

A MONGE-AMPÈRE TYPE FULLY NONLINEAR EQUATION ON HERMITIAN MANIFOLDS

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ABSTRACT. We study a fully nonlinear equation of complex Monge-Ampère type on Hermitian manifolds. We establish the *a priori* estimates for solutions of the equation up to the second order derivatives with the help of a subsolution.

1. Introduction. Let (M^n, ω) be a compact Hermitian manifold of dimension greater than 1, with smooth boundary ∂M (which may be empty), and χ a smooth real $(1, 1)$ form on M . Define

$$\chi_u = \chi + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u \text{ and } [\chi] = \{\chi_u : u \in C^2(M)\}.$$

In this paper we are concerned with the equation

$$\chi_u^n = \psi \chi_u^{n-1} \wedge \omega, \quad \chi_u > 0 \text{ on } M. \quad (1.1)$$

When M is closed, both ω, χ are Kähler and ψ is constant, equation (1.1) was introduced by Donaldson [5] in the setting of moment maps. Donaldson observed that in this case the solution of equation (1.1) is unique (up to a constant) if exists, and that a necessary condition for the existence of solution is $[n\chi - \psi\omega] > 0$. He remarked that a natural conjecture would be that this is also a sufficient condition.

Donaldson's problem was studied by Chen [3], Weinkove [13], [14], Song and Weinkove [10] using parabolic methods as limit of the J -flow introduced by Donaldson [5] and Chen [2]. In [10] Song and Weinkove gave a necessary and sufficient condition for convergence of the J -flow. Later on Fang, Lai and Ma [6] extended their approach and solved the equation

$$\chi_u^n = c_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha, \quad \chi_u > 0 \text{ on } M. \quad (1.2)$$

for all $1 \leq \alpha < n$.

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A key ingredient in solving elliptic or parabolic fully nonlinear equations is to derive *a priori* estimates up to the second order derivatives. For the complex Monge-Ampère equation on closed Kähler manifolds, these estimates were established by Yau [15] and Aubin [1]. Their results and techniques had far-reaching influences in both geometry and to the theory of nonlinear PDEs on manifolds. In 1987, Cherrier [4] studied the complex Monge-Ampère equation on Hermitian manifolds. He established the estimates for second order derivatives in the general case, and extended Yau's zeroth order estimate under an additional assumption on the Hermitian metric. Recently, Tosatti and Weinkove [12] were able to carry out Yau's estimate on general closed Hermitian manifolds.

There has been increasing interest to study fully nonlinear elliptic and parabolic equations other than the complex Monge-Ampère equation on Kähler or Hermitian manifolds, both from geometric problems such as Donaldson's problem mentioned above, and from the PDE point of view. In this paper our main interest is to seek general technical methods in establishing *a priori* estimates. We shall confine ourselves to $\alpha = 1$ in (1.2) but our method works for more general equations and in particular for all $\alpha < n$. We shall treat the other cases in separate papers.

Our first result for equation (1.1) is the following

Theorem 1.1. *Let $u \in C^4(M)$ a solution of equation (1.1) and set $C_0 = \sup_M u - \inf_M u$. Assume that there exists a function $\underline{u} \in C^2(M)$ satisfying*

$$\chi_{\underline{u}}^n \geq \psi \chi_{\underline{u}}^{n-1} \wedge \omega, \quad \chi_{\underline{u}} > 0 \quad \text{on } M. \quad (1.3)$$

Then there are constants C_1, C_2 , depending on $C_0, |\underline{u}|_{C^2(M)}$, the positive lower bound of $\chi_{\underline{u}}$, and $\inf_M \psi > 0$ as well as other known data, such that

$$\max_M |\nabla u| \leq C_1(1 + \max_{\partial M} |\nabla u|), \quad |\Delta u| \leq C_2((1 + \max_{\partial M} |\Delta u|)) \quad \text{on } M. \quad (1.4)$$

We remark that both C_1 and C_2 in Theorem 1.1 depend on C_0 , but the estimate for Δu is independent of the gradient bound. (i.e. C_2 is independent of C_1 .) Apparently, assumption (1.3) is a trivial necessary condition for the solvability of equation (1.1). Following the literature we shall call the function \underline{u} a *subsolution* of equation (1.1). It seems worthwhile to remark that the subsolution \underline{u} plays key roles in our proof of both estimates in (1.4); see Sections 3-4 for details. This appears to us a rather new phenomena, and we are not clear how to derive these estimates without using \underline{u} . We also remark that the gradient estimate seems new even in the Kähler case.

