac{m}{2}$ has been chosen such that the laboratory clock and a clock comoving with the particle remain synchronized whenever the particle is at rest $\left(\frac{dz}{d\tau} = \frac{dx}{d\tau} = \frac{dy}{d\tau} = 0\right)$ in the lab frame.

Dynamics as Geometrical Optics – The integral of motion, Eq. (7), has a property that leads directly to a geometrical optics formulation of the dynamics of the particle: Rewrite the integral in the form

$$\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 = n^2(t,z) \tag{8}$$

where

$$n(t,z) = \sqrt{1 + \left(\frac{p_x}{m} - \frac{q}{m}A_x(t,z)\right)^2 + \left(\frac{p_y}{m} - \frac{q}{m}A_y(t,z)\right)^2} \tag{9}$$

The integral, Eq.(8), holds along every spacetime trajectory. This suggests that one should replace the proper time τ of the charge with its *longitudinal* proper time[1]

$$\int d\tilde{\tau} = \int n\left(t(\tau), z(\tau)\right) d\tau \tag{10}$$

as the world line parameter. Such a replacement yields $\frac{d\tilde{\tau}}{d\tau} = n\ (> 0)$. Consequently, the integral of motion becomes

$$\left(\frac{dt}{d\tilde{\tau}}\right)^2 - \left(\frac{dz}{d\tilde{\tau}}\right)^2 = 1 \tag{11}$$

and Eqs. (2)-(3) become the *optical master equations*

$$\frac{d}{d\tilde{\tau}}n\frac{dt}{d\tilde{\tau}} = +\frac{\partial n}{\partial t} \tag{12}$$
$$\frac{d}{d\tilde{\tau}}n\frac{dz}{d\tilde{\tau}} = -\frac{\partial n}{\partial z}\ , \tag{13}$$

or more succinctly,

$$\frac{d}{d\tilde{\tau}}n\frac{dx^A}{d\tilde{\tau}} = \eta^{AB}\frac{\partial n}{\partial x^B}\ ;\ \ \eta^{AB} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\ ;\ \ \left.\begin{matrix} A \\ B \end{matrix}\right\} = 0,1\ . \tag{14}$$

Compare these equations with those for a light ray through a medium with refractive index $n(x^1, x^2, x^3)$ in geometrical optics[11],

$$\frac{d}{ds}n\frac{dx^i}{ds} = \frac{\partial n}{\partial x^i}\ ;\ \ i = 1,2,3\ , \tag{15}$$

where s is the geodesic length parameter along the ray. One sees that if $n(x^1, x^2, x^3)$ is the familiar Euclidean index of refraction for a medium in Euclidean space, then Eq.(9) is the *Lorentzian index of refraction* for the e.m.-induced medium in Lorentzian spacetime. The former determines the ray trajectories $x^i(s)$; the latter determines the world lines $x^A(\tilde{\tau})$.

The optical master equations are a local expression of $\delta\int n(t,z)\sqrt{dt^2 - dz^2} = 0$, which is what in Euclidean space corresponds to Fermat's principle of least time, $\delta\int n(x^1,x^2,x^3)\sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2} = 0$.

All world lines are determined by the optical master Eq.(12)-(13). A cursory examination reveals that Eq.(11) together with Eq.(12) (resp. Eq.(13)) implies Eq.(13) (resp. Eq.(12)). Consequently, one readily finds that the world lines satisfy the following single differential equation,

$$\frac{d^2z}{dt^2} = -\left[1 - \left(\frac{dz}{dt}\right)^2\right]\frac{1}{2n^2}\left[\frac{\partial n^2}{\partial t}\frac{dz}{dt} + \frac{\partial n^2}{\partial z}\right] \tag{16}$$

[1] This time is measured by a clock moving along fixed x and y coordinates, but comoving (and accelerating) with the charge strictly along the z-direction.

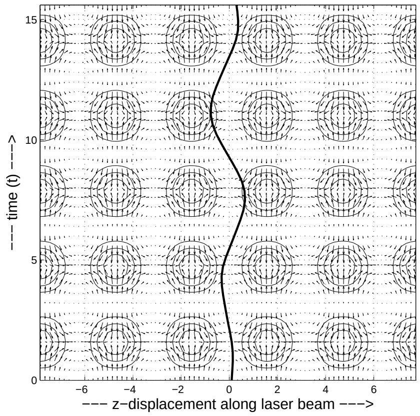

FIG. 1: World line of a charge through the periodic spacetime lattice of the electromagnetic standing wave field of a laser. The periodic index of refraction, Eq.(19), has maxima which are surrounded by the oval shaped isograms. The gradient vector field is (Lorentz) orthogonal to these isograms. Furthermore, this vector field pushes the charge and thereby imparts to it the acceleration given by Eq.(12)-(13). For $\eta = 1.5$ and for the depicted initial conditions a numerical integration of these equations yields the world line reproduced in this figure.

