On Graded Lie Algebras of Characteristic Three With Classical Reductive Null Component

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0 Introduction

Abstract

Irreducible transitive graded Lie algebras $L = \sum_{i=-q}^{r} L_i$ which have a classical reductive null component L_0 and which are finite dimensional over algebraically closed fields of characteristic three are considered. We show that if the depth q is greater than one, then the representation, induced by L's adjoint representation, of L_0' on L_{-1} must be restricted.

The classification of finite-dimensional simple Lie algebras over algebraically closed fields $\mathbf{F}$ of characteristic $p > 0$ depends to a great extent on the classification of the finite-dimensional irreducible transitive graded Lie algebras $L = \bigoplus_{i=-q}^{r} L_i$ of depth $q \geqq 1$ with classical reductive null component L_0. We recall some of the progress that has been made in the classification of such Lie algebras L. In the case in which L_{-1} is not only irreducible but also restricted as an L_0-module, such Lie algebras are described by the Recognition Theorem of Kac [K] for $p > 5$. (See also [BGP].) In [BG] it is shown that for $p > 5$, L_{-1} is necessarily a restricted L_0'-module. (The assertion is also true for $p = 5$ [BGP].)

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When $p = 3$, the situation is more complicated. In characteristic three, there are series of simple graded Lie algebras which satisfy the conditions of Kac's Recognition Theorem, but which are neither classical Lie algebras nor Lie algebras of Cartan type. (See [B], [Sk], [St].) Moreover, for $q = 1$, examples exist in which L_{-1} is a non-restricted L_0'−module. All simple depth-one graded Lie algebras of characteristic three with non-restricted L_0'-module L_{-1} were determined in [BKK]. In [BGK], two-graded (i.e., depth-two, graded) Lie algebras were examined, and it was proved that when $p = 3$ and $q = 2$, the L_0'−module L_{-1} must be restricted. For $q = 3$, the corresponding statement was proved in [GK]. It was conjectured in [BGK] that a non-restricted L_0'−module L_{-1} can exist only in Lie algebras of depth one. The present paper completes the proof of this conjecture. One has, of course, to exclude the case of $H(2 : \mathbf{n}, \omega)$ with the reverse gradation; however, this example does not satisfy condition (D) of the Main Theorem below (for depth greater than one). One has also to exclude the sum of a degenerate Lie algebra (in the sense of Theorem 1.3 below) and a (simple, classical) Lie algebra which resides in the null component and acts non-restrictedly on the minus-one component. It follows that all graded Lie algebras with non-restricted L_0'−module L_{-1} are known. We note, as in [BGK], that because there are only finitely many irreducible restricted modules for the derived algebra of a classical reductive Lie algebra, what needs to be considered in classifying graded Lie algebras over algebraically closed fields of characteristic three is reduced.

In this paper, we prove the following theorem, which we will henceforth call the "Main Theorem."

Theorem 0.1 (Main Theorem) *Let* $L = L_{-q} \oplus L_{-q+1} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_r$, $q > 1$, $r > 0$, *be a finite-dimensional graded Lie algebra over an algebraically closed field* $\mathbf{F}$ *of characteristic* $p = 3$ *such that*

(A) L_0 *is classical reductive;*

(B) L_{-1} *is an irreducible* L_0*-module (i.e.,* L *is irreducible);*

(C) *for all* $j \geqq 0$, *if* $x \in L_j$ *and* $[x, L_{-1}] = (0)$, *then* $x = 0$ *(i.e.,* L *is transitive);*

(D) $L_{-i} = [L_{-i+1}, L_{-1}]$ *for all* $i > 1$;

(E) $L_{-2} \nsubseteq M(L)$, *where* $M(L)$ *is the largest ideal of* L *contained in the sum of the negative gradation spaces. (See Theorem 1.3 below.)*

Then L_{-1} is a restricted module for L_0' under the adjoint action, except when $L = L' + I_0$, where L' is degenerate (in the sense of Theorem 1.3 below) and where the representation (induced by the adjoint representation of L) of I_0, a summand of the null component of L, on L_{-1} is not restricted.

To help to motivate hypothesis (E) above, we offer the following

Example 0.2 *Consider the irreducible transitive graded Lie algebra*

$$R \overset{def}{=} \mathcal{O}(2:(1,\,1)) \oplus H(2:(1,\,1)) = \bigoplus_{i=-2-2(p-1)}^{2p-5} R_i,$$

where $R_i = H(2:(1,\,1))_i$ for $i \geqq -1$, and $R_i = \mathcal{O}(2:(1,\,1))_{i+2p}$ for $-2p = -2-2(p-1) \leqq i \leqq -2$. Here, the divided-power algebra $\mathcal{O}(2:(1,\,1))$ is an abelian ideal of R, and $H(2:(1,\,1))$ has its usual Lie algebra multiplication and action on $\mathcal{O}(2:(1,\,1))$, except that $[D_{x_1},\, D_{x_2}] = x_1^{p-1}x_2^{p-1} \in R_{-2}$. Then $R/M(R) \cong R/\mathcal{O}(2:(1,\,1))$ (See Theorem 1.3 below.) has depth one. In general, if we consider the free Lie algebra generated by the local part of any depth-one graded Lie algebra L, and take a co-finite-dimensional subideal C of the maximal ideal D in the negative part (See [BW].), then $M(L \oplus D/C) = D/C$, and $(L \oplus D/C)/M(L \oplus D/C) \cong L$ has depth one.

We have noted that the Main Theorem has been proved for $q = 2$ in [BGK] and for $q = 3$ in [GK]. When we refer to the Main Theorem to substantiate certain claims below, it will be for the cases already proved.

1 Preliminaries

Recall that a classical Lie algebra over a field $\mathbf{F}$ of characteristic $p > 0$ can be obtained from a $\mathbb{Z}$-form (the so-called Chevalley basis) of a complex simple Lie algebra by reducing the scalars modulo p and extending them to $\mathbf{F}$. This process may result in a Lie algebra with a non-zero center; such a Lie algebra is still referred to as "classical" as is the quotient of such a Lie algebra by its center. For example, the Lie algebras $\mathfrak{gl}(pk)$ and $\mathfrak{pgl}(pk)$ are both considered to be classical Lie algebras. Thus, a classical Lie algebra $\mathfrak{g}$ may have a nontrivial center $Z(\mathfrak{g})$ as do the Lie algebras $\mathfrak{gl}(pk), \mathfrak{sl}(pk)$, and, if $p = 3$, E_6. It could also happen that a classical Lie algebra has a noncentral ideal, as do the Lie algebras $\mathfrak{gl}(pk)$, $\mathfrak{pgl}(pk)$, and, if $p = 3$, G_2. In characteristic three, G_2 contains an ideal I isomorphic to $\mathfrak{psl}(3)$, and $G_2/I \cong \mathfrak{psl}(3)$, as well.

A classical reductive Lie algebra $\mathfrak{g}$ is the sum of commuting ideals $\mathfrak{g}_j$ which are classical Lie algebras, and an at-most-one-dimensional center $\mathfrak{z}(\mathfrak{g})$:

$$\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_k + \mathfrak{z}(\mathfrak{g}) \tag{1.1}$$

For any classical Lie algebra $\mathfrak{g}_j$, the derived algebra $\mathfrak{g}_j'$ has a natural p-structure such that $e_\alpha^{[p]} = 0$, and $h_i^{[p]} = h_i$ for any Chevalley basis $\{e_\alpha,\ h_i \mid \alpha \in R,\ i = 1, \ldots, \operatorname{rank}(\mathfrak{g}_j')\}$ of $\mathfrak{g}_j'$, where R is the root system of the corresponding complex simple Lie algebra. For a classical reductive Lie algebra $\mathfrak{g}$, we will consider only the $p-$structure on $\mathfrak{g}' = \mathfrak{g}_1' + \cdots + \mathfrak{g}_k'$ which is the natural p-structure on each classical summand.

Let $\pi \longrightarrow \mathfrak{gl}(n)$ be a finite-dimensional irreducible representation of a restricted Lie algebra L. The *character* χ of π is a linear functional on L such that $\chi(y)^p I = \pi(y)^p - \pi(y^{[p]})$ for all $y \in L$. The representation is restricted when the character $\chi = 0$.

Lemma 1.2. *(See [BG, Lemma 1].) Assume that L is a graded Lie algebra satisfying conditions (A)-(D) of the Main Theorem. If χ is the character of L_0 on L_{-1}, then L_0 has character $-j\chi$ on L_j for all j.*

The following theorem of Weisfeiler [W] plays a fundamental rôle in the study of graded Lie algebras.

Theorem 1.3. *Let $L = L_{-q} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_r$ be a graded Lie algebra such that conditions (B)-(D) of the Main Theorem hold. Let $M(L)$ denote the largest ideal of L contained in $L_{-q} \oplus \cdots \oplus L_{-1}$. Then*

(i) $L/M(L)$ is semisimple and contains a unique minimal ideal $I = S \otimes \mathcal{O}(n : \mathbf{1})$, where S is a simple Lie algebra, n is a non-negative integer, and $\mathcal{O}(n : \mathbf{1}) = \mathbf{F}[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$. The ideal I is graded and $I_i = (L/M(L))_i$ for all $i < 0$.

(ii) If $I_1 = (0)$, then for some κ, $1 \leqq \kappa \leqq n$, the algebra $\mathcal{O}(n : \mathbf{1})$ is graded by setting $\deg(x_i) = -1$ for $1 \leqq i \leqq \kappa$ and $\deg(x_i) = 0$ for $\kappa < i \leqq n$. Then $I_i = S \otimes \mathcal{O}(n : \mathbf{1})_i$ for all i, $L_2 = (0)$, $I_0 = [L_{-1}, L_1]$, and $L_1 \subseteq \{D \in 1 \otimes \operatorname{Der}\mathcal{O}(n : \mathbf{1}) |\deg(D) = 1\}$.

(iii) If $I_1 \neq (0)$, then S is graded and $I_i = S_i \otimes \mathcal{O}(n : \mathbf{1})$ for all i. Moreover, $(0) \neq [L_{-1}, L_1] \subseteq I_0$.

We will make use of the following results from [BGK]. For definitions of the Lie algebras $L(\epsilon)$, M, $H(2 : \underline{n}, \omega)$, and $CH(2 : \mathbf{n}, \omega)$ mentioned in the conclusion of Proposition 1.4 below, see, for example, Section 2 of [BGK].

When we make use of certain properties of these Lie algebras in later sections, we will explicitly state the properties we need.

Proposition 1.4. *(See Lemma 2.12 of [BGK].) Let* $L = L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_r$ *be a graded Lie algebra satisfying conditions (A), (B), and (C) of the Main Theorem, and suppose that* $L_1 \neq 0$*. If* L_{-1} *is a nonrestricted* L_0'*−module, then either* L *is isomorphic to one of the Lie algebras* $L(\epsilon)$ *or* M*, or* L *is a Hamiltonian Lie algebra such that* $H(2 : \mathbf{n}, \omega) \subseteq L \subseteq CH(2 : \mathbf{n}, \omega)$*, where* $\mathbf{n} = (1, n_2)$*,* $\omega = (\exp x^{(3)}) dx \wedge dy$*, and the grading is of type (0,1).*

Corollary 1.5. *(See Corollary 2.13 of [BGK].) Under the assumptions of Proposition 1.4,* $L_0' \cong \mathfrak{sl}(2)$*,* L_1 *is an irreducible 3-dimensional* L_0'*−module, and* $[L_1, L_1] = 0$*. In addition,* $[L_{-1}, L_1] = \mathfrak{sl}(2)$ *if and only if* L *is a Hamiltonian Lie algebra.*

Lemma 1.6. *(See Lemma 2.14 of [BGK].) Let* $L = L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_q$ *be one of the Lie algebras* $L(\epsilon)$*,* M*, or* $H(2 : \mathbf{n}, \omega)$ *with* $\mathbf{n} = (1, n_2)$*, let* χ *be the nonzero character of the* L_0*-module* L_{-1}*, and let* V *be an* L*-module such that* $l^3 \cdot V = (0)$ *for any* $l \in L_{-1} \cup L_1$*. Suppose that* W *is an irreducible* L_0*-submodule of* V *with character* $\chi_W = \zeta\chi$*,* $\zeta \in \mathbf{F}^\times$*, and suppose that* $L_1 \cdot W = (0)$*. Then* $L_{-1}^2 \cdot W \neq (0)$*. Similarly, if* $L_{-1} \cdot W = (0)$*, then* $L_1^2 \cdot W \neq (0)$*.*

In what follows, all Lie algebras will be finite-dimensional over an algebraically closed field $\mathbf{F}$ *of characteristic* $p = 3$*. The commutator ideal* $[L, L]$ *of a Lie algebra* L *will be denoted by* L'*, and the* i^{th} *commutator* $(\operatorname{ad} X)^{i-1} X$ *of any set* X *will be written as* X^i*. The annihilator of an* L_0*−module* $M \subseteq L$ *in an* L_0*−module* $N \subseteq L$ *will be denoted by* $\operatorname{Ann}_N M$*. Set*

$$L_{<0} \stackrel{def}{=} \bigoplus_{i=-q}^{-1} L_i,$$

and

$$L_{>0} \stackrel{def}{=} \bigoplus_{i=1}^{r} L_i.$$

2 Properties of irreducible transitive graded Lie algebras

We begin this section by quoting a few results from [BG]. Let L, $M(L)$, I, and $S = \sum_{i=-q}^{s} S_i$ be as in Theorem 1.3. Throughout this section, we make the following two blanket assumptions:

(i) $M(L) = 0$

(ii) $I = S$.