Theorem 1.1 still holds under the following assumption which is slightly weaker than (1.3)

$$(n\chi_{\underline{u}} - (n-1)\psi\omega) \wedge \chi_{\underline{u}}^{n-2} > 0, \quad \chi_{\underline{u}} > 0 \quad \text{on } M. \quad (1.5)$$

When M is Kähler and ψ is a constant, this condition was first given by Song and Weinkove [10] and proved to be necessary and sufficient for the solvability (1.1) on closed Kähler manifolds.

The estimates in Theorem 1.1 enable us to treat the Dirichlet problem for equation (1.1) on Hermitian manifolds with boundary. More precisely we can prove the following existence result under the assumption of existence of a subsolution.

Theorem 1.2. *Let (M^n, ω) be a compact Hermitian manifold with smooth boundary ∂M , $\psi \in C^\infty(\bar{M})$, $\psi > 0$, where $\bar{M} = M \cup \partial M$ and $\varphi \in C^\infty(\partial M)$. Suppose there*

exists a subsolution $\underline{u} \in C^2(\bar{M})$ satisfying

$$\begin{cases} \chi_{\underline{u}}^n \geq \psi \chi_{\underline{u}}^{n-1} \wedge \omega, \chi_{\underline{u}} > 0 \text{ on } \bar{M} \\ \underline{u} = \varphi \text{ on } \partial M. \end{cases} \quad (1.6)$$

Then equation (1.1) admits a unique solution $u \in C^\infty(M)$ with $u = \varphi$ on ∂M .

In order to prove Theorem 1.2 we need to establish *a priori* boundary estimates. The gradient estimate on the boundary follows immediately from a barrier argument. The proof for the second order boundary estimates is similar to the Monge-Ampère equation case in [7] and will be omitted here. We shall come back to the issue for more general equations including (1.2) in our forthcoming papers where we shall also discuss the existence questions for the closed manifold case. In this paper we will just present the global *a priori* estimates up to the second order derivatives of the solutions to equation (1.1).

The rest of this paper is organized as follows. In section 2 we fix some notations and introduce some fundamental formulas in Hermitian geometry, which will be used throughout the paper. We also establish a crucial lemma in this section that will be applied to deriving the estimates in the following sections. Section 3 and Section 4 will be devoted to establishing the global gradient estimates and the estimates for the second order derivatives respectively. In both sections we make the important use of the existence of a *subsolution*.

We dedicate this article with sincere respect and admiration to Professor Avner Friedman on the occasion of his 80th birthday. We wish to thank Wei Sun for pointing out several mistakes in previous versions. Part of this work was done while the first author was in Xiamen University in summer 2011.

2. Preliminaries. We shall follow the notations in [7] where the reader can also find a brief introduction to the background materials for Hermitian manifolds. In particular, g and ∇ will denote the Riemannian metric and Chern connection of (M, ω) . The torsion and curvature tensors of ∇ are defined by

$$\begin{aligned} T(u, v) &= \nabla_u v - \nabla_v u - [u, v], \\ R(u, v)w &= \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w, \end{aligned} \quad (2.1)$$

respectively. In local coordinates $z = (z_1, \dots, z_n)$,

$$\begin{cases} g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right), \quad \{g^{i\bar{j}}\} = \{g_{i\bar{j}}\}^{-1}, \\ T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k = g^{k\bar{l}} \left(\frac{\partial g_{j\bar{l}}}{\partial z_i} - \frac{\partial g_{i\bar{l}}}{\partial z_j} \right), \\ R_{i\bar{j}k\bar{l}} = -g_{m\bar{l}} \frac{\partial \Gamma_{ik}^m}{\partial \bar{z}_j} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z_i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j}. \end{cases} \quad (2.2)$$

Let $v \in C^4(M)$. For convenience we write in local coordinates

$$v_{i\bar{j}} = \nabla_{\bar{j}} \nabla_i v, v_{i\bar{j}k} = \nabla_k v_{i\bar{j}}, \text{ etc.}$$