Given any laser-induced refractive index, Eq.(9), this *master equation* governs all properties of all possible electron trajectories resulting from a generic plane wave laser environment.

One of the properties of a spacetime trajectory is how it gets bent (a.k.a. accelerated) by the refractive index gradient. The right hand side of Eq.(16) is proportional to the directional derivative along the comoving z-axis. Indeed, given that, in light of Eq.(11), the unit tangent to the 2-d world line is $\left(\frac{dt}{d\tilde{\tau}}, \frac{dz}{d\tilde{\tau}}\right)$, the unit vector along the comoving z-axis is $\left(\frac{dz}{d\tilde{\tau}}, \frac{dt}{d\tilde{\tau}}\right) \equiv \hat{z}$. It follows that

$$\frac{d}{d\tilde{\tau}}\left(\frac{dz}{dt}\right) = -\left[1 - \left(\frac{dz}{dt}\right)^2\right] \partial_{\hat{z}} \ln n \qquad (17)$$

where $\partial_{\hat{z}} \ln n = \frac{dz}{dt}\partial_t \ln n + \frac{dt}{dt}\partial_z \ln n$. Thus, if n increases along the positive (resp. negative) z-direction, the charge will accelerate into the negative (resp. positive) z-direction. In other words, opposite to the Euclidean circumstance, regions of higher Lorentzian index of refraction accelerate charges into a direction where the index is smaller. *A charge gets expelled.*

A second property arises for a particularly interesting case, a charge driven by the laser field of a standing wave[1, 6, 7] linearly polarized along, say, the x-axis. For such a wave the e.m. vector potential is

$$A_x(t,z) = \frac{E_0}{\omega} \sin\omega t \sin\omega z \ , \ A_y(t,z) = 0 \ . \qquad (18)$$

and the squared index of refraction is

$$n^2(t,z) = 1 + (p_x/m - \eta \sin\omega t \sin\omega z)^2 \ . \qquad (19)$$

Such a refractive index $n(t,z)$ forms a two-dimensional optical lattice of what in Euclidean space are convex and and concave lenses. Referring to Figure 1 (*Note:* In this figure we have set $p_x = 0$. This suppresses certain interesting features associated with *averaged* motion, which have been identified in[6, 7].), one infers from the depicted trajectory that there exists a linear stack of of convex "Lorentzian" lenses all localized around $z = 0$ half way between two adjacent maxima of the refractive index $n(t,z)$. In Figure 1 they are separated ("vertically") by an amount $L = \pi$. These lenses guide the center of the oscillating trajectory. Moreover, each lense has a *temporal focal length.* For a laser beam having a standing wave field of frequency $\frac{\omega}{2\pi}$, this *focal length* is

$$\frac{c}{\pi\omega}\frac{1}{\eta^2} \equiv F \qquad (20)$$

It is a property pertaining only to "paraxial" trajectories, i.e. those for which the lab velocity of the charge is non-relativistic: $\left(\frac{dz}{dt}\right)^2 \ll 1$.

The mathematical reasoning that leads to Eq.(20). is taken directly from geometrical optics. It starts with Eq.(16) combined with Eq.(19). The idea is to calculate for a single lens

$$\Delta\left(\frac{dz}{dt}\right) \equiv \int_0^{2\pi/\omega} \frac{d^2z}{dt^2}dt \ ,$$

the change in particle velocity over time $2\pi/\omega$ for trajectories near $z = 0$ that start with $\frac{dz}{dt} = 0$. The result for such paraxial spacetime trajectories is

$$\Delta\left(\frac{dz}{dt}\right) = -\frac{z}{F} \ ,$$

where F, the temporal focal length, is given by Eq.(20).

Stable and Unstable Motion – The existence of a focal length and the periodic structure encountered by a paraxial particle trajectory direct attention to the possibility of parametric instability in its oscillatory motion. Recall that from geometrical optics one knows that if neighboring convex lenses of a periodic stack are separated by the same amount L, then such a lens system accomodates stable paraxial trajectories if and only if[12, 13]

$$0 < \frac{(Separation\ length)}{(Focal\ length)} \equiv \frac{L}{F} < 4$$

From Figure 1 one sees that the separation between consecutive lenses is

$$L = \pi\frac{c}{\omega}$$

In light of Eq.(20) one finds that paraxial spacetime trajectories oscillate around $z = 0$, i.e. are stable if and only if the laser impulse parameter satisfies

$$\eta^2 < \frac{4}{\pi^2} \ , \qquad (21)$$

which is to say, the laser intensity should satisfy $I_{rms} < .56 \cdot (10,000\text{Å}/\lambda)^2 \times 10^{18}\text{watts/cm}^2$. If this inequality is violated one has a possible type of parametric resonance, which we alluded near the beginning of this article.