Lemma 2.1. *(See [BG, Lemma 6].) For any x in $L \backslash L_{-q}$, $[L_{-1}, x] \neq 0$.*

Lemma 2.2. *(See [BG, Lemma 7].) $S_j = (\mathrm{ad}\, S_{-1})^{s-j} S_s$ for all j, $-q \leqq j \leqq s$. If $q(t-1) \leqq s$, then $(\mathrm{ad}\, S_{-q})^t S = 0$ if and only if $(\mathrm{ad}\, S_{-q})^t S_i = 0$ for some i, $q(t-1) \leqq i \leqq s$.*

Lemma 2.3. *(See [BG, Lemma 8].) $[L_{-q}, L_i] \neq 0$ for all $i = 0, \dots r$. In addition, $S_j = (\mathrm{ad}\, L_{-1})^{r-j} L_r$ for all j, $-q \leqq j \leqq r-1$, so that $s = r-1$ or r. S_s is an irreducible S_0-module.*

Lemma 2.4. *(See [BG, Lemma 9].) S_{-q} is an irreducible S_0-module. In particular, L_{-q} is an irreducible L_0-module.*

Lemma 2.5. *(See [BG, Lemma 10].) $\mathrm{Ann}_{L_0} L_i \cap \mathrm{Ann}_{L_0} V_{i+1} = 0$ for all $i = -q, \dots, r-1$, where V_{i+1} is any non-zero L_0-submodule of L_{i+1}.*

Lemma 2.6. *(See [BG, Lemma 11].) $\mathrm{Ann}_{L_i} L_{-q} \cap \mathrm{Ann}_{L_i} V_{-q+1} = 0$ for all $i = 0, \dots, r$, where V_{-q+1} is any non-zero L_0-submodule of L_{-q+1}.*

Lemma 2.7. *(See [BG, Lemma 12].) $\mathrm{Ann}_{L_{q-1}} L_{-q+1} = 0$.*

Lemma 2.8. *(See [BG, Lemma 13].) If $r \geqq q$, then $L_{-q+i} = [L_{-q}, L_i]$ for $i = 0, 1, \dots q-1$.*

Lemma 2.9. *(See [BG, Lemma 14].) Let U, V be L_0-submodules of L such that $[U, V] \subseteq L_0$ and $[U, [U, V]] = 0$. Then $\{\mathrm{ad}\,[u, v] | u \in U, v \in V\}$ is weakly closed (in the sense of [J, p.31]); consequently, if $(\mathrm{ad}\,[u, v])^i M = 0$, $u \in U$, $v \in V$, for some $i > 1$, and L_0-module M, then $\mathrm{ad}_M [U, V]$ is "associative nilpotent." (See Theorem II.2.1 of [J].)*

Lemma 2.10. *(See [BG, Lemma 15].) If $r \geqq q$, then* $(\operatorname{ad} L_{-q})^2 L \neq (0)$.

Corollary 2.11. *If $s \geqq q$, then $[S_{-q}, [S_{-q}, S_q]] \neq (0)$. In particular, if $s \geqq q$, then $[L_{-q}, [L_{-q}, L_q]] \neq (0)$.*

Proof. Lemma 2.2, and Lemma 2.10 applied to the Lie algebra $S + L_0$.
□

Lemma 2.12. *(See [BG, Lemma 16].) Let V be an L_0-submodule of L_{-q+i} for some i, where $0 < i \leqq \frac{q}{2}$, and suppose that $[V, L_{q-i-1}] = 0 = [V, [V, L_{q-i}]]$. Suppose further that L_{-q+i-1} is an irreducible L_0-submodule of L, and that $[L_{-q+i-1}, L_{q-i}] \neq 0$ (so that it equals L_{-1}). Then $V = 0$.*

Lemma 2.13. *(See [BG, Lemma 17].) Suppose that V is an irreducible L_0-submodule of L_{-q+i} for some i, where $0 < i < \frac{q-1}{2}$, such that $[V, L_{q-i-1}] \neq 0$ (so that it equals L_{-1}). Then L_{-q+i} is an irreducible L_0-module; i.e., it equals V.*

From our observations at the beginning of this section, we have that $S \subseteq L \subseteq \operatorname{Der} S$ where $S = S_{-q} \oplus S_{-q+1} \oplus \cdots \oplus S_{-1} \oplus S_0 \oplus S_1 \oplus \cdots \oplus S_s$ is a simple Lie algebra with $S_i = L_i$ for $i < 0$. Since S_1 is an L_0-submodule of L_1, it follows that if L_1 is an irreducible L_0-module, then $S_1 = L_1$. If, in addition, L is generated by its local part $L_{-1} \oplus L_0 \oplus L_1$, then for $i \geqq 1$, we have

$$L_i = L_1^i = S_1^i \subseteq S_i \subseteq L_i,$$

so that $S_i = L_i$ for $i > 0$, and L could differ from S only in the null component. In particular, s would equal r.

In the lemmas that follow, we will consider graded Lie algebras $L = L_{-q} \oplus L_{-q+1} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_r$ satisfying assumptions (i) and (ii) below. Other assumptions will be noted in the statements of the results for which we use them. Note, for example, that, as noted in the paragraph above, assumption (iii) follows from assumptions (iv) and (v) (and assumptions (i) and (ii), of course). Also, assumption (viii) can be assumed whenever the previous assumptions are true, since if they hold, we can reverse the gradation, and have that all of the hypotheses of the Main Theorem continue to be true for the reversed gradation; in this regard, see Lemma 2.14 below, as well as Lemma 1.2 above.

(i) L satisfies conditions (A)-(D) of the Main Theorem.

(ii) $L \subseteq \mathrm{Der} S$ where $S = S_{-q} \oplus S_{-q+1} \oplus S_{-1} \oplus S_0 \oplus S_1 \oplus \cdots \oplus S_s$ is a simple graded Lie algebra.

(iii) $L_i = S_i,\ i \neq 0$.

(iv) $L_1 = S_1$ is an irreducible L_0-module.

(v) $L_{i+1} = [L_i, L_1]$ for $i > 0$.

(vi) If x is a non-zero element in L_{-i} for some $i \geqq 0$, then $[L_1, x] \neq (0)$.

(vii) The character χ of L_0' on L_{-1} is non-zero.

(viii) $r \geqq q$.

Lemma 2.14. *If assumptions (iv) and (v) hold, and* $S_1 \neq 0$, *then* $\mathrm{Ann}_{L_0} L_1 = 0$.

Proof. Suppose, on the contrary, that $A_0 \stackrel{\text{def}}{=} \mathrm{Ann}_{L_0} L_1 \neq 0$. Then (as in [BG, Lemma 18])

$$[L_{-1},\, L_1] = [[L_{-1},\, A_0],\, L_1] = [[L_{-1},\, L_1],\, A_0] \subset A_0,$$

so that by transitivity (C),

$$0 \neq [S_{-1},\, L_1] \subseteq A_0 \cap S_0 \subseteq \mathrm{Ann}_{S_0} S_1.$$

Since for $i > 0$, we have by assumptions (iv) and (v) that $S_i = S_1^i$, and since $J \stackrel{\text{def}}{=} S_{-q} \oplus S_{-q+1} \oplus \cdots \oplus S_{-1} + A_0 \cap S_0$ is invariant under $\mathrm{ad}\, S_i,\ -q \leqq i \leqq s$, it follows from our assumption that $S_1 \neq 0$ that J is a proper ideal of the simple Lie algebra S; i.e., we have obtained a contradiction. Thus, we must conclude that $\mathrm{Ann}_{L_0} L_1 = 0$. □

Lemma 2.15. *If assumption (vi) holds, then* $[V_{-2},\, L_1] = L_{-1}$ *for any non-zero* L_0*−submodule* V_{-2} *of* L_{-2}.

Proof. This lemma follows from assumptions (vi) and (B). □

Lemma 2.16. *If assumption (iv) holds, then* $\mathrm{Ann}_{L_i} L_{-q} = 0$ *for all* $i > 0$.

Proof. Consider first the case in which $i = 1$. If $\mathrm{Ann}_{L_1} L_{-q} \neq 0$, then, since we are assuming (iv) that L_1 is an irreducible L_0-module, we would have $\mathrm{Ann}_{L_1} L_{-q} = L_1$. But then

$$[L_{-q},\, L_1] = [L_{-q},\, \mathrm{Ann}_{L_1} L_{-q}] = 0,$$

to contradict Lemma 2.3. Consequently, $\mathrm{Ann}_{L_1} L_{-q} = 0$. Now, if $Q_i \overset{\text{def}}{=} \mathrm{Ann}_{L_i} L_{-q} \neq 0$ for some $i > 1$, then by transitivity (C), we would have

$$0 \neq (\mathrm{ad}\, L_{-1})^{i-1} Q_i \subset \mathrm{Ann}_{L_1} L_{-q},$$

to contradict what we just showed. Thus, $\mathrm{Ann}_{L_i} L_{-q} = 0$ for all $i > 0$, which is what we wanted to show. □

Lemma 2.17. *If assumptions (vi) and (viii) hold, then* $L_{-q} = [L_{-q+i},\, L_{-i}]$, $0 \leqq i \leqq q$.

Proof. By Lemmas 2.3 and 2.4, $[L_0,\, L_{-q}] = L_{-q}$, so the lemma is true for $i = 0$ and $i = q$. For $i = 1$, we use Lemmas 2.1 and 2.4. Now note that for $1 \leqq i \leqq q-1$, we have

$$\begin{aligned} [(\mathrm{ad}\, L_1)^i L_{-q},\, (\mathrm{ad}\, L_1)^{q-i} L_{-q}] &= [(\mathrm{ad}\, L_1)^i L_{-q},\, [L_1,\, (\mathrm{ad}\, L_1)^{q-(i+1)} L_{-q}] \\ &= [(\mathrm{ad}\, L_1)^{i+1} L_{-q},\, (\mathrm{ad}\, L_1)^{q-(i+1)} L_{-q}], \end{aligned}$$

so that in view of (vi), (B), and Lemmas 2.1 and 2.4

$$\begin{aligned} L_{-q} &= [(\mathrm{ad}\, L_1)^1 L_{-q},\, (\mathrm{ad}\, L_1)^{q-1} L_{-q}] \\ &= [(\mathrm{ad}\, L_1)^i L_{-q},\, (\mathrm{ad}\, L_1)^{q-i} L_{-q}],\ 1 \leqq i \leqq q. \end{aligned}$$

Then

$$L_{-q} = [(\mathrm{ad}\, L_1)^i L_{-q},\, (\mathrm{ad}\, L_1)^{q-i} L_{-q}] \subseteq [L_{-q+i},\, L_{-i}] \subseteq L_{-q},\ 1 \leqq i \leqq q.$$

□

Lemma 2.18. *Let V be any (non-zero) irreducible L_0-submodule of L_{-q+i}. If assumptions (v) and (viii) hold and $0 \leqq i < \frac{q-1}{2}$, then $[V,\, L_{q-(i+1)}] \neq 0$, so that $[V,\, L_{q-(i+1)}] = L_{-1}$. Moreover, L_{-q+i} is an irreducible L_0-module.*

Proof. If $[V,\, L_{q-(i+1)}] = 0$, then, since the positive gradation spaces are assumed (v) to be generated by L_1,

$$\begin{aligned}[V,\, L_{q-i}] &= [V,\, [L_1,\, L_{q-(i+1)}]] \\ &= [[V,\, L_1],\, L_{q-(i+1)}] \\ &\subseteq [L_{-q+(i+1)},\, L_{q-(i+1)}].\end{aligned}$$

Consequently, since i is assumed to be less than $\frac{q-1}{2}$, so that $2i+1<q$,

$$\begin{aligned}[V,\, [V,\, L_{q-i}]] &\subseteq [V,\, [L_{-q+(i+1)},\, L_{q-(i+1)}]] \\ &= [L_{-q+(i+1)},\, [V,\, L_{q-(i+1)}]] \\ &= [L_{-q+(i+1)},\, 0] = 0.\end{aligned}$$

Now, $L_{-q} = S_{-q}$ is irreducible by Lemma 2.4, so we can assume by induction that L_{-q+i-1} is an irreducible L_0-module. Also, since $[L_{-q},\, L_{q-1}] = L_{-1}$ by Lemma 2.3 and (B), we can assume by induction that $[L_{-q+i-1},\, L_{q-i}] = L_{-1}$. But then Lemma 2.12 would imply that $V=0$, contrary to assumption. Thus, $[V,\, L_{q-(i+1)}]$ is a non-zero L_0-submodule of L_{-1}, so that by irreducibility (B), $[V,\, L_{q-(i+1)}] = L_{-1}$. The last assertion follows from Lemma 2.13. □

Lemma 2.19. *Suppose that assumptions (vi) and (viii) hold. Then for any* $i, 1 \leqq i \leqq q$, *we have and* $[L_{-q},\, [L_{-i},\, L_i]] = L_{-q}$; *in particular,* $[L_{-i},\, L_i] \neq 0$.

Proof. The lemma will follow from Lemma 2.4 once we show that $[L_{-q},\, [L_{-i},\, L_i]] \neq 0$. For $i=q$, the lemma follows from Corollary 2.11. Let $1 \leqq i < q$. Then by Lemma 2.17, we have $L_{-q} = [L_{-q+i},\, L_{-i}]$, and by Lemma 2.8, we have $L_{-q+i} = [L_{-q},\, L_i]$. Then we have

$$L_{-q} = [L_{-q+i},\, L_{-i}] = [L_{-q+i},\, [L_{-q},\, L_i]] = [L_{-q}, [L_{-i},\, L_i]],$$

so that $[L_{-q}, [L_{-i},\, L_i]] \neq 0$, as required. □

Lemma 2.20. *Suppose that assumptions (iv), (v), (and therefore, (iii)), (vi), and (viii) hold and that* $0 < i < \frac{q-3}{2}$. *Then* $[L_{-q+i},\ L_{q-i+1}] = L_1$.

Proof. Suppose that $[L_{-q+i},\ L_{q-i+1}] = 0$. Then, since by (iii) and Lemma 2.3, $L_j = S_j = [S_{j+1},\, S_{-1}] = [L_{j+1},\, L_{-1}] \subseteq L_j$ for all $j, 0 < j < r$, we have

$$\begin{aligned}[L_{-q+i},\, L_{q-i}] &= [L_{-q+i},\, [L_{q-i+1},\, L_{-1}]] \\ &= [L_{q-i+1},\, [L_{-q+i},\, L_{-1}]] \\ &= [L_{q-i+1},\, L_{-q+i-1}],\end{aligned}$$

so that (since $i < \frac{q-3}{2}$ implies that $2i + 3 < q$, so that *a fortiori* $2i - 1 < q$)

$$\begin{aligned}[L_{-q+i},\ [L_{-q+i},\, L_{q-i}]] &= [L_{-q+i},\ [L_{q-i+1},\, L_{-q+i-1}]] \\ &= [L_{-q+i-1},\ [L_{-q+i},\, L_{q-i+1}]] \\ &= [L_{-q+i-1},\ 0] = 0.\end{aligned}$$

Let $v \in L_{-q+i}$ and $u \in L_{q-i}$. Then

$$\begin{aligned}2(\operatorname{ad}[v,\, u])^2 L_{-q+i+1} &= (\operatorname{ad} v)^2 (\operatorname{ad} u)^2 L_{-q+i+1} \\ &\subseteq (\operatorname{ad} v)^2 L_{q-i+1} \\ &\subseteq (\operatorname{ad} v)[L_{-q+i},\ L_{q-i+1}] = (\operatorname{ad} v)0 = 0.\end{aligned}$$

Consequently, $\operatorname{ad}_{L_{-q+i+1}}[L_{-q+i},\, L_{q-i}]$ is a nilpotent set of linear transformations by Lemma 2.9. Since we are assuming that $i < \frac{q-3}{2}$, we have $i + 1 < \frac{q-1}{2}$, so we can apply Lemma 2.18 to conclude that L_{-q+i+1} is an irreducible L_0-module. It follows that $\operatorname{ad}_{L_{-q+i+1}}[L_{-q+i},\, L_{q-i}]$ annihilates L_{-q+i+1}. Thus (since $2i + 1 < 2i + 3 < q$)

$$0 = [[L_{-q+i},\, L_{q-i}],\, L_{-q+i+1}] = [L_{-q+i},\, [L_{q-i},\, L_{-q+i+1}]].$$