Recall $v_{i\bar{j}} = v_{\bar{j}i} = \partial_i \bar{\partial}_j v$ and $v_{i\bar{j}k} = \partial_k v_{i\bar{j}} - \Gamma_{ki}^l v_{l\bar{j}}$. It follows that

$$\begin{cases} v_{i\bar{j}k} - v_{k\bar{j}i} = T_{ik}^l v_{l\bar{j}}, \\ v_{i\bar{j}k} - v_{i\bar{k}j} = \overline{T_{jk}^l} v_{i\bar{l}}. \end{cases} \quad (2.3)$$

We calculate

$$\begin{aligned}
v_{i\bar{j}k\bar{l}} &= \bar{\partial}_l v_{i\bar{j}k} - \overline{\Gamma_{lj}^q} v_{i\bar{q}k} \\
&= \bar{\partial}_l (\partial_k v_{i\bar{j}} - \Gamma_{ki}^p v_{p\bar{j}}) - \overline{\Gamma_{lj}^q} v_{i\bar{q}k} \\
&= \bar{\partial}_l \partial_k v_{i\bar{j}} - \bar{\partial}_l \Gamma_{ki}^p v_{p\bar{j}} - \Gamma_{ki}^p (v_{p\bar{j}\bar{l}} + \overline{\Gamma_{lj}^q} v_{p\bar{q}}) - \overline{\Gamma_{lj}^q} v_{i\bar{q}k} \\
&= \partial_k \bar{\partial}_l v_{i\bar{j}} + g^{p\bar{q}} R_{k\bar{l}i\bar{q}} v_{p\bar{j}} - \Gamma_{ki}^p v_{p\bar{j}\bar{l}} - \overline{\Gamma_{lj}^q} v_{i\bar{q}k} - \Gamma_{ki}^p \overline{\Gamma_{lj}^q} v_{p\bar{q}},
\end{aligned} \tag{2.4}$$

$$v_{i\bar{j}l\bar{k}} = \overline{v_{j\bar{i}l\bar{k}}} = \bar{\partial}_l \partial_k v_{i\bar{j}} - \overline{\Gamma_{lj}^q} \Gamma_{ki}^q v_{q\bar{p}} - \overline{\Gamma_{lj}^p} v_{i\bar{p}k} - \Gamma_{ki}^q v_{j\bar{q}\bar{l}} + g^{q\bar{p}} R_{k\bar{l}j\bar{q}} v_{i\bar{p}}.$$

Therefore,

$$\begin{cases} v_{i\bar{j}k\bar{l}} - v_{i\bar{j}l\bar{k}} = g^{p\bar{q}} R_{k\bar{l}i\bar{q}} v_{p\bar{j}} - g^{p\bar{q}} R_{p\bar{l}k\bar{j}} v_{i\bar{q}}, \\ v_{i\bar{j}k\bar{l}} - v_{k\bar{l}i\bar{j}} = g^{p\bar{q}} (R_{k\bar{l}i\bar{q}} v_{p\bar{j}} - R_{i\bar{j}k\bar{q}} v_{p\bar{l}}) + T_{ik}^p v_{p\bar{j}\bar{l}} + \overline{T_{jl}^q} v_{i\bar{q}k} - T_{ik}^p \overline{T_{jl}^q} v_{p\bar{q}}. \end{cases} \tag{2.5}$$

The second identity in (2.5) follows from (2.4), (2.3) and

$$\Gamma_{ki}^p \overline{\Gamma_{lj}^q} - \Gamma_{ik}^p \overline{\Gamma_{jl}^q} + T_{ik}^p \overline{\Gamma_{jl}^q} + \Gamma_{ik}^p \overline{T_{jl}^q} = T_{ik}^p \overline{T_{jl}^q}.$$

It can also be derived as follows.