Snell's Law[14] – A fundamental manifestation of a refraction process is the manner in which a ray propagates across the boundary between two regions having different indices of refraction. In Euclidean space this propagation is expressed by Snell's law. What is its form for a spacetime medium whose index of refraction varies (in the limit) discontinuously, as in Figure 2?

Consider the world line of a charge as it leaves a spacetime medium with index n_- and enters another one with index n_+. Fig. 2 gives a close-up view. One introduces the "null" coordinates

$$u = t - z \quad \text{“retarded time” coordinate} \tag{22}$$
$$v = t + z \quad \text{“advanceded time” coordinate} \tag{23}$$

and lets the boundary be the locus of events where

$$v = 0 \quad \text{“history of pulse discontinuity”} \ . \tag{24}$$

It is along characteristics like this where solutions to the wave equation

$$\frac{\partial^2}{\partial u \partial v}\begin{pmatrix} A_x(u,v) \\ A_y(u,v) \end{pmatrix} = 0 \tag{25}$$

are allowed to be discontinuous, and hence where the index of refraction, Eq. (9), is allowed to be discontinuous.

A charge which crosses such a boundary will experience a refractive index of the form

$$n(u,v) = \begin{cases} n_- & v < 0 \\ n_+ & 0 < v \end{cases} \tag{26}$$

The indices are different but constant on either side of the boundary.

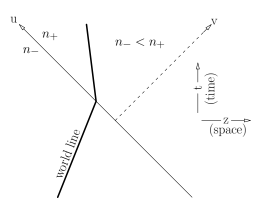

FIG. 2: Refraction of the world line of a charge

The problem is indicated in Figure 2. One must establish the relationship between the slopes

$$\frac{dv}{du} = \frac{d(t+z)}{d(t-z)} = \frac{1 + dz/dt}{1 - dz/dt} \tag{27}$$

on either side of the "null" boundary ("characteristic" of the wave equation) $v = 0$. The solution consists of the statement that

$$\left.\frac{dv}{du}n^2\right|_+ = \left.\frac{dv}{du}n^2\right|_- \ , \tag{28}$$

where the "+" and "-" refer to $0 < v$ and $v < 0$ repectively. This is the Lorentzian version of what is Snell's law in Euclidean space.

The slope dv/du is well known. It is the square of the *Doppler factor ("rapidity factor")* e^θ for a particle with z-velocity $\frac{dz}{dt} = \tanh\theta \equiv (e^\theta - e^{-\theta})/(e^\theta + e^{-\theta})$. The quantity θ is generally known as the particle's *rapidity*. The differencies and similarities between the Lorentzian and the Euclidean versions of Snell's law become particularly perspicuous when one introduces this Doppler factor. In terms of it Snell's law takes the form[14]

$$\left. e^\theta n \right|_{v>0} = \left. e^\theta n \right|_{v<0} \tag{29}$$

across a left-travelling pulse ($v = 0$). This is to be contrasted with the Euclidean version of Snell's law, which we recall, is

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{30}$$

The validity of Eq. (28), and hence of Eq. (29), follows directly from Eq.(11) and the optical master Eqs. (12)-(13) recast in terms of the "null" coordinates u and v:

$$\frac{du}{d\tilde{\tau}}\frac{dv}{d\tilde{\tau}} = 1 \tag{31}$$

$$\frac{d}{d\tilde{\tau}} n \frac{du}{d\tilde{\tau}} = 2\frac{\partial n(u,v)}{\partial v} \quad \text{and} \quad \frac{d}{d\tilde{\tau}} n \frac{dv}{d\tilde{\tau}} = 2\frac{\partial n(u,v)}{\partial u} \ . \tag{32}$$

By stipulating that $\frac{\partial n}{\partial u} = 0$ as in Figure 2, one can replace $\tilde{\tau}$ with v as a worldline parameter. With the help of the second equation of Eq. (32), Eq. (31) leads directly to Eq. (28).

Conclusion – Relativistic particle dynamics driven by generic plane wave laser radiation of arbitrary intensity is expressed in terms of the Lorentzian ("spacetme") version (i) of the optical master equations, (ii) Fermat's principle of least time, (iii) of Snell's law, (iv) of the focal lengths of Lorentzian lenses, which make up the periodic spacetime lattice of a standing wave, and (v) of the associated stability criterion for particles moving through this lattice.

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