If $[L_{q-i},\, L_{-q+i+1}] \neq 0$, then, since L_1 is assumed (iv) to be irreducible, $[L_{q-i},\, L_{-q+i+1}]$ would have to equal L_1, and the above-displayed formula would imply a lack of $\{1\}-$transitivity (vi) of L in its negative part. It follows that $[L_{q-i},\, L_{-q+i+1}] = 0$. Then, in view of our initial assumption that $[L_{-q+i},\ L_{q-i+1}] = 0$, we would have

$$0 = [[L_{-q+i},\, L_1],\, L_{q-i}] = [[L_{-q+i},\, L_{q-i}],\, L_1],$$

to contradict Lemma 2.14, in view of Lemma 2.19. Thus, it must be that $[L_{-q+i},\ L_{q-i+1}] \neq 0$, so that by the assumed irreducibility of L_1, $[L_{-q+i},\ L_{q-i+1}] = L_1$, as required. □

Lemma 2.21. *Let $q > 5$, and suppose that assumptions (iv), (v) (and therefore (iii)), (vi), and (viii) hold. If q is even, then $L_{-2}^{\frac{q}{2}} = L_{-q}$, while if q is odd, then $L_{-2}^{\frac{q-1}{2}} = L_{-q+1}$.*

Proof. We have by Lemma 2.8 that $L_{-2} = [L_{-q}, L_{q-2}]$ and by Lemma 2.18 (since $q > 3$) that $L_{-1} = [L_{-q+1}, L_{q-2}]$. Thus, for any j, $1 < j < q-1$, we have by $\{-1\}-$transitivity (Lemma 2.1) that

$$0 \neq [V_{-j}, L_{-1}] = [V_{-j}, [L_{-q+1}, L_{q-2}]] = [L_{-q+1}, [V_{-j}, L_{q-2}]].$$

where V_{-j} is any non-zero L_0-submodule of L_{-j}. Consequently, $[V_{-j}, L_{q-2}] \neq 0$. Then by Lemma 2.16 when $j < q-2$, or, when $j = q-2$, by Lemmas 2.18 and 2.19,

$$\begin{aligned} 0 \neq [L_{-q}, [V_{-j}, L_{q-2}]] &= [V_{-j}, [L_{-q}, L_{q-2}]] \\ &= [V_{-j}, L_{-2}] \end{aligned}$$

by Lemma 2.8. If we successively let $V_{-2j} \stackrel{\text{def}}{=} L_{-2}^{j}$, we can conclude that $L_{-2}^{\frac{q}{2}} \neq 0$ if q is even and greater than two, and $L_{-2}^{\frac{q-1}{2}} \neq 0$ if q is odd and greater than three. Then, by Lemmas 2.4 (See (iii).) and 2.18, respectively, $L_{-2}^{\frac{q}{2}} = L_{-q}$ and $L_{-2}^{\frac{q-1}{2}} = L_{-q+1}$. □

Lemma 2.22. *Let $q > 5$, and suppose that assumptions (iv), (v) (and therefore (iii)), (vi) and (viii) hold. Then L_{-2} is an irreducible L_0-module.*

Proof. Let V_{-2} be any irreducible L_0-submodule of L_{-2}. Since $[L_{-q+1}, [V_{-2}, L_q]] = [V_{-2}, [L_{-q+1}, L_q]] = [V_{-2}, L_1] = L_{-1}$ by Lemmas 2.20 and 2.15, it follows that for any j, $0 < j < \frac{q}{2}$ (i.e., $0 < j \leqq \frac{q-1}{2}$) for which $V_{-2}^{j} \neq 0$, we have by transitivity (Lemma 2.1) that

$$[L_{-q+1}, [V_{-2}^{j}, [V_{-2}, L_q]]] = [V_{-2}^{j}, [V_{-2}, [L_{-q+1}, L_q]] = [V_{-2}^{j}, L_{-1}] \neq 0,$$

so we conclude that $[V_{-2}^{j}, [V_{-2}, L_q]] \neq 0$; i.e., $(\operatorname{ad} V_{-2})^{j+1} L_q \neq 0$. Thus, so long as $2(j+1) < q$ (i.e., $j < \frac{q}{2} - 1$), we have by Lemma 2.16 that

$$\begin{aligned} 0 &\neq [L_{-q}, [V_{-2}^{j}, [V_{-2}, L_q]]] \\ &= [V_{-2}^{j}, [V_{-2}, [L_{-q}, L_q]]] \\ &\subseteq V_{-2}^{j+1}. \end{aligned}$$

Thus, $V_{-2}^{j} \neq 0$ for all j, $0 < j \leqq \frac{q-1}{2}$, and $(\operatorname{ad} V_{-2})^{j} L_q \neq 0$ for all j, $0 < j \leqq \frac{q+1}{2}$. If q is odd, then, since $q > 5$, we have by Lemma 2.18 that $V_{-2}^{\frac{q-1}{2}} = L_{-q+1}$, while if q is even, we have $V_{-2}^{\frac{q}{2}-1} = L_{-q+2}$.

In the case of odd q, we have, by the irreducibility (B) of L that $L_{-1} = (\operatorname{ad} V_{-2})^{\frac{q+1}{2}} L_q$, so that

$$\begin{aligned} L_{-2} &= [L_{-1},\, L_{-1}] \\ &= [L_{-1},\, (\operatorname{ad} V_{-2})^{\frac{q+1}{2}} L_q] \\ &\subseteq [V_{-2},\, [L_{-q},\, L_q]] + [V_{-2},\, L_0] \\ &\subseteq V_{-2}. \end{aligned}$$

Thus, when q is odd, we see that L_{-2} is irreducible.

In the case of even q, we have by Lemma 2.18 (since $q > 5$) that $L_{-1} = [L_{-q+2},\, L_{q-3}] = [V_{-2}^{\frac{q}{2}-1},\, L_{q-3}] \subseteq (\operatorname{ad} V_{-2})^{\frac{q}{2}-1} L_{q-3} \subseteq L_{-1}$. By (D) and transitivity (Lemma 2.1),

$$\begin{aligned} L_{-q+1} &= [L_{-1},\, L_{-q+2}] \\ &= [(\operatorname{ad} V_{-2})^{\frac{q}{2}-1} L_{q-3},\, L_{-q+2}] \\ &= [(\operatorname{ad} V_{-2})^{\frac{q}{2}-1} L_{q-3},\, V_{-2}^{\frac{q}{2}-1}] \\ &\subseteq (\operatorname{ad} V_{-2})^{q-2} L_{q-3} \\ &\subseteq L_{-q+1}, \end{aligned}$$

so that (See also Lemma 2.18.) $(\operatorname{ad} V_{-2})^{q-2} L_{q-3} = L_{-q+1}$. Then, by Lemma 2.17,

$$\begin{aligned} L_{-q} &= [L_{-1},\, L_{-q+1}] \\ &= [L_{-1},\, (\operatorname{ad} V_{-2})^{q-2} L_{q-3}] \\ &\subseteq [V_{-2},\, L_{-q+2}] + [(\operatorname{ad} V_{-2})^{q-2} L_{-1},\, L_{q-3}] \\ &\subseteq [V_{-2},\, V_{-2}^{\frac{q}{2}-1}] + [0,\, L_{q-3}] \\ &\subseteq V_{-2}^{\frac{q}{2}} \end{aligned}$$

Now, by Lemma 2.16 and irreducibility (B), we have

$$L_{-1} = [L_{-q},\, L_{q-1}] = [V_{-2}^{\frac{q}{2}},\, L_{q-1}] \subseteq (\operatorname{ad} V_{-2})^{\frac{q}{2}} L_{q-1} \subseteq L_{-1}.$$

Consequently, we have

$$\begin{aligned} L_{-2} &= [L_{-1},\, L_{-1}] \\ &= [L_{-1},\, (\operatorname{ad} V_{-2})^{\frac{q}{2}} L_{q-1}] \\ &= [(\operatorname{ad} V_{-2})^{\frac{q}{2}} L_{-1},\, L_{q-1}] + [V_{-2},\, L_0] \\ &\subseteq [0,\, L_{q-1}] + V_{-2} \\ &\subseteq V_{-2} \end{aligned}$$

as required. □

Lemma 2.23. *If* $q > 2$ *and assumptions (vi) and (vii) hold, then* $\operatorname{Ann}_{L_1} L_{-2} = 0$.

Proof. Set $A_1 = \operatorname{Ann}_{L_1} L_{-2}$, and suppose that $A_1 \neq 0$. Since

$$[L_{-2},\, [L_{-q+1},\, A_1]] = [L_{-q+1},\, [L_{-2},\, A_1]] = 0,$$

we have

$$\begin{aligned} 0 &= [L_{-2},\, [L_{-q+1},\, A_1]] \\ &\supseteq [[L_{-3},\, L_1],\, [L_{-q+1},\, A_1]] = [L_{-3},\, [L_1,\, [L_{-q+1},\, A_1]]] \\ &\supseteq [[L_{-4},\, L_1],\, [L_1,\, [L_{-q+1},\, A_1]]] = [L_{-4},\, [L_1,\, [L_1,\, [L_{-q+1},\, A_1]]]] \\ &\cdots \\ &\supseteq [L_{-q+1},\, (\operatorname{ad} L_1)^{q-3} [L_{-q+1},\, A_1]]. \end{aligned}$$

Now, if $[L_{-q+1},\, A_1] \neq 0$, then by (vi) and irreducibility (B), we would have $(\operatorname{ad} L_1)^{q-3}\, [L_{-q+1},\, A_1] = L_{-1}$, so that $[L_{-q+1},\, L_{-1}] = 0$, to contradict transitivity (Lemma 2.1). Thus, we must have $[L_{-q+1},\, A_1] = 0$.

Now, since $[L_{-q},\, [A_1,\, A_1]] \subseteq [L_{-q+1},\, A_1] = 0$, and, clearly, $[L_{-q+1},\, [A_1,\, A_1]] = 0$, it follows from Lemma 2.6 that $[A_1,\, A_1] = 0$. Then $L^{\dagger} \stackrel{\text{def}}{=} (L_{-q} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus A_1)/(L_{-q} \oplus \cdots \oplus L_{-2})$ is a depth-one Lie algebra which satisfies conditions (A) through (C) of the Main Theorem. Consequently, by Proposition 1.4, $(L^{\dagger})'$ is one of the Lie algebras enumerated in the hypothesis of Lemma 1.6. If we set $V = L_{-q} \oplus L_{-q+1}$, then the $(L^{\dagger})'$−module V satisfies the hypotheses of Lemma 1.6. If we set $W = L_{-q+1}$, we have that $[A_1,\, L_{-q+1}] = 0$, so that we must conclude that $\zeta = q - 1 \equiv 0 \bmod 3$. However, if we then set $W = L_{-q}$, we have that $[L_{-1},\, L_{-q}] = 0$, so that we must conclude that $\zeta = q \equiv 0 \mod 3$. Since both $q-1$ and q cannot be equivalent to zero modulo three, we have arrived at a contradiction. We therefore conclude that $\mathrm{Ann}_{L_1} L_{-2} = A_1 = 0$, as required. $\square$

Lemma 2.24. *If $i < \frac{q}{2}$ and $[L_{-q+i-1},\, L_{q-i}] \neq 0$, then $L_{-q+i} = [L_{-q+i-1},\, L_1]$.*

Proof. Since $[L_{-q+i-1},\, L_{q-i}] \neq 0$, it follows from the irreducibility (B) of L that $[L_{-q+i-1},\, L_{q-i}] = L_{-1}$. Then by (D) we have

$$\begin{aligned} L_{-q+i} &= [L_{-q+i+1},\, L_{-1}] \\ &= [L_{-q+i+1},\, [L_{-q+i-1},\, L_{q-i}]] \\ &= [L_{-q+i-1},\, [L_{-q+i+1},\, L_{q-i}]] \\ &\subseteq [L_{-q+i-1},\, L_1] \end{aligned}$$

$\square$

Lemma 2.25. *Let L be as in the statement of the Main Theorem, and suppose that $L_2 \neq 0$, that $[L_{-2},\, L_1] = 0 = [L_{-2},\, L_2]$, and that assumption (vii) holds. Let $\tilde{L}$ be the Lie subalgebra of L generated by L_{-1}, L_0, and L_1. If $M(\tilde{L})$ is as in Theorem 1.3, then $\tilde{L}/M(\tilde{L})$ is Hamiltonian, and we have $[L_{-1},\, L_1] \cong \mathfrak{sl}(2)$.*

Proof. Let $\tilde{\tilde{L}}$ be the Lie subalgebra of L generated by L_{-1}, L_0, L_1, and L_2. Since $[L_{-2},\, L_1] = 0 = [L_{-2},\, L_2]$, we have $M(\tilde{L}) = L_{-q} \oplus \cdots \oplus L_{-2} = M(\tilde{\tilde{L}})$. But the depth $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ is then one, so by Proposition 1.4, $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ is either Hamiltonian (i.e., between $H(2 : \underline{n},\, \omega)$ and $CH(2 : \underline{n},\, \omega)$) or is isomorphic to a Lie algebra of type $L(\epsilon)$ or M. However, the height of the latter two Lie algebras is one, and, since $L_2 \neq 0$, the height of $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ is at least two. Thus, $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ must be Hamiltonian, so $[L_{-1},\, L_1] \cong \mathfrak{sl}(2)$. It now follows from Corollary 1.5 that $\tilde{L}/M(\tilde{L})$ is Hamiltonian, as well. $\square$

Lemma 2.26. *Let*

$$A_1 \stackrel{def}{=} \mathrm{Ann}_{L_1} L_{-2}.$$

If condition (vi) of holds, then either $A_1 = 0$, or $q = 2$ and $\mathrm{Ann}_L L_{-2}/L_{-2}$ is Hamiltonian.