$$\begin{aligned}
v_{i\bar{j}k\bar{l}} - v_{k\bar{l}i\bar{j}} &= (v_{i\bar{j}k\bar{l}} - v_{k\bar{j}i\bar{l}}) + (v_{k\bar{j}i\bar{l}} - v_{k\bar{j}l\bar{i}}) \\
&\quad + (v_{k\bar{j}l\bar{i}} - v_{k\bar{l}j\bar{i}}) + (v_{k\bar{l}j\bar{i}} - v_{k\bar{l}i\bar{j}}) \\
&= \nabla_{\bar{l}} (T_{ik}^p v_{p\bar{j}}) + g^{p\bar{q}} R_{i\bar{l}k\bar{q}} v_{p\bar{j}} - g^{p\bar{q}} R_{i\bar{l}p\bar{j}} v_{k\bar{q}} \\
&\quad + \nabla_i (\overline{\Gamma_{jl}^q} v_{k\bar{q}}) - g^{p\bar{q}} R_{i\bar{j}k\bar{q}} v_{p\bar{l}} + g^{p\bar{q}} R_{i\bar{j}p\bar{l}} v_{k\bar{q}} \\
&= \nabla_{\bar{l}} T_{ik}^p v_{p\bar{j}} + T_{ik}^p v_{p\bar{j}\bar{l}} + g^{p\bar{q}} R_{i\bar{l}k\bar{q}} v_{p\bar{j}} - g^{p\bar{q}} R_{i\bar{l}p\bar{j}} v_{k\bar{q}} \\
&\quad + \nabla_i \overline{\Gamma_{jl}^q} v_{k\bar{q}} + \overline{T_{jl}^q} v_{k\bar{q}i} - g^{p\bar{q}} R_{i\bar{j}k\bar{q}} v_{p\bar{l}} + g^{p\bar{q}} R_{i\bar{j}p\bar{l}} v_{k\bar{q}} \\
&= g^{p\bar{q}} (R_{k\bar{l}i\bar{q}} v_{p\bar{j}} - R_{i\bar{j}k\bar{q}} v_{p\bar{l}}) + T_{ik}^p v_{p\bar{j}\bar{l}} + \overline{T_{jl}^q} v_{k\bar{q}i} \\
&= g^{p\bar{q}} (R_{k\bar{l}i\bar{q}} v_{p\bar{j}} - R_{i\bar{j}k\bar{q}} v_{p\bar{l}}) + T_{ik}^p v_{p\bar{j}\bar{l}} + \overline{T_{jl}^q} v_{i\bar{q}k} - T_{ik}^p \overline{T_{jl}^q} v_{p\bar{q}}.
\end{aligned} \tag{2.6}$$

Let $u \in C^4(M)$ be a solution of equation (1.1). As in [7], we denote $\mathbf{g}_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$, $\{\mathbf{g}^{i\bar{j}}\} = \{\mathbf{g}_{i\bar{j}}\}^{-1}$ and let $W = \text{tr}\chi + \Delta u$. Assume that $g_{i\bar{j}} = \delta_{ij}$ and $\mathbf{g}_{i\bar{j}}$ is diagonal at a fixed point $p \in M$. Then

$$u_{i\bar{i}k\bar{k}} - u_{k\bar{k}i\bar{i}} = R_{k\bar{k}i\bar{p}} u_{p\bar{i}} - R_{i\bar{i}k\bar{p}} u_{p\bar{k}} + 2\Re\{\overline{T_{ik}^j} u_{i\bar{j}k}\} - T_{ik}^p \overline{T_{ik}^q} u_{p\bar{q}}, \tag{2.7}$$

and therefore,

$$\mathbf{g}_{i\bar{i}k\bar{k}} - \mathbf{g}_{k\bar{k}i\bar{i}} = R_{k\bar{k}i\bar{i}} \mathbf{g}_{i\bar{i}} - R_{i\bar{i}k\bar{k}} \mathbf{g}_{k\bar{k}} + 2\Re\{\overline{T_{ik}^j} \mathbf{g}_{i\bar{j}k}\} - |T_{ik}^j|^2 \mathbf{g}_{j\bar{j}} - G_{i\bar{i}k\bar{k}} \tag{2.8}$$

where

$$G_{i\bar{i}k\bar{k}} = \chi_{k\bar{k}i\bar{i}} - \chi_{i\bar{i}k\bar{k}} + R_{k\bar{k}i\bar{p}} \chi_{p\bar{i}} - R_{i\bar{i}k\bar{p}} \chi_{p\bar{k}} + 2\Re\{\overline{T_{ik}^j} \chi_{i\bar{j}k}\} - T_{ik}^p \overline{T_{ik}^q} \chi_{p\bar{q}}. \tag{2.9}$$