Proof. If $q > 2$, it follows from Lemma 2.23 that $A_1 = 0$. On the other hand, if $q = 2$ and $A_1 \neq 0$, then $\mathfrak{A} \stackrel{\text{def}}{=} (L_{-2} \oplus L_{-1} \oplus L_0 \oplus \sum_{i=1}^{\infty} A_1^i)/L_{-2}$ satisfies the conditions of Proposition 1.4. Consequently, $\mathfrak{A}$ is congruent to M or $L(\epsilon)$, or $\mathfrak{A}$ is Hamiltonian; i.e., between $H(2:\ (1,\ 1),\ \omega)$ and $CH(2:\ (1,\ 1),\ \omega)$. Note that in each of these cases, L_1 is an irreducible $L_0'-$ module, so $L_1 = A_1$. Since $q = 2$, L_{-2} and L_{-1} commute, so $[L_{-2},\ [L_{-1},\ A_1]] = 0$, and we have $\operatorname{Ann}_{L_0} L_{-2} \supseteq [L_{-1},\ A_1] = [L_{-1},\ L_1]$. In M and $L(\epsilon)$ however, $[L_{-1},\ L_1]$ contains the center, which has non-zero bracket with L_{-1} and consequently with $L_{-2} = [L_{-1},\ L_{-1}]$. Thus $\mathfrak{A}/L_{-2}$ must be Hamiltonian, as required. □

Lemma 2.27. *If M_1 is a non-zero L_0-submodule of L_1 such that $[[L_{-1},\ M_1],\ M_1] = 0$, then $[L_{-2},\ M_1] \neq 0$.*

Proof. Suppose $[[L_{-1},\ M_1],\ M_1] = 0$ and $[L_{-2},\ M_1] = 0$. Then if $v \in L_{-1}$ and $m \in M_1$, we have

$$(\operatorname{ad} m)^2 (\operatorname{ad} v)^2 [L_{-1},\ M_1] \subseteq (\operatorname{ad} m)[M_1,\ L_{-2}] = 0$$

so

$$0 = (\operatorname{ad} m)^2 (\operatorname{ad} v)^2 [L_{-1},\ M_1] = 2(\operatorname{ad} [m,\ v])^2 [L_{-1},\ M_1]$$

Consequently, by Lemma 2.9, $\operatorname{ad}_{[L_{-1}, M_1]} [L_{-1},\ M_1]$ is "associative nilpotent". Thus, $[L_{-1},\ M_1]$ is a nilpotent Lie algebra. Since by transitivity, $[L_{-1},\ M_1] \neq 0$, it follows from (i) that $[L_{-1},\ M_1]$ is the center of L_0. But the center acts as a non-zero scalar on L_{-1} so, since it"acts as zero" on L_0, it must act as the negative of that non-zero scalar on M_1, contrary to assumption. □

Lemma 2.28. *If M_1 is a non-zero L_0-submodule of L_1 such that* $\operatorname{Ann}_{L_0} M_1 \neq 0$, *then* $[[L_{-1}, M_1],\ M_1] = 0$, *and* $[M_1,\ M_1] = 0$.

Proof. Set $X \stackrel{\text{def}}{=} \operatorname{Ann}_{L_0} M_1 \neq 0$, and suppose that $X \neq 0$. Then by transitivity (C) and irreducibility (B), $[L_{-1},\ X] = L_{-1}$, so

$$[L_{-1},\ M_1] = [[L_{-1},\ X],\ M_1] = [[L_{-1},\ M_1], X] \subseteq X.$$

Thus, $[L_{-1}, [M_1,\ M_1]] \subseteq [X,\ M_1] = 0$, so, by transitivity (C), $[M_1,\ M_1] = 0$.
□

Lemma 2.29. *If M_1 is a non-zero L_0-submodule of L_1 such that $[[L_{-1}, M_1], M_1] = 0$, and N_1 is another L_0-submodule of L_1 such that $[M_1, N_1] = 0$, then $[N_1, N_1] = 0$; i.e., N_1 must be abelian.*

Proof. We have by Lemma 2.27 that $[L_{-2}, M_1] \neq 0$. Consequently, by irreducibility (B), we have $[L_{-2}, M_1] = L_{-1}$. Then

$$\begin{aligned}
[N_1, [M_1, L_{-1}]] &= [N_1, [M_1, [M_1, L_{-2}]] \\
&= [M_1, [M_1, [N_1, L_{-2}]] \\
&\subseteq [[L_{-1}, M_1], M_1] \\
&= 0.
\end{aligned}$$

Thus, $[M_1, L_{-1}]$ annihilates N_1 as well as M_1. The result follows from Lemma 2.28 above. □

Lemma 2.30. *If M_1 and N_1 are L_0-submodules of L_1 such that* $\mathrm{Ann}_{L_0} M_1 \cap \mathrm{Ann}_{L_0} N_1 \neq 0$, *then* $[M_1, N_1] = 0$.

Proof. Set $X \stackrel{\text{def}}{=} Ann_{L_0} M_1 \cap \mathrm{Ann}_{L_0} N_1$, and suppose that $X \neq 0$. Then by irreducibility (B) and transitivity (C), we have $[L_{-1}, X] = L_{-1}]$. Then

$$[L_{-1}, M_1] = [[L_{-1}, X], M_1] = [[L_{-1}, M_1], X] \subseteq X.$$

Similarly, $[L_{-1}, N_1] \subseteq X$. Then

$$[L_{-1}, [M_1, N_1]] \subseteq [[L_{-1}, N_1], M_1] + [M_1, [L_{-1}, N_1]] = 0,$$

so by transitivity (C), $[M_1, N_1] = 0$, as required.

3 A factor algebra of a subalgebra of the even part of L

In this and the following two sections, we assume that ((i) and (ii), of course), (iv), (v) (and therefore (iii)), (vi), and (viii) of the previous section hold, so that, in particular, by Lemma 2.22, L_{-2} is an irreducible L_0-module.

We begin this section by forming the irreducible, transitive Lie algebra $B(L_{-2})$. (See, for example, Section 3 of [BG].) Indeed, consider the subalgebra

$$E = E_{-\lfloor \frac{q}{2} \rfloor} \oplus \cdots \oplus E_0 \oplus \cdots \oplus E_{\lfloor \frac{r}{2} \rfloor}$$

of L consisting of the gradation spaces $E_i = L_{-2}^{-i}$ for $i < 0$, and $E_i = L_{2i}$ for $i \geqq 0$. Set $T_0 = \operatorname{Ann}_{E_0} E_{-1} = \operatorname{Ann}_{L_0} L_{-2}$, and for $i = 1, 2, \ldots$, let

$$T_i = \{x \in E_i \mid [x, E_{-1}] \subseteq T_{i-1}\}.$$

Then

$$\mathcal{T} = T_0 \oplus T_1 \oplus \ldots \oplus T_{\lfloor \frac{r}{2} \rfloor}$$

is an ideal of E, and the factor algebra

$$G = E/\mathcal{T} = G_{-\lfloor \frac{q}{2} \rfloor} \oplus \cdots \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus \cdots \oplus G_{\lfloor \frac{r}{2} \rfloor}$$

is a transitive graded Lie algebra (See [BG, Lemma 3].) Thus, the Lie algebra $B(L_{-2}) \overset{\text{def}}{=} G$ satisfies conditions (A)-(D) of the Main Theorem. (It is shown in, for example, [BGP], that the process of forming $B(L_{-2})$ preserves condition (A).)

Lemma 3.1 . *Let L be as in the statement of the Main Theorem, and suppose that assumptions (i) - (vi) and (viii) hold. If $q \geqq 6$, then $B(L_{-2})$ is an irreducible, transitive graded Lie algebra of height greater than zero and depth greater than 1, and $B(L_{-2})_{-2} \not\subseteq M(B(L_{-2}))$. Consequently, since the depth of $B(L_{-2})$ is no greater than half of the depth of L, we can, using induction, apply the Main Theorem to conclude that the character of the representation of $B(L_{-2})_0'$ on $B(L_{-2})_{-1}$ is equal to zero. Then the character of L_0' on L_{-1} is also zero.*

Proof. By Lemma 2.22, L_{-2} is an irreducible L_0-module, and by Lemma 2.21, the depth of $B(L_{-2})$ is greater than one.

To show that the height of $B(L_{-2})$ is positive, we begin by proving that since $q \geqq 6$, we have $[L_{-q+1},\, L_2] \neq 0$. Indeed, we have by Lemma 2.20 and $\{1\}-$transitivity (vi) that

$$\begin{aligned} [L_{-q+1},\, L_2] &\supseteq [L_{-q+1},\, [L_{-q+2},\, L_q]] \\ &= [L_{-q+2},\, [L_{-q+1},\, L_q]] \\ &= [L_{-q+2},\, L_1] \\ &\neq 0, \end{aligned}$$

as required, so that if q is odd, we have by Lemma 2.21 that

$$0 \neq [L_{-q+1},\, L_2] \subseteq (\operatorname{ad} L_{-2})^{\frac{q-1}{2}} L_2.$$

On the other hand, if q is even, we have by Lemma 2.3 and Lemma 2.21 that

$$0 \neq [L_{-q},\, L_2] \subseteq (\operatorname{ad} L_{-2})^{\frac{q}{2}} L_2.$$

In either case,

$$[L_{-2},\, [L_{-2},\, L_2]] \neq 0, \tag{3.2}$$

so $[L_{-2},\, L_2] \nsubseteq T_0$. Thus, $T_1 \neq L_2$, so $G_1 \neq (0)$, and the height of $B(L_{-2})$ is positive.

We must now verify hypothesis (E) of the Main Theorem for the Lie algebra $B(L_{-2})$; that is, we must show that $[L_{-2},\, L_{-2}]$ is not contained in $M(B(L_{-2}))$. Thus, suppose that $[L_{-2},\, L_{-2}]$ is contained in $M(B(L_{-2}))$. We will arrive at a contradiction by successively considering the two cases: 1) even q, and 2) odd q.

1) Suppose first that q is even. Then by Lemma 2.21, $L_{-q} = L_{-2}^{\frac{q}{2}} = (\operatorname{ad} L_{-2})^{\frac{q}{2}-2}[L_{-2},\, L_{-2}]$, so that $L_{-q} \in M(B(L_{-2}))$. Then by Lemma 2.8, $B(L_{-2})_{-1} = L_{-2} = [L_{-q},\, L_{q-2}] \subseteq M(B(L_{-2}))$, so that we would have

$$[L_{-2},\, L_2] \subseteq L_0 \cap M(B(L_{-2})) = 0,$$

to contradict, for example, (3.2) above. Thus, q cannot be even.

2) Next suppose that q is odd. Then by Lemma 2.21 again, $L_{-q+1} = L_{-2}^{\frac{q-1}{2}} = (\operatorname{ad} L_{-2})^{\frac{q-1}{2}-2}[L_{-2},\, L_{-2}]$, so that $L_{-q+1} \in M(B(L_{-2}))$. Now, by Lemma 2.18, $[L_{-q+2},\, L_{q-3}] = L_{-1}$. If $[L_{-q+1},\, L_{q-3}]$ were equal to zero, then we would have

$$\begin{aligned} 0 &= [L_{-q+2},\, 0] \\ &= [L_{-q+2},\, [L_{-q+1},\, L_{q-3}]] \\ &= [L_{-q+1},\, [L_{-q+2},\, L_{q-3}]] = [L_{-q+1},\, L_{-1}] \end{aligned}$$

to contradict transitivity (Lemma 2.1). We therefore conclude that $[L_{-q+1},\, L_{q-3}] \neq 0$. Since by Lemma 2.22, L_{-2} is an irreducible L_0-module, it follows that $L_{-2} = [L_{-q+1},\, L_{q-3}] \subseteq M(B(L_{-2}))$. But we saw at the conclusion of 1) above that L_{-2} cannot be contained in $M(B(L_{-2}))$. This second contradiction shows that $B(L_{-2})_{-2} = [L_{-2},\, L_{-2}]$ is in fact not contained in $M(B(L_{-2}))$, no matter what the parity of q is.

Consequently, we can conclude that $B(L_{-2})$ satisfies hypothesis (E), and therefore all of the hypotheses of the Main Theorem, which we can now apply to conclude that the character χ of $B(L_{-2})_0'$ on $B(L_{-2})_{-1}$ is zero. Then $\frac{1}{2}\chi$, which is the character of L_0' on L_{-1}, must be zero as well, and Lemma 3.1 is proved. $\square$

We now address the depth-four and depth-five cases individually.

4 The depth-four case

Suppose

$$L = L_{-4} \oplus L_{-3} \oplus \cdots \oplus L_r$$

satisfies conditions (i) through (viii), in particular, the assumption (iv) that L_1 is an irreducible L_0-module.

By Lemma 2.15,

$$[L_{-2}, L_1] = L_{-1}.$$

Now suppose that $[L_{-2}, L_{-2}] = 0$. Then

$$[L_{-3}, L_{-1}] = [L_{-3}, [L_{-2}, L_1]] = [L_{-2}, [L_{-3}, L_1]] \subseteq [L_{-2}, L_{-2}] = 0,$$

to contradict $\{-1\}-$transitivity (Lemma 2.1). Thus, it follows that

$$[L_{-2}, L_{-2}] \neq 0.$$

It now follows from Lemma 2.4 and (iii) that $[L_{-2}, L_{-2}] = L_{-4}$. Furthermore, from Lemma 2.3 we have $[L_{-4},\, L_2] \neq 0$. Then

$$0 \neq [L_{-4}, L_2] = [[L_{-2}, L_{-2}], L_2] \subseteq [L_{-2},\, [L_{-2}, L_2]].$$

Now let V_{-2} be any irreducible L_0- submodule of L_{-2}. If $[V_{-2},\, L_3] = 0$, then by Lemma 2.3 and irreducibility (B), $0 = [L_{-4},\, [V_{-2},\, L_3]] = [V_{-2},\, [L_{-4},\, L_3]] = [V_{-2},\, L_{-1}]$ to contradict transitivity (Lemma 2.1). Thus, we can assume that $[V_{-2},\, L_3] = L_1$, since we are assuming that L_1 is irreducible (iv).

Then by Lemma 2.15, $L_{-1} = [V_{-2}, [V_{-2}, L_3]]$. Then we have by condition (D) of the Main Theorem that

$$\begin{aligned} L_{-2} &= [L_{-1}, L_{-1}] \\ &= [L_{-1}, [V_{-2}, [V_{-2}, L_3]]] \\ &\subseteq [[L_{-1}, V_{-2}], [V_{-2}, L_3]] + [V_{-2}, L_0] \\ &= [V_{-2}, [[L_{-1}, V_{-2}], L_3]] + [V_{-2}, L_0] \\ &\subseteq [V_{-2}, L_0] \\ &\subseteq V_{-2} \end{aligned}$$

so that L_{-2} is an irreducible L_0-module.

Thus, in the depth-two irreducible, transitive Lie algebra $B(L_{-2})$, we have by (4.1) that $B(L_{-2})_1 \neq 0$. Furthermore, it follows again from (4.1) above that

$$B(L_{-2})_{-2} = L_{-4} = [L_{-2}, L_{-2}] \not\subseteq M(B(L_{-2})),$$

so that hypothesis (E) of the Main Theorem is satisfied for $B(L_{-2})$, as are the other hypotheses of the Main Theorem. Then the Main Theorem (proved for the case $q = 2$ in [BGK]) applies to show that the representation of $B(L_{-2})'_0$ on $B(L_{-2})_{-1}$ is restricted. Consequently, the character χ of L'_0 on L_{-2} is zero, as must be the character $\frac{1}{2}\chi$ of L'_0 on L_{-1}.

5 The depth-five case

Suppose

$$L = L_{-5} \oplus L_{-4} \oplus \cdots \oplus L_r$$

satisfies conditions (i) through (viii), in particular, the assumption (iv) that L_1 is an irreducible L_0-module.

Since L is transitive (C), $[L_{-1}, L_5] \neq 0$.

$$[L_{-5}, [L_{-1}, L_5]] = [L_{-1}, [L_{-5}, L_5]]$$

is also non-zero by Lemma 2.3 and transitivity (Lemma 2.1), so that, by irreducibility (B), $[L_{-5}, [L_{-1}, L_5]] = L_{-1}$.

Now suppose that $[L_{-4}, L_5] = 0$. If $[L_{-3}, [L_{-3}, L_5]] = 0$, then we would have

$$0 = [L_{-1},\, [L_{-3},\, [L_{-3},\, L_5]]] = [L_{-3},\, [L_{-3},\, [L_{-1},\, L_5]]].$$

However, since $[L_{-5},\, [L_{-1},\, L_5]] = L_{-1}$, we have by transitivity (Lemma 2.1) that

$$0 \neq [L_{-1},\, L_{-3}] = [L_{-3},\, [L_{-5},\, [L_{-1},\, L_5]]] = [L_{-5},\, [L_{-3},\, [L_{-1},\, L_5]]]$$

so that $[L_{-3},\, [L_{-1},\, L_5]] \neq 0$. Since L_1 is assumed (iv) to be irreducible, we must have $[L_{-3},\, [L_{-1},\, L_5]] = L_1$. Then by $\{1\}-$transitivity (vi),

$$0 \neq [L_{-3},\, L_1] = [L_{-3},\, [L_{-3},\, [L_{-1},\, L_5]]],$$

contrary to what was derived above. Thus, we can assume that $[L_{-3},\, [L_{-3},\, L_5]] \neq 0$. But then, by the irreducibility (B) of L, we must have $[L_{-3},\, [L_{-3},\, L_5]] = L_{-1}$. Then, by $\{-1\}-$transitivity (Lemma 2.1),

$$0 \neq [L_{-4},\, L_{-1}] = [L_{-4},\, [L_{-3},\, [L_{-3},\, L_5]]] = [L_{-3},\, [L_{-3},\, [L_{-4},\, L_5]]],$$

so that $[L_{-4},\, L_5] \neq 0$.

Since we are assuming (iv) that L_1 is irreducible, it follows that

$$[L_{-4},\, L_5] = L_1. \tag{5.1}$$

Now let V_{-2} be any non-zero L_0-submodule of L_{-2}. Then we have by (5.1) and $\{1\}-$transitivity (vi) that

$$0 \neq [V_{-2},\, L_1] = [V_{-2},\, [L_{-4},\, L_5]] = [[L_{-4},\, [V_{-2},\, L_5]],$$

so

$$[L_{-4},\, [V_{-2},\, L_5]] = L_{-1},$$

by the irreducibility (B) of L. Then, by $\{-1\}-$transitivity (Lemma 2.1),

$$0 \neq [V_{-2},\, L_{-1}] = [V_{-2},\, [L_{-4},\, [V_{-2},\, L_5]]] = [L_{-4},\, [V_{-2},\, [V_{-2},\, L_5]]],$$

so that $[V_{-2},\, [V_{-2},\, L_5]] \neq 0$. Thus, by the assumed irreducibility (iv) of L_1, we must have $[V_{-2},\, [V_{-2},\, L_5]] = L_1$. Then, as above, by the $\{1\}-$transitivity (vi) of L, we have

$$0 \neq [V_{-2},\, L_1] = [V_{-2},\, [V_{-2},\, [V_{-2},\, L_5]]],$$

so that

$$[V_{-2},\, [V_{-2},\, [V_{-2},\, L_5]]] = L_{-1}, \tag{5.2}$$

by the irreducibility (B) of L. Then, since the negative gradation spaces are generated (D) by L_{-1}, we have

$$L_{-2} = [L_{-1},\, L_{-1}] = [L_{-1},\, [V_{-2},\, [V_{-2},\, [V_{-2},\, L_5]]]] \subseteq [V_{-2},\, L_0] \subseteq V_{-2},$$

so that L_{-2} is an irreducible L_0-module.

Now, by Lemma 2.17, $L_{-5} = [L_{-2},\, L_{-3}]$. Consequently, in view of (D) and (5.2) above

$$\begin{aligned} L_{-5} &= [L_{-2},\, L_{-3}] \\ &= [L_{-2},\, [L_{-1},\, L_{-2}]] \\ &= [L_{-2},\, [L_{-2},\, [L_{-2},\, [L_{-2},\, [L_{-2},\, L_5]]]. \end{aligned}$$

Then by $\{1\}-$transitivity (vi), we have

$$0 \neq [L_1,\, L_{-5}] = [L_1,\, [L_{-2},\, [L_{-2},\, [L_{-2},\, [L_{-2},\, [L_{-2},\, L_5]]] \subseteq [L_{-2},\, [L_{-2},\, L_0]],$$

so that $[L_{-2},\, L_{-2}] \neq 0$. Furthermore, since the negative gradation spaces are generated (D) by L_{-1}, we have by (5.2) above that

$$\begin{aligned} L_{-4} &= [L_{-1}, L_{-3}] \\ &= [L_{-1}, [L_{-1}, L_{-2}]] \\ &= [L_{-1},\, [L_{-2},\, [L_{-2},\, [L_{-2},\, [L_{-2},\, L_5]]]]] \\ &\subseteq [L_{-2},\, L_{-2}], \end{aligned}$$

so

$$L_{-4} = [L_{-2},\, L_{-2}]. \tag{5.3}$$

Now suppose that $[L_{-4},\, L_2] = 0$, and suppose further that $[L_{-3},\, L_2] = 0$. Then we would have by (C) and (D) that

$$0 = [L_{-4},\, L_2] = [[L_{-3},\, L_{-1}],\, L_2] = [L_{-3},\, [L_{-1},\, L_2]] = [L_{-3},\, L_1]$$

by the assumed irreducibility (iv) of L_1, to contradict $\{1\}-$transitivity (vi). Thus, $[L_{-3},\, L_2] \neq 0$, so that by the irreducibility (B) of L, $[L_{-3},\, L_2] = L_{-1}$. Then

$$[[L_{-3},\, [L_{-4},\, L_2]] = [[L_{-4},\, [L_{-3},\, L_2]] = [L_{-4},\, L_{-1}] \neq 0,$$

by $\{-1\}-$transitivity (Lemma 2.1). Thus, it must be true that

$$[L_{-4},\, L_2] \neq 0, \tag{5.4}$$

so that

$$0 \neq [L_{-4},\, L_2] = [[L_{-2},\, L_{-2}],\, L_2] \subseteq [L_{-2},\, [L_{-2},\, L_2]]. \tag{5.5}$$

Thus, $B(L_{-2})$ is a transitive, irreducible depth-two Lie algebra. By (5.5) above, $B(L_{-2})_1 \neq 0$. By (5.3) and (5.4) above, $B(L_{-2})$ satisfies hypothesis (E) of the Main Theorem. Therefore, as in the depth-four case above, it follows from the Main Theorem (proved for the case $q = 2$ in [BGK]) that the character of $B(L_{-2})'_0$ on $B(L_{-2})_{-1} = L_{-2}$ is zero. Consequently, the character of L'_0 on L_{-1} is zero, as well, and L_{-1} is a restricted L'_0-module.

6 Conclusion of the proof of the Main Theorem

Let L be as in the statement of the Main Theorem. In this section, we will construct from L a Lie algebra $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ in which properties (i) through (vi) of Section 2 hold. We will show that the conclusion of the Main Theorem holds for $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$. Since it will be clear that the conclusion of the Main Theorem can hold for $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ only if it holds for L, we will have succeeded in thereby proving the Main Theorem for L, as required.

Since property (viii) may not hold in all the Lie algebras we consider, it may be necessary for us to consider "opposite" gradations. We will therefore have carefully to analyze the irreducible L'_0-submodules of L_1.

Consider the Lie algebra $\tilde{L} \subseteq L$ generated by the local part $L_{-1} \oplus L_0 \oplus L_1$ of L. We will now show that if (as we are assuming) the depth of $L?M(L)$ is greater than one, then the depth of $\tilde{(L)}/M(\tilde{L})$ cannot be one. This is the content of the next three lemmas.

Suppose, then, that the representation of L'_0 on L_{-1} is not restricted and that the depth of $\tilde{L}/M(\tilde{L})$ is one. In each of the three possible cases, corresponding to the three lemmas below, respectively, we proceed to obtain a contradiction. We first observe that (in light of the assumptions in the previous sentence) we can apply Proposition 1.4 to conclude that $\tilde{L}/M(\tilde{L})$ is either between $H(2:\ (1,1),\omega)$ and $CH(2:\ (1,1),\omega)$, or is equal to $L(\epsilon)$ or M. We will show that in each of these cases, $[L_{-2}, L_1] \neq 0$, so that $L_{-2} \not\subseteq M(\tilde{L})$, and the depth of $\tilde{L}/M(\tilde{L})$ is greater than one. For each of these cases, L_1 is an irreducible abelian L_0-module. Thus, we can from now on assume that (iv) holds, that

$$M(\tilde{L}) = L_{-q} \oplus L_{-q+1} \oplus \cdots \oplus L_{-2}$$

and that $[L_{-2}, L_1] = 0$.

By hypothesis (E) of the Main Theorem, $L_{-2} \not\subseteq M(L)$. Consequently, $L/M(L)$ has depth greater than one, and $[L_{>0}, L_{-2}] \neq 0$. Let j be the smallest natural number such that $[L_j, L_{-2}] \neq 0$. Our goal is thus to show that in fact $j = 1$ in each of the possibilities for $\tilde{L}/M(\tilde{L})$. We begin with the Hamiltonian case.

Lemma 6.1. *If*

$$H(2:\ (1,1),\omega) \subseteq \tilde{L}/M(\tilde{L}) \subseteq CH(2:\ (1,1),\omega),$$

then $[L_{-2}, L_1] \neq 0$; *i.e.*, $j = 1$.

Proof. Recall (See, for example, [BGK].) that $H(2:\ (1,1),\omega)$ has a realization as the divided power algebra $\mathcal{O}(2:(1,1))$, with Poisson bracket

$$\{f, g\} = \partial_y f(\partial_x g + g x^{(2)}) - \partial_y g(\partial_x f + f x^{(2)}).$$

In this realization, $H(2:\ (1,1),\omega)$ is a graded Lie algebra, with the homogeneous degree of a monomial equal to one less than its degree in the second variable, y.

We first note that $H(2:(1,1),\omega)$ is the sum (not direct, of course) of three three-dimensional simple Lie algebras

$$L_0 = \langle e_0 = \mathcal{D}_{H(2:(1,1),\omega)}(y),\ h = \mathcal{D}_{H(2:(1,1),\omega)}(xy),\ f_0 = \mathcal{D}_{H(2:(1,1),\omega)}(x^{(2)}y)\rangle$$

$$\langle e_{-1} = \mathcal{D}_{H(2:(1,1),\omega)}(2x^{(2)}),\ h,\ f_1 = \mathcal{D}_{H(2:(1,1),\omega)}(y^{(2)})\rangle$$

$$\langle e_1 = \mathcal{D}_{H(2:(1,1),\omega)}(xy^{(2)}),\ h,\ f_{-1} = \mathcal{D}_{H(2:(1,1),\omega)}(x)\rangle$$

and a self-normalizing three-dimensional abelian subalgebra

$$\langle z_{-1} = \mathcal{D}_{H(2:(1,1),\,\omega)}(1),\ h,\ z_1 = \mathcal{D}_{H(2:(1,1),\,\omega)}(x^{(2)}y^{(2)})\rangle.$$

(When we write, for example, $e_0 = \mathcal{D}_{H(2:(1,1),\,\omega)}(y)$, we are merely indicating that e_0 is the element of the Lie algebra $H(2 : (1,\, 1),\, \omega)$ which corresponds to the monomial y.)

In view of the universal property of the tensor product, we have a mapping

$$L_{-1} \otimes L_{-1} \longrightarrow L_{-2}$$

given by

$$u \otimes v \mapsto [u,\, v]$$

Since the symmetric tensors map to zero, it must be that L_{-2} is isomorphic to $\wedge^2(L_{-1})$. We will write

$$e_{-2} = [z_{-1},\, e_{-1}]$$

$$z_{-2} = [e_{-1},\, f_{-1}]$$

$$f_{-2} = [f_{-1},\, z_{-1}]$$

In all this notation, the subscripts refer to the gradation space of the element; in addition, e_i, $i = -2, -1, 0, 1$ all have $h-$eigenvalue two, f_i, $i = -2, -1, 0, 1$ all have $h-$eigenvalue minus two, and h, z_1, z_{-1} and z_{-2} all have $h-$eigenvalue zero. To assist the reader in the calculations, we provide the following bracket-operation table:

H	$[,]$	f_0	h	e_0	f_{-1}	h	e_1	f_1	h	e_{-1}	z_{-1}	h	z_1
$x^{(2)}y$	f_0	0	$2f_0$	$2h$	$2e_{-1}$	$2f_0$	0	e_1	$2f_0$	0	0	$2f_0$	0
xy	h	f_0	0	$2e_0$	f_{-1}	0	$2e_1$	f_1	0	$2e_{-1}$	0	0	0
y	e_0	h	e_0	0	z_{-1}	e_0	f_1	$2z_1$	e_0	$2f_{-1}$	$2e_{-1}$	e_0	e_1
x	f_{-1}	e_{-1}	$2f_{-1}$	$2z_{-1}$	0	$2f_{-1}$	$2h$	$2e_0$	$2f_{-1}$	$2z_{-2}$	f_{-2}	$2f_{-1}$	$2f_0$
xy	h	f_0	0	$2e_0$	f_{-1}	0	$2e_1$	f_1	0	$2e_{-1}$	0	0	0
$xy^{(2)}$	e_1	0	e_1	$2f_1$	h	e_1	0	0	e_1	f_0	0	e_1	0
$y^{(2)}$	f_1	$2e_1$	$2f_1$	z_1	e_0	$2f_1$	0	0	$2f_1$	$2h$	f_0	$2f_1$	0
xy	h	f_0	0	$2e_0$	f_{-1}	0	$2e_1$	f_1	0	$2e_{-1}$	0	0	0
$2x^{(2)}$	e_{-1}	0	e_{-1}	f_{-1}	z_{-2}	e_{-1}	$2f_0$	h	e_{-1}	0	$2e_{-2}$	e_{-1}	0
1	z_{-1}	0	0	e_{-1}	$2f_{-2}$	0	0	$2f_0$	0	e_{-2}	0	0	0
xy	h	f_0	0	$2e_0$	f_{-1}	0	$2e_1$	f_1	0	$2e_{-1}$	0	0	0
$x^{(2)}y^{(2)}$	z_1	0	0	$2e_1$	f_0	0	0	0	0	0	0	0	0

In addition, we have

$$[e_0,\, e_{-2}] = 2f_{-2}, \quad [e_0,\, f_{-2}] = z_{-2}, \quad [e_0,\, z_{-2}] = 2e_{-2},$$

and

$$[f_0,\, e_{-2}] = 0, \quad [f_0,\, f_{-2}] = e_{-2}, \quad [f_0,\, z_{-2}] = 0.$$