In local coordinates, equation (1.1) can be written in the form

$$\mathbf{g}^{i\bar{j}} g_{i\bar{j}} = \frac{n}{\psi}. \tag{2.10}$$

Differentiating this equation twice gives at p

$$F^{i\bar{i}}(u_{i\bar{i}k} u_{k\bar{i}} + u_k u_{i\bar{i}\bar{k}}) = 2\Re\{f_k u_{k\bar{i}} - F^{i\bar{i}} \chi_{i\bar{i}k} u_{k\bar{i}}\}. \tag{2.11}$$

$$F^{i\bar{i}} \mathbf{g}_{i\bar{i}k\bar{k}} - (F^{i\bar{i}} \mathbf{g}^{j\bar{j}} + F^{j\bar{j}} \mathbf{g}^{i\bar{i}}) \mathbf{g}_{i\bar{j}k} \mathbf{g}_{j\bar{i}\bar{k}} = f_{k\bar{k}} \tag{2.12}$$

where $F^{i\bar{i}} = (\mathfrak{g}^{i\bar{i}})^2$ and $f = -n\psi^{-1}$. Note that $\sum F^{i\bar{i}} \leq (\sum \mathfrak{g}^{i\bar{i}})^2 \leq C$; we shall use this fact without further reference. By (2.8) and (2.12) we have

$$F^{i\bar{i}}W_{i\bar{i}} = F^{i\bar{i}}\mathfrak{g}_{j\bar{j}i\bar{i}} \geq \sum_k F^{i\bar{i}}\mathfrak{g}^{j\bar{j}}(|\mathfrak{g}_{i\bar{j}k} - T_{ik}^j\mathfrak{g}_{j\bar{j}}|^2 + |\mathfrak{g}_{i\bar{j}k}|^2) - CW. \quad (2.13)$$

Finally, we note that since $\underline{u} \in C^2(M)$ and $\chi_{\underline{u}} > 0$ on M ,

$$\epsilon\omega \leq \chi_{\underline{u}} \leq \epsilon^{-1}\omega \quad (2.14)$$

for some $\epsilon > 0$. Consequently,

$$\sum \mathfrak{g}^{i\bar{i}}(\chi_{i\bar{i}} + \underline{u}_{i\bar{i}}) \geq \epsilon \sum \mathfrak{g}^{i\bar{i}}. \quad (2.15)$$

Let $\lambda_1(\chi_{\underline{u}}), \dots, \lambda_n(\chi_{\underline{u}})$ denote the eigenvalues of $\{\chi_{i\bar{j}} + \underline{u}_{i\bar{j}}\}$. Then (1.3) is equivalent to

$$\sum \frac{1}{\lambda_i(\chi_{\underline{u}})} \leq \frac{n}{\psi} \quad (2.16)$$

while (1.5) is equivalent to

$$\sum_{i \neq k} \frac{1}{\lambda_i(\chi_{\underline{u}})} < \frac{n}{\psi} \text{ for each } k = 1, \dots, n. \quad (2.17)$$

It is clear that at a point where $g_{i\bar{j}} = \delta_{ij}$ in local coordinates,

$$\sum \frac{1}{\chi_{i\bar{i}} + \underline{u}_{i\bar{i}}} \leq \sum \frac{1}{\lambda_i(\chi_{\underline{u}})} \leq \frac{n}{\psi}. \quad (2.18)$$

We conclude this section with the following inequality which will play a crucial role in both the gradient and second order estimates in the sections below.

Lemma 2.1. *There exist $\theta > 0$ and $N \geq n$ depending on ϵ such that if $W \geq N$ then*

$$F^{i\bar{j}}(\chi_{i\bar{j}} + \underline{u}_{i\bar{j}}) \geq \frac{n + \theta}{\psi}. \quad (2.19)$$