If, as we may assume, $[L_{-2},\, L_1] = 0$, we would have $[L_1,\, [z_{-1}, e_{-2}]] = 0$. Thus, $[z_{-1}, e_{-2}]$ would be an element of $\mathrm{Ann}_{L_{-3}} L_1$, which is an L_0-module. It would then follow that $[f_0,\, [e_0,\, [e_0, [z_{-1}, e_{-2}]]]] \in \mathrm{Ann}_{L_{-3}} L_1$, as well. However,

$$\begin{aligned}
[f_0,\, [e_0,\, [e_0, [z_{-1}, e_{-2}]]]] &= [f_0,\, [e_0,\, 2([e_{-1},\, e_{-2}] + [z_{-1},\, f_{-2}])]] \\
&= [f_0,\, [f_{-1}, e_{-2}] + 2[e_{-1}, f_{-2}] + 2[z_{-1}, z_{-2}]] \\
&= [e_{-1}, e_{-2}]
\end{aligned}$$

and $[e_{-1}, e_{-2}] \notin \mathrm{Ann}_{L_{-3}} L_1$, since it has non-zero bracket with f_1. This contradiction shows that $[L_{-2},\, L_1] \neq 0$, as required. □

Lemma 6.2. *If $\tilde{L}/M(\tilde{L}) = L(\epsilon)$, then $j = 1$.*

Proof. Recall (See, for example, [BGK].) that $L(\epsilon)$ and M, as subalgebras of contact Lie algebras, have realizations as divided power algebras of $\mathcal{O}(2:(1,\,1))$, with Poisson bracket

$$\{f,\,g\} = \Delta f \partial_z g - \Delta g \partial_z f + \partial_x f \partial_y g - \partial_y f \partial_x g,$$

where

$$\Delta f = 2f = x\partial_x f - y\partial y f.$$

We first note that $L(\epsilon)$ is the sum (again not direct) of four three-dimensional simple Lie algebras. In what follows, we will simply write, for example, $e_0 = y$ to indicate that e_0 is the element of $L(\epsilon)$ corresponding to y. We set $c = 2(z + xy)$, so that $\operatorname{ad} c$ is the "degree derivation."

$$L_0 = \langle e_0 = y,\, h = \frac{1}{\epsilon - 1}(xy + \epsilon z)),\, f_0 = \frac{1}{\epsilon - 1}((1+\epsilon)x^{(2)}y + \epsilon xz)\rangle$$

$$\langle e_{-1} = x^{(2)},\, h' = \frac{2}{\epsilon - 1}(h - \epsilon c),\, f_1 = y^{(2)}\rangle$$

$$\langle e_1 = (1+\epsilon)xy^{(2)} - \epsilon yz,\, h'' = \frac{1}{\epsilon - 1}((1+\epsilon)h + \epsilon c),\, f_{-1} = x)\rangle$$

$$\langle z_{-1} = 1,\, h''' = \frac{2\epsilon^2}{\epsilon - 1}(h - c),\, z_1 = \epsilon(\epsilon+1)x^{(2)}y^{(2)} + \epsilon^2 z^{(2)}\rangle.$$

As before, in the proof of Lemma 6.1, L_{-2} is isomorphic to $\wedge^2(L_{-1})$, and we set $e_{-2} = [z_{-1},\, e_{-1}]$, $z_{-2} = [e_{-1},\, f_{-1}]$, and $f_{-2} = [f_{-1},\, z_{-1}]$.

Set

$$\epsilon_{ijk}^{lmn} \stackrel{\text{def}}{=} \frac{l\epsilon^2 + m\epsilon + n}{i\epsilon^2 + j\epsilon + k},$$

so that, for example, $\epsilon_{111}^{101} = \frac{\epsilon^2+1}{(\epsilon-1)^2}$. Then we have

$L(\epsilon)$	e_0	h	f_0	e_{-1}	h'	f_1	e_1	h''	f_{-1}	z_{-1}	h'''	z_1
e_0	0	e_0	h	$2f_{-1}$	$\epsilon_{012}^{002}e_0$	0	$2f_1$	$\epsilon_{012}^{011}e_0$	$2z_{-1}$	0	$\epsilon_{012}^{200}e_0$	$\epsilon_{001}^{020}e_1$
h	$2e_0$	0	$\epsilon_{012}^{002}f_0$	$\epsilon_{012}^{001}e_{-1}$	0	$\epsilon_{012}^{002}f_1$	$\epsilon_{012}^{011}e_1$	0	$\epsilon_{012}^{022}f_{-1}$	$\epsilon_{012}^{010}z_{-1}$	0	$\epsilon_{012}^{020}z_1$
f_0	$2h$	$\epsilon_{012}^{001}f_0$	0	0	$\epsilon_{111}^{002}f_0$	$\epsilon_{012}^{001}e_1$	$\epsilon_{012}^{002}z_1$	$\epsilon_{111}^{011}f_0$	$\epsilon_{012}^{001}e_{-1}$	$\epsilon_{012}^{020}f_{-1}$	$\epsilon_{111}^{200}f_0$	0
e_{-1}	f_{-1}	$\epsilon_{012}^{002}e_{-1}$	0	0	$\epsilon_{111}^{121}e_{-1}$	h'	$\epsilon_{001}^{012}f_0$	$\epsilon_{111}^{202}e_{-1}$	z_{-2}	$2e_{-2}$	$\epsilon\epsilon_{111}^{100}e_{-1}$	0
h'	$\epsilon_{012}^{001}e_0$	0	$\epsilon_{111}^{001}f_0$	$\epsilon_{111}^{212}e_{-1}$	0	$\epsilon_{111}^{121}f_1$	$\epsilon_{111}^{112}e_1$	0	$\epsilon_{111}^{221}f_{-1}$	$\epsilon_{111}^{200}z_{-1}$	0	$\epsilon_{111}^{100}z_1$
f_1	0	$\epsilon_{012}^{001}f_1$	$\epsilon_{012}^{002}e_1$	$2h'$	$\epsilon_{111}^{212}f_1$	0	0	$\epsilon_{111}^{221}f_1$	$2e_0$	0	$\epsilon\epsilon_{111}^{200}f_1$	0
e_1	f_1	$\epsilon_{012}^{022}e_1$	$\epsilon_{012}^{001}z_1$	$\epsilon_{001}^{021}f_0$	$\epsilon_{111}^{221}e_1$	0	0	$\epsilon_{111}^{122}e_1$	h''	$\epsilon_{001}^{020}e_0$	$\epsilon_{111}^{200}e_1$	0
h''	$\epsilon_{012}^{022}e_0$	0	$\epsilon_{111}^{022}f_0$	$\epsilon_{111}^{101}e_{-1}$	0	$\epsilon_{111}^{112}f_1$	$\epsilon_{111}^{211}e_1$	0	$\epsilon_{111}^{122}f_{-1}$	$\epsilon_{111}^{020}z_{-1}$	0	$\epsilon_{111}^{010}z_1$
f_{-1}	z_{-1}	$\epsilon_{012}^{011}f_{-1}$	$\epsilon_{012}^{002}e_{-1}$	$2z_{-2}$	$\epsilon_{111}^{112}f_{-1}$	e_0	$2h''$	$\epsilon_{111}^{211}f_{-1}$	0	f_{-2}	$\epsilon_{111}^{100}f_{-1}$	$\epsilon_{001}^{210}f_0$
z_{-1}	0	$\epsilon_{012}^{020}z_{-1}$	$\epsilon_{012}^{010}f_{-1}$	e_{-2}	$\epsilon_{111}^{100}z_{-1}$	0	$\epsilon_{001}^{010}e_0$	$\epsilon_{111}^{010}z_{-1}$	$2f_{-2}$	0	$\epsilon\epsilon_{111}^{220}z_{-1}$	h'''
h'''	$\epsilon_{012}^{100}e_0$	0	$\epsilon_{111}^{100}f_0$	$\epsilon\epsilon_{111}^{200}e_{-1}$	0	$\epsilon\epsilon_{111}^{100}f_1$	$\epsilon_{111}^{100}e_1$	0	$\epsilon_{111}^{200}f_{-1}$	$\epsilon\epsilon_{111}^{110}z_{-1}$	0	$\epsilon\epsilon_{111}^{220}z_1$
z_1	$\epsilon_{001}^{010}e_1$	$\epsilon_{012}^{010}z_1$	0	0	$\epsilon_{111}^{200}z_1$	0	0	$\epsilon_{111}^{020}z_1$	$\epsilon_{001}^{120}f_0$	$2h'''$	$\epsilon\epsilon_{111}^{110}z_1$	0

Then, since we are assuming that $[L_{-2}, L_1] = 0$, we have

$$\begin{aligned}
0 &= [e_1, z_{-2}] \\
&= [e_1, [e_{-1}, f_{-1}]] \\
&= 2(\epsilon - 1)[f_0, f_{-1}] + \frac{1}{\epsilon - 1}[e_{-1}, (1+\epsilon)h + \epsilon c] \\
&= 2e_{-1} + \frac{1}{\epsilon - 1}((1+\epsilon)\frac{2}{\epsilon - 1} + \epsilon)e_{-1} \\
&\equiv \frac{4}{(\epsilon - 1)^2}e_{-1},
\end{aligned}$$

which is, of course, absurd. Thus, it must be true that $[L_{-2}, L_1] \neq 0$ in the case of $L(\epsilon)$. □

Lemma 6.3. *If $\tilde{L}/M(\tilde{L}) = M$, then $j = 1$.*

Proof. We note that M, too, is the non-direct sum of the four three-dimensional simple Lie algebras

$$L_0 = \langle e_0 = xz,\ h = z - xy,\ f_0 = y + x^{(2)}z \rangle$$

$$\langle e_{-1} = x^{(2)},\ h' = c + h,\ f_1 = y^{(2)} - xz(2) \rangle$$

$$\langle e_1 = z^{(2)},\ h'' = c - h,\ f_{-1} = 1 \rangle$$

$$\langle z_{-1} = x,\ h''' = 2c,\ z_1 = yz + z^{(2)}x^{(2)}) \rangle.$$

Here, again, $\operatorname{ad} c$ is the degree derivation, with $c = 2(z + xy)$.

Again, L_{-2} is isomorphic to $\wedge^2(L_{-1})$, and we again define e_{-2}, z_{-2}, and f_{-2} as in the proof of Lemma 6.1. Then

M	e_0	h	f_0	e_{-1}	h'	f_1	e_1	h''	f_{-1}	z_{-1}	h'''	z_1
e_0	0	e_0	h	0	e_0	z_1	0	$2e_0$	z_{-1}	e_{-1}	0	$2e_1$
h	$2e_0$	0	f_0	$2e_{-1}$	0	f_1	$2e_1$	0	f_{-1}	0	0	0
f_0	$2h$	$2f_0$	0	$2z_{-1}$	$2f_0$	e_1	z_1	f_0	e_{-1}	$2z_{-1}$	0	f_1
e_{-1}	0	e_{-1}	z_{-1}	0	$2e_{-1}$	h'	0	0	z_{-2}	$2e_{-2}$	$2e_{-1}$	e_0
h'	$2e_0$	0	f_0	e_{-1}	0	$2f_1$	0	0	0	$2z_{-1}$	0	z_1
f_1	$2z_1$	$2f_1$	$2e_1$	$2h'$	f_1	0	0	0	e_0	$2f_0$	f_1	0
e_1	0	e_1	$2z_1$	0	0	0	0	e_1	h''	$2e_0$	e_1	0
h''	e_0	0	$2f_0$	0	0	0	$2e_1$	0	f_1	$2z_{-1}$	0	z_1
f_{-1}	$2z_{-1}$	$2f_{-1}$	$2e_{-1}$	$2z_{-2}$	0	$2e_0$	$2h''$	$2f_1$	0	f_{-2}	$2f_{-1}$	$2f_0$
z_{-1}	$2e_{-1}$	0	z_{-1}	e_{-2}	z_{-1}	f_0	e_0	z_{-1}	$2f_{-2}$	0	$2z_{-1}$	h'''
h'''	0	0	0	e_{-1}	0	$2f_1$	$2e_1$	0	f_{-1}	z_{-1}	0	$2z_1$
z_1	e_1	0	$2f_1$	$2e_0$	$2z_1$	0	0	$2z_1$	f_0	$2h'''$	z_1	0

and we have

$$[e_0,\, e_{-2}] = 0, \quad [e_0,\, f_{-2}] = 2z_{-2}, \quad [e_0,\, z_{-2}] = 2e_{-2},$$

and

$$[f_0,\, e_{-2}] = z_{-2}, \quad [f_0,\, f_{-2}] = 2e_{-2}, \quad [f_0,\, z_{-2}] = f_{-2}.$$

If, as we are assuming, $[L_{-2},\, L_1] = 0$, we would have (since $[h', e_{-2}] = 0$) that $[L_1,\, [e_{-1}, e_{-2}]] = 0$. Thus, $[e_{-1}, e_{-2}]$ would be an element of $\operatorname{Ann}_{L_{-3}} L_1$, and it would then follow that $[f_0,\, [e_0,\, [e_0, [e_{-1}, e_{-2}]]]] \in \operatorname{Ann}_{L_{-3}} L_1$, as well. However,

$$\begin{aligned} [f_0,\, [f_0, [e_{-1}, e_{-2}]]] &= [f_0,\, 2([z_{-1},\, e_{-2}] + [e_{-1},\, z_{-2}])] \\ &= [f_{-1}, e_{-2}] + [z_{-1}, z_{-2}] + [e_{-1}, f_{-2}]. \end{aligned}$$

But

$$[z_1,\, [f_{-1}, e_{-2}] + [z_{-1}, z_{-2}] + [e_{-1}, f_{-2}] = [f_0,\, e_{-2}] + 2\cdot 2[c,\, z_{-2}] + 2[e_0,\, f_{-2}] \equiv z_{-2},$$

so $[f_{-1}, e_{-2}] + [z_{-1}, z_{-2}] + [e_{-1}, f_{-2}] \notin \mathrm{Ann}_{L_{-3}} L_1$. This contradiction shows that $[L_{-2},\ L_1] \neq 0$ in the case of M, as well. $\square$

It is now clear that when $\tilde{L}/M(\tilde{L})$ is Hamiltonian or is isomorphic either to $L(\epsilon)$ or to M, we must have $j = 1$, so we may assume in what follows that $[L_{-2},\ L_1] \neq 0$.