Proof. We may assume $g_{i\bar{j}} = \delta_{ij}$ and $\{\mathfrak{g}_{i\bar{j}}\}$ is diagonal. Suppose that $\mathfrak{g}_{1\bar{1}} \geq \dots \geq \mathfrak{g}_{n\bar{n}}$. By Schwarz inequality, (2.10), (2.18) and (2.14) we have

$$\begin{aligned} \sum_{i \geq 2} (\mathfrak{g}^{i\bar{i}})^2 (\chi_{i\bar{i}} + \underline{u}_{i\bar{i}}) &\geq \left(\sum_{i \geq 2} \mathfrak{g}^{i\bar{i}} \right)^2 / \sum_{i \geq 2} \frac{1}{\chi_{i\bar{i}} + \underline{u}_{i\bar{i}}} \\ &\geq \left(\frac{n}{\psi} - \mathfrak{g}^{1\bar{1}} \right)^2 / \left(\frac{n}{\psi} - \frac{1}{\chi_{1\bar{1}} + \underline{u}_{1\bar{1}}} \right) \\ &\geq \frac{(n - \psi \mathfrak{g}^{1\bar{1}})^2}{n\psi} \left(1 + \frac{\psi}{n(\chi_{1\bar{1}} + \underline{u}_{1\bar{1}})} \right) \\ &\geq \frac{(n - \psi \mathfrak{g}^{1\bar{1}})^2}{n\psi} \left(1 + \frac{\epsilon\psi}{n} \right) \\ &\geq \frac{n + \theta}{\psi} \end{aligned} \quad (2.20)$$

provided $\mathfrak{g}_{1\bar{1}}$ is sufficiently large. \square

Remark 1. One can replace assumption (1.3) by (1.5) in Lemma 2.1. This is clear from the proof.

3. Gradient estimates. Let $u \in C^4(M)$ be a solution of equation (1.1). The primary goal of this section is to establish the *a priori* gradient estimates.

Proposition 1. *There exists a uniform constant $C > 0$ such that*

$$\max_M |\nabla u| \leq C(1 + \max_{\partial M} |\nabla u|). \quad (3.1)$$

Proof. Let $\phi = Ae^\eta$ where $\eta = \underline{u} - u$ and A is a positive constant to be determined later. Suppose the function $e^\phi |\nabla u|^2$ attains its maximum at an interior point $p \in M$. We choose local coordinate around p such that $g_{i\bar{j}} = \delta_{ij}$ and $\mathbf{g}_{i\bar{j}}$ is diagonal at p where, unless otherwise indicated, the computations below are evaluated.

For each $i = 1, \dots, n$, we have

$$\frac{(|\nabla u|^2)_i}{|\nabla u|^2} + \phi_i = 0, \quad \frac{(|\nabla u|^2)_{\bar{i}}}{|\nabla u|^2} + \phi_{\bar{i}} = 0 \quad (3.2)$$

and

$$\frac{(|\nabla u|^2)_{i\bar{i}}}{|\nabla u|^2} - \frac{|(|\nabla u|^2)_i|^2}{|\nabla u|^4} + \phi_{i\bar{i}} \leq 0. \quad (3.3)$$

A straightforward calculation shows that

$$(|\nabla u|^2)_i = u_k u_{i\bar{k}} + u_{ki} u_{\bar{k}}, \quad (3.4)$$

$$\begin{aligned} (|\nabla u|^2)_{i\bar{i}} &= u_{k\bar{i}} u_{i\bar{k}} + u_{ki} u_{\bar{k}\bar{i}} + u_{k\bar{i}\bar{i}} u_{\bar{k}} + u_k u_{i\bar{k}\bar{i}} \\ &= u_{ki} u_{\bar{k}\bar{i}} + u_{i\bar{i}k} u_{\bar{k}} + u_{i\bar{i}\bar{k}} u_k + R_{i\bar{i}k\bar{l}} u_l u_{\bar{k}} \\ &\quad + \sum_k |u_{i\bar{k}} - T_{i\bar{l}}^k u_{\bar{l}}|^2 - \sum_k |T_{i\bar{l}}^k u_{\bar{l}}|^2. \end{aligned} \quad (3.5)$$

It follows that

$$F^{i\bar{i}}(|\nabla u|^2)_{i\bar{i}} \geq F^{i\bar{i}} u_{ki} u_{\bar{k}\bar{i}} + \sum_k F^{i\bar{i}} |u_{i\bar{k}} - T_{i\bar{l}}^k u_{\bar{l}}|^2 - C(1 + |\nabla u|^2). \quad (3.6)$$

By (3.2) and (3.4),

$$|(|\nabla u|^2)_i|^2 = |u_{\bar{k}} u_{ki}|^2 - 2|\nabla u|^2 \Re\{u_k u_{i\bar{k}} \phi_{\bar{i}}\} - |u_k u_{i\bar{k}}|^2. \quad (3.7)$$

Combining (3.3), (3.7) and (3.6), we obtain

$$|\nabla u|^2 F^{i\bar{i}} \phi_{i\bar{i}} + 2F^{i\bar{i}} \Re\{u_k u_{i\bar{k}} \phi_{\bar{i}}\} \leq C(1 + |\nabla u|^2). \quad (3.8)$$

Now,

$$\phi_i = \phi \eta_i, \quad \phi_{i\bar{i}} = \phi(\eta_i \eta_{\bar{i}} + \eta_{i\bar{i}}).$$

We have

$$\begin{aligned} 2\phi^{-1} F^{i\bar{i}} \Re\{u_k u_{i\bar{k}} \phi_{\bar{i}}\} &= 2\mathbf{g}^{i\bar{i}} \Re\{u_i \eta_{\bar{i}}\} - 2F^{i\bar{i}} \Re\{\chi_{i\bar{k}} u_k \phi_{\bar{i}}\} \\ &\geq -\frac{1}{2} |\nabla u|^2 F^{i\bar{i}} \eta_i \eta_{\bar{i}} - C \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} \phi^{-1} F^{i\bar{i}} \phi_{i\bar{i}} &= F^{i\bar{i}} \eta_i \eta_{\bar{i}} + F^{i\bar{i}} \eta_{i\bar{i}} \\ &\geq F^{i\bar{i}} \eta_i \eta_{\bar{i}} + F^{i\bar{i}} (\underline{u}_{i\bar{i}} + \chi_{i\bar{i}}) - \frac{n}{\psi}. \end{aligned} \quad (3.10)$$

Therefore, by (3.8),

$$\frac{1}{2} F^{i\bar{i}} \eta_i \eta_{\bar{i}} + F^{i\bar{i}} (\underline{u}_{i\bar{i}} + \chi_{i\bar{i}}) - \frac{n}{\psi} \leq C(\phi^{-1} + |\nabla u|^{-2}). \quad (3.11)$$

We consider two cases separately: (a) $W > N$ for some N sufficiently large, and (b) $W \leq N$.

In case (a) we have

$$F^{i\bar{i}}(\underline{u}_{i\bar{i}} + \chi_{i\bar{i}}) - \frac{n}{\psi} \geq \frac{\theta n}{\psi},$$

by Lemma 2.1 for some $\theta > 0$. Therefore from (3.11) we obtain a bound $|\nabla u| \leq C$ when A is chosen sufficiently large.

Suppose now that $W \leq N$. Then, using equation (2.10),

$$\begin{aligned} F^{i\bar{i}}\eta_i\eta_{\bar{i}} + F^{i\bar{i}}(\underline{u}_{i\bar{i}} + \chi_{i\bar{i}}) &\geq |\nabla\eta|^2 \min_i F^{i\bar{i}} + \epsilon \sum F^{i\bar{i}} \\ &\geq n|\nabla\eta|^{\frac{2}{n}} \epsilon^{\frac{n-1}{n}} (\det \mathbf{g}^{i\bar{j}})^{\frac{2}{n}} \\ &\geq \epsilon^{\frac{n-1}{n}} |\nabla\eta|^{\frac{2}{n}} \left(\frac{\psi}{W^{n-1}} \right)^{\frac{2}{n}} \\ &\geq c_0 |\nabla\eta|^{\frac{2}{n}}. \end{aligned} \quad (3.12)$$

Plugging this back in (3.11) we derive a bound $|\nabla\eta| \leq C$ which in turn implies a bound $|\nabla u| \leq C$. \square

4. The second order estimates. In this section we derive second order estimates for solutions of equation (1.1).

Proposition 2. *Let $u \in C^4(M)$ be a solution of equation (1.1). There exists a constant $C > 0$ depending on ϵ , $\sup \psi^{-1}$, $\sup_M u - \inf_M u$, the C^2 norms of χ , \underline{u} and ψ , and the geometric quantities of M , such that*

$$\max_M |\Delta u| \leq C(1 + \max_{\partial M} |\Delta u|).$$

Proof. Consider $\Phi = e^\phi W$, where as aforementioned $W = \text{tr}\chi + \Delta u$ and ϕ is a function to be determined. Suppose Φ achieves its maximum at a point $p \in M$. Choose local coordinates such that $g_{i\bar{j}} = \delta_{ij}$ and $\mathbf{g}_{i\bar{j}}$ is diagonal at p . We have (all calculations below are done at p)

$$\frac{W_i}{W} + \phi_i = 0, \quad \frac{W_{\bar{i}}}{W} + \phi_{\bar{i}} = 0, \quad (4.1)$$

$$\frac{W_{i\bar{i}}}{W} - \frac{|W_i|^2}{W^2} + \phi_{i\bar{i}} \leq 0. \quad (4.2)$$