Now let V_1 be any irreducible L_0-submodule of L_1. Because of the $\{1\}-$transitivity of the negative part of $\tilde{L}/M(\tilde{L})$, and because we have shown that $[L_{-2},\ L_1] \neq 0$, we can apply Lemma 2.26 above to $\tilde{L}/M(\tilde{L})$ to conclude either that $[L_{-2},\ V_1] \neq 0$, or that the depth of $\tilde{L}/M(\tilde{L})$ is two and $\tilde{L}/M(\tilde{L})$ modulo its minus-two component is Hamiltonian. However, in the latter case, L_1 is an irreducible $L_0'-$module, so $V_1 = L_1$. Since we are assuming that $[L_{-2},\ L_1] \neq 0$, we have $[L_{-2},\ V_1] \neq 0$ in this latter case, also. Consequently, if $\tilde{\tilde{L}}$ is the Lie algebra generated by $L_{-1} \oplus L_0 \oplus V_1$, then the depth $\tilde{\tilde{q}}$ of $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ (Again, see Theorem 1.3.) is (also) greater than one.

Let $\tilde{\tilde{r}}$ be the height of $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$. Suppose first that the depth $\tilde{\tilde{q}}$ is less than $\tilde{\tilde{r}}$. If $\tilde{\tilde{q}}$ is less than q, then the Main Theorem follows by induction; otherwise, it follows from Sections 4 and 5 and Lemma 3.1.

From now on, then, we will assume that $\tilde{\tilde{r}}$ is less than or equal to $\tilde{\tilde{q}}$. Suppose first that $\tilde{\tilde{r}} > 1$. Then, by definition of $\tilde{\tilde{L}}$, it must be that $[V_1,\ V_1] \neq 0$, so that by Lemma 2.28, $\mathrm{Ann}_{L_0} V_1 = 0$. Clearly, $B(V_1)$ (See Section 3.) satisfies the conditions of the Main Theorem. (Condition (E), for example, follows from the transitivity (C) of L, which shows that actually $M(B(V_1)) = 0$.) Consequently, we can conclude from that theorem that the representation of L_0' on V_1 is restricted, and that the representation of L_0' on $B(V_1)_1 = L_{-1}$ is restricted, as well (since $B(V_1)_1 \subseteq \mathrm{Hom}\,(V_1,\ L_0)$).

Thus, we may assume that $\tilde{\tilde{r}} = 1$; i.e., that $[V_1,\ V_1] = 0$. If $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ is not degenerate (i.e., if we are not in case (iii) of Theorem 1.3), then we can apply Proposition 1.4 to $B((\tilde{\tilde{L}}/M(\tilde{\tilde{L}}))_1) = B(V_1)$ (i.e., we can, as in the previous paragraph, reverse the gradation of $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$) to conclude that $B((\tilde{\tilde{L}}/M(\tilde{\tilde{L}}))_1)$ is isomorphic to $L(\epsilon)$ or M, or is Hamiltonian (i.e., between $H(2 : \mathbf{n}, \omega)$ and $CH(2 : \mathbf{n}, \omega)$). But in those cases $(B((\tilde{\tilde{L}}/M(\tilde{\tilde{L}}))_1))_1 = (\tilde{\tilde{L}}/M(\tilde{\tilde{L}}))_{-1}$ is abelian; i.e., $[L_{-1} + M(\tilde{\tilde{L}}),\ L_{-1} + M(\tilde{\tilde{L}})] \subseteq M(\tilde{\tilde{L}})$, so that by (D), $L_{-2} = [L_{-1},\ L_{-1}] \subseteq M(\tilde{\tilde{L}})$, so that $[L_{-2},\ V_1] = 0$, contrary to what we have proved. Thus, if $\tilde{\tilde{r}} = 1$, $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ must be degenerate.

We will therefore assume in the rest of this section that $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ is degenerate for all choices of irreducible L_0-submodules V_1 in the socle of

L_1. In particular, we can assume that any L_0-submodule of L_1 contains an L_0-submodule V_1 such that $[V_1, [V_1, L_{-1}]] = 0$, and the subalgebra of L generated by L_{-1}, $[L_{-1}, V_1]$, and V_1 is degenerate. We will consider two cases, $1 < r < \tilde{\tilde{q}}$ and $\tilde{\tilde{q}} \leqq r$, and obtain a contradiction in each case. We will thereby easily be able to complete the proof of the Main Theorem.

Suppose first that $1 < r < \tilde{\tilde{q}}$. Then $L_{-r-1} \neq 0$. Since $\tilde{\tilde{L}}/M(\tilde{\tilde{L}})$ is degenerate, we have

$$\tilde{\tilde{L}}_{-i} = [V_1, \tilde{\tilde{L}}_{-i-1}],\ 1 \leqq i \leqq \tilde{\tilde{q}} - 1.$$

Since L_{-1} is an irreducible L_0-module, it follows that

$$L_{-1} = (\operatorname{ad} V_1)^{\tilde{\tilde{q}}-1} L_{-\tilde{\tilde{q}}}.$$

We have

$$0 = [[L_{-1}, V_1], V_1] \supseteq [[[L_{-r-1}, L_r], V_1], V_1] = [[[L_{-r-1}, V_1], V_1]\, L_r].$$

If $r = 2$, this says that $[L_{-1}, L_r] = 0$, to contradict transitivity (C). If $r \geqq 3$, then

$$0 = (\operatorname{ad} V_1)^{r-2}[[[L_{-r-1}, V_1], V_1], L_r] = [(\operatorname{ad} V_1)^r L_{-r-1}, L_r] = [L_{-1}, L_r],$$

to again contradict transitivity, and prove that r cannot be strictly between one and $\tilde{\tilde{q}}$.

Now suppose that $\tilde{\tilde{q}} \leqq r$. In this situation, we will find the following two lemmas useful.

Lemma 6.4. *Let L satisfy hypothesis (A) of the Main Theorem, and let i be a non-zero integer. Further, let M_{-i} and N_i be L_0-submodules of L_{-i} and L_i, respectively, such that $[M_{-i}, N_i] \neq 0$, and $[M_{-i}, [M_{-i}, N_i]] = 0 = [[[M_{-i}, N_i], N_i], N_i]$. Then $[M_{-i}, N_i]$ lies in the center of L_0.*

Proof. By hypothesis,

$$[M_{-i}, [M_{-i}, [N_i, [N_i, [M_{-i}, N_i]]]]] \subseteq [M_{-i}, [M_{-i}, 0]] = 0.$$

Consequently, we have, for $m \in M_{-i}$ and $n \in N_i$,

$$0 = (\operatorname{ad} m)^2 (\operatorname{ad} n)^2 [M_{-i}, N_i] = 2(\operatorname{ad}\, [m, n])^2 [M_{-i}, N_i].$$

It follows from Lemma 2.9 that $[M_{-i},\, N_i]$ is nilpotent; however, L_0 is assumed to be classical reductive, so $[M_{-i},\, N_i]$ must be in the center of L_0. □

Corollary 6.5. *Under the hypotheses of the above lemma, if, in addition, $[M_{-i},\, N_i]$ is not in the center of L_0, then either $[M_{-i},\, [M_{-i},\, N_i]] \neq 0$ or $[[[M_{-i},\, N_i],\, N_i],\, N_i] \neq 0$, so that, in particular, either $B(M_{-i})_1 \neq 0$ or $B(N_i)_1 \neq 0$.*

Lemma 6.6. *Let L be as in the statement of the Main Theorem. If $r > 1$, and all L_0-submodules of L_1 contain an irreducible L_0-submodule V_1 such that $\tilde{\tilde{L}}$, as defined above, is degenerate, then*

$$[L_{-i},\, L_i] \neq 0,\ 1 \leqq i \leqq \tilde{\tilde{q}}.$$

Proof. Since L is transitive and r is assumed to be greater than one, so that $L_2 \neq 0$, we have from above that

$$\begin{aligned} 0 &\neq [L_{-1},\, L_2] = [(\operatorname{ad} V_1)^{\tilde{\tilde{q}}-1} L_{-\tilde{\tilde{q}}},\, L_2] \\ &\subseteq [[(\operatorname{ad} V_1)^{\tilde{\tilde{q}}-2} L_{-\tilde{\tilde{q}}},\, L_2],\, V_1] + [(\operatorname{ad} V_1)^{\tilde{\tilde{q}}-2} L_{-\tilde{\tilde{q}}},\, [V_1,\, L_2]] \end{aligned}$$

Now, focusing on the first term in the above sum (and bearing in mind that $[[L_{-1},\, V_1],\, V_1] = 0$), we have

$$\begin{aligned} [[(\operatorname{ad} V_1)^{\tilde{\tilde{q}}-2} L_{-\tilde{\tilde{q}}},\, L_2],\, V_1] &= [[[V_1,\, L_2],\, (\operatorname{ad} V_1)^{\tilde{\tilde{q}}-3} L_{-\tilde{\tilde{q}}}],\, V_1] \\ &= [[[V_1,\, [V_1,\, L_2]],\, (\operatorname{ad} V_1)^{\tilde{\tilde{q}}-4} L_{-\tilde{\tilde{q}}}],\, V_1] \\ &= \cdots = [[(\operatorname{ad} V_1)^{\tilde{\tilde{q}}-2} L_2,\, L_{-\tilde{\tilde{q}}}],\, V_1] \end{aligned}$$

Consequently, if this (first) term is not zero, then the conclusion is true. Suppose, then, that this (first) term is zero. Then the second term must be non-zero, and we have (since the first term is zero) that

$$0 \neq [(\operatorname{ad} V_1)^{\tilde{\tilde{q}}-2} L_{-\tilde{\tilde{q}}},\, [V_1,\, L_2]] = \cdots = [L_{-\tilde{\tilde{q}}},\, (\operatorname{ad} V_1)^{\tilde{\tilde{q}}-1} L_2]$$

so that for all i between one and $\tilde{\tilde{q}}$ inclusive,

$$[L_{-i},\, [L_i,\, V_1]] \neq 0.$$

However, if we then assume that $[L_{-i},\, L_i] = 0$, we have, for all natural numbers j (since we are assuming that any non-zero L_0-submodule of L_1

contains an L_0-submodule V_1 as described in the statement of the lemma, so that if necessary, we can replace our old V_1 with a new one) that

$$\begin{array}{rcl}(\mathrm{ad}\, L_i)^j(\mathrm{ad}\, L_{-i})^j V_1 & = & (\mathrm{ad}\, L_i)^{j-1}(\mathrm{ad}\, L_{-i})^{j-1}[L_i,\, [L_{-i},\, V_1]] \\ & \supseteq & (\mathrm{ad}\, L_i)^{j-1}(\mathrm{ad}\, L_{-i})^{j-1}V_1 \\ & \supseteq & \cdots \supseteq V_1 \neq 0,\end{array}$$

But that would imply that $(\mathrm{ad}\, L_{-i})^j V_1 \neq 0$ for all natural numbers j. Since L is assumed to be finite-dimensional, we have arrived at a contradiction. Thus, $[L_{-i},\, L_i]$ must be non-zero, as claimed. $\square$

Note that in the above proof, $L_{-\tilde{\tilde{q}}}$ could be replaced by any L_0-submodule K_{-i} of L_{-i}, $2 \leqq i \leqq \tilde{\tilde{q}}$, such that $(K_{-i} + M(\tilde{\tilde{L}})_{-i})/M(\tilde{\tilde{L}})_{-i} \neq 0$ to demonstrate that $[K_{-i'},\, L_i'] \neq 0$, $1 \leqq i' \leqq i$; in addition, note that L_2 could be replaced by any non-zero L_0-submodule K_2 of L_2, so that, for example, $[K_{-2},\, K_2] \neq 0$ for any such choices of K_{-2} and K_2.

Now suppose that $\mathrm{Ann}_{L_{-2}} V_1 \neq 0$, as it would, if, for example, $M(\tilde{\tilde{L}})_{-2} \neq 0$. Let X_{-2} be an irreducible L_0-submodule of $\mathrm{Ann}_{L_{-2}} V_1$. Then either $[X_{-2},\, [L_{-1},\, V_1]] = X_{-2}$, or $[X_{-2},\, [L_{-1},\, V_1]] = 0$. In the latter case, we have by the transitivity (C) and irreducibility (B) of L that $[X_{-2},\, [X_{-2},\, [L_1,\, V_1]]] = [X_{-2},\, [L_{-1},\, V_1]] = 0$. Since L_{-1} can be assumed to be a non-restricted L_0-module, it follows (See Lemma 2.9.) that the representation of $B(X_{-2})_0 \cong L_0/\mathrm{Ann}_{L_0} X_{-2}$ on $B(X_{-2})_{-1} \cong X_{-2}$ is also not restricted. Then (using the Main Theorem and induction on q) $B(X_{-2})$ must be a depth-one graded Lie algebra isomorphic to one of the Lie algebras listed in Proposition 1.4. In particular, $[[L_1,\, V_1],\, [L_1,\, V_1]] = 0$. Then

$$\begin{aligned}&[X_{-2},\, [X_{-2},\, [[L_1,\, V_1],\, [[L_1,\, V_1],\, [X_{-2},\, [L_1,\, V_1]]]]]] \\ &\quad\subseteq [X_{-2},\, [X_{-2},\, [[L_1,\, V_1],\, [L_1,\, V_1]]]] \\ &\quad\subseteq [X_{-2},\, [X_{-2},\, 0]] \\ &\quad= 0.\end{aligned}$$

Then if $w \in X_{-2}$ and $d \in [L_1,\, V_1]$, we would have

$$\begin{array}{rcl}0 & = & (\mathrm{ad}\, w)^2(\mathrm{ad}\, d)^2[X_{-2},\, [L_1,\, V_1]] \\ & = & 2(\mathrm{ad}\, [w,\, d])^2[X_{-2},\, [L_1,\, V_1]]\end{array}$$

so by Lemma 2.9 $[X_{-2},\, [L_1,\, V_1]]$ is nilpotent. Then (See Lemma 6.6.) $[X_{-2},\, [L_1,\, V_1]]$ would be in the center of L_0. But (again, as we observed above) by the definition of X_{-2}, the fact that $\tilde{L}$ is 1-transitive in the negative part, and the irreducibility of L, $[X_{-2}, [L_1, V_1]] = [L_{-1}, V_1]$. Then $[L_{-1}, V_1]$ would be in the center of L_0. However (See Corollary 6.5.), since the adjoint endomorphism of any non-zero element z of the center of L_0 must (by Schur's Lemma, and the irreducibility (B) and transitivity (C) of L) act as a non-zero scalar on $L_{-1} = [L_{-1}, [L_{-1}, V_1]]$, it follows that $0 \neq [z, V_1] \subseteq [[L_{-1}, V_1], V_1]$, contrary to hypothesis. This contradiction shows that $[X_{-2}, [L_{-1}, V_1]] = X_{-2}$. But the representation (induced by the adjoint representation of L on itself) of $[L_{-1}, V_1]$ on L_{-1} is restricted (by the definition of a degenerate Lie algebra), as must then be the representation of $[L_{-1}, V_1]$ on $X_{-2} \subseteq [L_{-1}, L_{-1}]$. Consequently, $B(X_{-2})_0$ contains at least two non-trivial ideals, one isomorphic to $[L_{-1}, V_1]$ and one isomorphic to the ideal whose representation on L_{-1} is not restricted. However, it would then follow from [Ku2] that $B(X_{-2})$ is classical, a contradiction. Thus, we can conclude that $\mathrm{Ann}_{L_{-2}} V_1 = 0$.