By (2.3) and (4.1),

$$\begin{aligned} |W_i|^2 &= \left| \sum_j \mathbf{g}_{j\bar{j}i} \right|^2 = \left| \sum_j (\mathbf{g}_{i\bar{j}j} - T_{ij}^j \mathbf{g}_{j\bar{j}}) + \lambda_i \right|^2 \\ &\leq \left| \sum_j (\mathbf{g}_{i\bar{j}j} - T_{ij}^j \mathbf{g}_{j\bar{j}}) \right|^2 + 2 \sum_j (\Re\{(\mathbf{g}_{i\bar{j}j} - T_{ij}^j \mathbf{g}_{j\bar{j}}) \bar{\lambda}_i\}) + |\lambda_i|^2 \\ &= \left| \sum_j (\mathbf{g}_{i\bar{j}j} - T_{ij}^j \mathbf{g}_{j\bar{j}}) \right|^2 - 2W \Re\{\phi_i \bar{\lambda}_i\} - 2|\lambda_i|^2 \end{aligned} \quad (4.3)$$

where

$$\lambda_i = \sum_j (\chi_{j\bar{j}i} - \chi_{i\bar{j}j} + T_{ij}^l \chi_{l\bar{j}}).$$

By Schwarz inequality,

$$\left| \sum_j (\mathfrak{g}_{i\bar{j}j} - T_{ij}^j \mathfrak{g}_{j\bar{j}}) \right|^2 \leq W \sum_j \mathfrak{g}^{j\bar{j}} |\mathfrak{g}_{i\bar{j}j} - T_{ij}^j \mathfrak{g}_{j\bar{j}}|^2. \quad (4.4)$$

It therefore follows from (4.2), (4.3), (4.4) and (2.13) that

$$0 \geq F^{i\bar{i}} W_{i\bar{i}} - F^{i\bar{i}} \frac{|W_i|^2}{W} + W F^{i\bar{i}} \phi_{i\bar{i}} \geq W F^{i\bar{i}} \phi_{i\bar{i}} + 2F^{i\bar{i}} \Re\{\phi_i \bar{\lambda}_i\} - CW. \quad (4.5)$$

Let $\phi = e^{A\eta}$ where $\eta = \underline{u} - u + \sup_M(u - \underline{u})$ and A is a positive constant. We see that $\phi_i = A\phi\eta_i$ and $\phi_{i\bar{i}} = A\phi\eta_{i\bar{i}} + A^2\phi\eta_i\eta_{\bar{i}}$. By Schwarz inequality,

$$2AF^{i\bar{i}} \Re\{\eta_i \bar{\lambda}_i\} \geq -A^2 F^{i\bar{i}} |\eta_i|^2 - C. \quad (4.6)$$

By Lemma 2.1 we see that

$$\begin{aligned} \phi^{-1} F^{i\bar{i}} \phi_{i\bar{i}} &= AF^{i\bar{i}} \eta_{i\bar{i}} + A^2 F^{i\bar{i}} \eta_i \eta_{\bar{i}} \\ &= AF^{i\bar{i}} (\chi_{i\bar{i}} + \underline{u}_{i\bar{i}}) - \frac{nA}{\psi} + A^2 F^{i\bar{i}} \eta_i \eta_{\bar{i}} \\ &\geq \frac{\theta A}{\psi} + A^2 F^{i\bar{i}} \eta_i \eta_{\bar{i}} \end{aligned} \quad (4.7)$$

provided W is sufficiently large. This combined with (4.6) and (4.5) gives

$$(\theta A - C\psi)W \leq C.$$

Choosing A large enough we derive a bound $W \leq C$. \square

By Proposition 2 equation (1.1) is uniformly elliptic for solutions u with $\chi_u > 0$. Therefore one can apply Evans-Krylov Theorem and the Schauder theory to derive higher order estimates.

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