Now let K_{-2} be any irreducible L_0-submodule of L_{-2}, and let K_2 be as in the remark following the proof of Lemma 6.4; i.e., K_2 is a (non-zero) irreducible L_0-submodule of L_2. Because we can assume that $M(\tilde{\tilde{L}})_{-2} \subseteq \mathrm{Ann}_{L_{-2}} V_1 = 0$, we can, in view of the aforementioned remark following the proof of Lemma 6.6, conclude that $[K_{-2}, K_2] \neq 0$, and (in view of the irreducibility (C) of L) that $[[K_{-2}, V_1], V_1] = [L_{-1}, V_1]$. Since (See Theorem 1.3.) $[[L_{-1}, V_1], [L_{-1}, V_1]] \neq 0$, we have

$$0 \neq [[[K_{-2}, V_1], V_1], [L_{-1}, V_1]] = [[[K_{-2}, [L_{-1}, V_1]], V_1], V_1].$$

It follows that K_{-2} isn't a trivial $[L_{-1}, V_1]-$module. Furthermore, since $[K_{-2}, K_2] \neq 0$, we have by Corollary 6.5 that if $[K_{-2}, [K_{-2}, K_2]] = 0$, then $[[[K_{-2}, K_2], K_2], K_2] \neq 0$. However, in that case, we would have by Theorem 1.3 that (in the Lie algebra generated by K_{-2}, L_0, and K_2) $B(K_2)$ is a degenerate Lie algebra, so that K_{-2} would be abelian, contrary to the definition (according to the same Theorem 1.3) of the bracket operation in $\tilde{\tilde{L}}$, unless, of course, $\dim V_1 = 1$ and $\tilde{\tilde{q}} = 2$; in this latter case, though, V_1 would be a trivial $L_0'-$module, to contradict Lemma 1.2 and our assumption that the χ of that lemma is not (identically) zero. (Of course, the commutativity of K_{-2} could also be deduced from the transitivity of $B(K_2)$, since $[K_2, [K_{-2}, K_{-2}]] \subseteq [K_{-2}, [K_{-2}, K_2]] = 0$.) It follows that $[K_{-2}, [K_{-2}, K_2]] \neq 0$; i.e., $B(K_{-2})_1 \neq 0$, so we can argue as above in the $X_{-2} \neq 0$ case to arrive at a contradiction; i.e., $B(K_{-2})$ cannot both be classical and have the representation of its null component on its minus-one component fail to be restricted.

Thus, it must be that $r = 1$, and

$$L = L_{-q} \oplus \cdots L_{-1} \oplus [L_{-1}, V_1] \oplus V_1$$

is a degenerate Lie algebra (in the sense of Theorem 1.3(ii)). But in that situation, the representation of $[L_{-1}, V_1]$ on L_{-1} is restricted. It cannot be, then, that $L_0 = [L_{-1}, V_1]$; there must be a summand $I_0 \subseteq L_0$ which acts non-restrictedly on L_{-1}.

It follows, as noted above, that the proof of the Main Theorem is now complete.

7 Remark

Recall that in the proof of Lemma 6.1, we noted that $L_{-2} \cong L_{-1} \wedge L_{-1}$. It turns out that a similar statement can be made for an irreducible L_0-submodule of L_2 in the cases in which $\tilde{L}/M(\tilde{L})$ is isomorphic either to $L(\epsilon)$ or to M, even though $[L_1, L_1] = 0$ in those cases. Let j be as in the previous section. By Lemma 2.25, $j \leqq 2$; furthermore,

$$\tilde{L}/M(\tilde{L}) \cong \tilde{L}_{-1} + \tilde{L}_0 + \tilde{L}_1 = L(\epsilon) \text{ or } M.$$

In this case

$$L_0 = \tilde{L}_0 = \mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus \langle w \rangle, \tag{7.1}$$

where $\langle w \rangle$ is the center of L_0. If we compose the bracket operation in L with the projection π of $\mathfrak{gl}(2)$ onto $\langle w \rangle$, we obtain a non-zero L_0-invariant pairing

$$L_{-1} \times L_1 \xrightarrow{[\,,]} L_0 \xrightarrow{\pi} \langle w \rangle.$$

Since L_{-1} and L_1 are irreducible L_0-modules, we have $L_1 \cong L_{-1}^*$. Let L_{-1} correspond to the point (χ, a) on the Zassenhaus variety of $\mathfrak{sl}(2)$; that is, χ is the character of the adjoint representation of L_0 on L_{-1} (See Lemma 1.2.), and a is the value of the Casimir operator on L_{-1}; i.e., $((-\operatorname{ad} h + I)^2 + (\operatorname{ad} e)(\operatorname{ad} f))|_{L_{-1}} = aI$, where I is the identity operator on L_{-1}. (See [BKK].) It can be shown that L_{-1}^* corresponds to the point $(-\chi, a)$.

Let $\check{L}_2$ be an irreducible L_0- submodule of L_2. Since L is transitive (C),

$$L_2 \subseteq \operatorname{Hom}(L_{-1}, L_1) = L_{-1}^* \otimes L_1.$$

Using the isomorphism $L_{-1}^* \cong L_1$, we have

$$L_2 \subseteq L_{-1}^* \otimes L_{-1}^* = \mathcal{B}(L_{-1}^*),$$

the L_0-module of bilinear forms on L_{-1}^*. Obviously, $\mathcal{B}(L_{-1}^*)$ has two natural L_0-submodules, namely, $S(L_{-1}^*)$, the L_0-submodule of symmetric bilinear forms, and $\Lambda^2(L_{-1}^*)$, the L_0-submodule of skew-symmetric bilinear forms. Recall that $\dim L_{-1} = 3$. Therefore, $\dim S(L_{-1}^*) = 6$, and $\dim \Lambda^2(L_{-1}^*) = 3$.

We first consider the case in which $\check{L}_2 \subseteq S(L_{-1}^*)$. For $l \in \check{L}_2$, the corresponding bilinear form $\psi \in \mathcal{B}(L_{-1})$ is defined by

$$\psi(u,\, v) = \pi([[l,\, u],\, v]),\; u, v \in L_{-1}.$$

Since $\psi \in S(L_{-1})$, it follows that $\pi([[l,\, u],\, v]) = \pi([[l,\, v],\, u])$ for any $u,\, v \in L_{-1}$, whence $[l,\, [u,\, v]] \in \mathfrak{sl}(2) \subset L_0$. Thus, since elements of the form $[u,\, v]$ span L_{-2}, we have that $[L_{-2},\, \tilde{L}_2] = \mathfrak{sl}(2)$.

In the notation of [BKK],

$$\tilde{L}_{-1} = L_{-1} = \langle 1,\, x,\, x^2 \rangle$$

for both $\tilde{L}/M(\tilde{L}) = L(\epsilon)$ and $\tilde{L}/M(\tilde{L}) = M$ (and for Hamiltonian $\tilde{L}/M(\tilde{L})$ as well, as we saw earlier). Let $\{v_0,\, v_1,\, v_2\}$ be the basis of L_{-1}^* which is dual to $\{1,\, x,\, x^2\}$.

Lemma 7.2. *If $\tilde{L}/M(\tilde{L}) = M$ then $\check{L}_2 \not\subseteq S(L_{-1}^*)$.*

Proof. When $\tilde{L}/M(\tilde{L}) = M$, $\langle v_0^2,\, v_0 v_1,\, v_1^2 + v_0 v_2 \rangle$ is the unique three-dimensional L_0-submodule of $S(L_{-1}^*)$. By Lemma 1.2, the character on Ľ2 is non-zero. Since the only non-restricted L_0-modules are three dimensional, it follows that

$$\check{L}_2 = \langle v_0^2,\, v_0 v_1,\, v_1^2 + v_0 v_2 \rangle.$$

It can be shown that the L_0-module $\langle v_0^2,\, v_0 v_1,\, v_1^2 + v_0 v_2 \rangle$ corresponds to the point $(\chi,\, 1)$ in the Zassenhaus variety, and that $L_{-2} = \Lambda^2(L_{-1})$ corresponds to the point $(-\chi,\, 0)$.

Suppose that $[\check{L}_2,\, \check{L}_2] = 0$ (or that $[L_{-2},\, L_{-2}] = 0$). Then (in the first case) we would have a one-graded Lie algebra

$$\cdots \oplus L_{-2} \oplus L_0' \oplus \check{L}_2$$

generated by L_{-2} and $\check{L}_2$, since as we noted above, $[L_{-2},\, \check{L}_2] = \mathfrak{sl}(2) = L_0'$. According to Corollary 1.5, such a Lie algebra is of type $H(2 : \mathbf{n}, \omega)$.

However, in $H(2:\mathbf{n},\omega)$, the $\mathfrak{sl}(2)-$module $H(2:\mathbf{n},\omega)_1$ is of type $(-\chi,\,1)$. Here, however, the rôle of $H(2:\mathbf{n},\omega)_1$ is being played by L_{-2}, and we saw above that L_{-2} is of type $(-\chi,\,0)$. This contradiction shows that $[\check{\mathrm{L}}_2,\,\check{\mathrm{L}}_2]\neq 0$. Similarly, $[L_{-2},\,L_{-2}]\neq 0$. We can now let the Lie algebra generated by L_{-2}, L_0', and $\tilde{L}_2$ play the rôle of $\tilde{L}/M(\tilde{L})$ in the above argument. Then $[\check{\mathrm{L}}_2,\,\check{\mathrm{L}}_2]$ would be of type $(-\chi,\,0)$ and $[L_{-2},\,L_{-2}]$ would be of type $(-\chi,\,1)$, and we could argue as above to conclude that $[[\check{\mathrm{L}}_2,\,\check{\mathrm{L}}_2],\,[\check{\mathrm{L}}_2,\,\check{\mathrm{L}}_2]]$ is non-zero and of type $(-\chi,\,1)$, and that $[[L_{-2},\,L_{-2}],\,[L_{-2},\,L_{-2}]]$ is likewise non-zero and of type $(-\chi,\,0)$. Continuing in this way, we conclude that $\tilde{L}_2^n\neq 0$ and $L_{-2}^n\neq 0$ for any positive integer n. Thus, since we are assuming that L is finite-dimensional, it cannot be that $\check{\mathrm{L}}_2\subseteq S(L_{-1}^*)$ when $\tilde{L}/M(\tilde{L})=M$. □

Lemma 7.3. *If* $\tilde{L}/M(\tilde{L})=L(\epsilon)$ *then* $\check{L}_2\not\subseteq S(L_{-1}^*)$.

Proof. Suppose now that $\tilde{L}/M(\tilde{L})=L(\epsilon)$, and (again) that either $[\check{\mathrm{L}}_2,\,\check{\mathrm{L}}_2]=0$ or $[L_{-2},\,L_{-2}]=0$. Then, as in the previous case where $\tilde{L}/M(\tilde{L})$ $=M$, we would have a one-graded Lie algebra, for example,

$$\cdots\oplus L_{-2}\oplus L_0'\oplus\check{\mathrm{L}}_2$$

with a semisimple character χ of $L_0'=\mathfrak{sl}(2)$ on $\check{\mathrm{L}}_2$. According to [BKK], such a situation is impossible. Thus, $[\check{\mathrm{L}}_2,\,\check{\mathrm{L}}_2]\neq 0$ and $[L_{-2},\,L_{-2}]\neq 0$, and both L_0-modules correspond to semisimple characters of $\mathfrak{sl}(2)$. We repeat our argument for $[\check{\mathrm{L}}_2,\,\check{\mathrm{L}}_2]$ and $[L_{-2},\,L_{-2}]$ to show that $(\check{\mathrm{L}}_2^2)^2\neq 0$ and that $(L_{-2}^2)^2\neq 0$. Continuing in this way, we see that $\dim L=\infty$, so that it cannot be that $\check{\mathrm{L}}_2\subseteq S(L_{-1}^*)$ when $\tilde{L}/M(\tilde{L})=L(\epsilon)$, either. □

More explicit analysis of the L_0-module $\mathcal{B}(L_{-1}^*)=S^2(L_{-1}^*)\oplus\Lambda^2(L_{-1}^*)$ in the case $\tilde{L}/M(\tilde{L})=M$ shows that $S^2(L_{-1}^*)$ is an indecomposable L_0-module, and that

$$soc(S^2(L_{-1}^*))=\langle v_0^2,\,v_0v_1,\,v_1^2+v_0v_2\rangle\subseteq\tilde{S}^2(L_{-1}^*),$$

where $\tilde{S}^2(L_{-1}^*)$ is the L_0-submodule of $S^2(L_{-1}^*)$ corresponding to the point $(\mu,\,1)$ of the Zassenhaus variety. (Note that $\Lambda^2(L_{-1}^*)$ corresponds to the point $(\mu,\,0)$.) It follows that

$$soc(\mathcal{B}(L_{-1}^*))=soc(S^2(L_{-1}^*)\oplus\Lambda^2(L_{-1}^*))=\tilde{S}^2(L_{-1}^*)\oplus\Lambda^2(L_{-1}^*).$$

Therefore, any non-trivial submodule of $\mathcal{B}(L_{-1}^*)$ contains either $\tilde{S}^2(L_{-1}^*)$ or $\Lambda^2(L_{-1}^*)$. We are focusing on L_2 as a submodule of $\mathcal{B}(L_{-1}^*)$. Therefore, either $\tilde{S}^2(L_{-1}^*) \subseteq L_2$ or $\Lambda^2(L_{-1}^*) \subseteq L_2$. In light of the previous lemma, we can conclude that $\mathrm{u} \cong \Lambda^2(L_{-1}^*) \cong \Lambda^2(L_1)$ in the M case.

We will now demonstrate that this is true more generally; i.e., that $\breve{\mathrm{L}} \cong \Lambda^2(L_1)$ both when $\tilde{L}/M(\tilde{L}) \cong L(\epsilon)$ and when $\tilde{L}/M(\tilde{L}) \cong M$. Since u_2 was assumed to be any irreducible L_0-submodule of L_2, and since (in light of the previous two lemmas) we may assume that $\breve{\mathrm{L}}_2$ is not contained in $S(L_{-1}^*)$, and since

$$\mathrm{u}_2 \subseteq L_2 \subseteq S(L_{-1}^*) \oplus \Lambda^2(L_{-1}^*),$$

it follows that

$$\mathrm{u}_2 \cap \Lambda^2(L_{-1}^*) \neq 0.$$

However, because of the irreducibility of $\breve{\mathrm{L}}_2$, it must be that

$$\mathrm{u}_2 \subseteq \Lambda^2(L_{-1}^*).$$

Then by hypothesis, we can assume that

$$\mathrm{u}_2 = \Lambda^2(L_{-1}^*)$$

for any irreducible submodule $\breve{\mathrm{L}}_2$ of L_2, as we wanted to show.

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