Algebraic Topology (topics course) **Spring 2010**
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Series 1

Exercise 1. Prove Proposition 1.

The following proposition is often the easiest way to establish an isomorphism between a space A and the limit of a diagram; it verifies that limits are unique up to isomorphism. Denote by Top the category of topological spaces and continuous functions.

Proposition 1. *Let $X\colon \mathsf{D}\longrightarrow\mathsf{Top}$ be a diagram such that $\lim_{\mathsf{D}} X$ exists. If A is a space, then $A\cong \lim_{\mathsf{D}} X$ if and only if there exists a collection $\{f_d\}$ of maps*

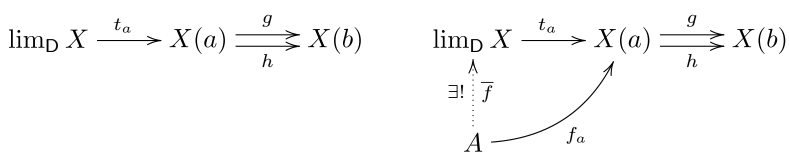
$$f_d\colon A\longrightarrow X(d),\qquad d\in\mathsf{D},$$

indexed on the objects $d\in\mathsf{D}$, such that $\{f_d\}$ is a cone into X which is terminal with respect to all cones into X.

Exercise 2. Let $\mathsf{D}=\{a\rightrightarrows b\}$. Then a diagram $X\colon \mathsf{D}\longrightarrow\mathsf{Top}$ has the form

$$X(a)\overset{g}{\underset{h}{\rightrightarrows}} X(b)$$

and the limit $\lim_{\mathsf{D}} X$ is a space with the following mapping properties: (1) there is a map t_a



such that $gt_a=ht_a$, (2) (universal property) for any space A and map f_a such that $gf_a=hf_a$, there exists a unique map $\overline{f}$ which makes the diagram commute. Prove that $\lim_{\mathsf{D}} X$ is isomorphic to the following subspace

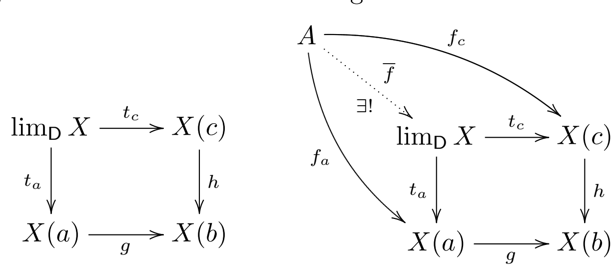
$$\lim_{\mathsf{D}} X\cong\{x\mid x\in X(a),\, g(x)=h(x)\}$$

of $X(a)$. In this case, the limit $\lim_{\mathsf{D}} X$ is called the *equalizer* of the pair of maps g,h.

Exercise 3. Let $\mathsf{D}=\{a\rightarrow b\leftarrow c\}$. Then a diagram $X\colon \mathsf{D}\longrightarrow\mathsf{Top}$ has the form

$$X(a)\xrightarrow{g} X(b)\xleftarrow{h} X(c) \qquad (1)$$

and the limit $\lim_{\mathsf{D}} X$ is a space with the following mapping properties: (1) there are maps t_a, t_c which make the left-hand diagram



commute, (2) (universal property) for any space A and maps f_a, f_c which make the right-hand outer diagram commute, there exists a unique map $\overline{f}$ which makes the diagram commute. Prove that $\lim_{\mathsf{D}} X$ is isomorphic to the following subspace

$$\lim_{\mathsf{D}} X \cong \{(x,y) \mid x \in X(a),\, y \in X(c),\, g(x) = h(y)\}$$

of the product $X(a) \times X(c)$. In this case, the limit $\lim_{\mathsf{D}} X$ is called the *pullback* of the diagram (1) and is usually denoted by $X(a) \times_{X(b)} X(c)$.

Exercise 4. Let D be the empty category. Then there is a unique diagram $X\colon \mathsf{D} \longrightarrow \mathsf{Top}$ (the empty diagram). Prove that $\lim_{\mathsf{D}} X \cong *$. Here, $*$ denotes a one point space.

A category D is *small* if its collection of objects forms a set, and *finite* if (1) its collection of objects forms a finite set and (2) D has only a finite number of morphisms between any two objects. A diagram $X\colon \mathsf{D} \longrightarrow \mathsf{C}$ is *small* (resp. *finite*) if the indexing category D is small (resp. finite), and a category C *has all small* (resp. *finite*) *limits* if $\lim_{\mathsf{D}} X$ exists for each small (resp. finite) diagram $X\colon \mathsf{D} \longrightarrow \mathsf{C}$.

Exercise 5. Prove that the category of topological spaces has all small limits. In other words, if $X\colon \mathsf{D} \longrightarrow \mathsf{Top}$ is a small diagram, prove that the limit $\lim_{\mathsf{D}} X$ exists.

Exercise 6. Prove Proposition 2.

The following proposition is often the easiest way to verify that a pair of maps into a limit are the same.

Proposition 2. *Let $X\colon \mathsf{D} \longrightarrow \mathsf{C}$ be a diagram such that $\lim_{\mathsf{D}} X$ exists and let the collection $\{t_d\}$ of maps*

$$t_d\colon \lim_{\mathsf{D}} X \longrightarrow X(d), \qquad d \in \mathsf{D},$$

indexed on the objects $d \in \mathsf{D}$, be the terminal cone into X. Consider any pair of maps $f, g\colon A \longrightarrow \lim_{\mathsf{D}} X$. Then f and g are the same if and only if their corresponding cones into X are identical; in other words, $f = g$ if and only if $t_d f = t_d g$ for every object $d \in \mathsf{D}$.

Remark 3. For instance, consider any diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[j]{} & \lim_{\mathsf{D}} X \end{array}$$

in C. Then the diagram commutes if and only if $t_d j i = t_d g f$ for every object $d \in \mathsf{D}$.

Exercise 7. Prove Proposition 4 below. The idea is to reformulate your construction of $\lim_{\mathsf{D}} X$ in Exercise 5 as the equalizer of a pair of maps.

Proposition 4. *Let C be a category with all equalizers and small (resp. finite) products. If $X\colon \mathsf{D} \longrightarrow \mathsf{C}$ is a small (resp. finite) diagram, then the limit $\lim_{\mathsf{D}} X$ exists and is isomorphic to an equalizer of the form*

$$\lim_{\mathsf{D}} X \cong \lim\Bigg(\prod_{d \in \mathsf{D}} X(d) \rightrightarrows \prod_{(\alpha\colon d \to d') \in \mathsf{D}} X(d') \Bigg)$$

in C. In particular, the category C has all small (resp. finite) limits.

An object $*$ of a category C is a *terminal object* if for each object $X \in \mathsf{C}$ there exists a unique map $X \longrightarrow *$. It follows that the limit of the empty diagram in C, if it exists, is a terminal object of C. Since the empty category is discrete, if C has all small (resp. finite) products, then C has a terminal object $*$.

Let C be a category with all pullbacks. Given a commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ B & \xrightarrow[h]{} & D \end{array} \tag{2}$$

in C, the induced map $A \longrightarrow B \times_D C$ is often called the *pullback corner map*; if it is an isomorphism, then (2) is called a *pullback diagram* and i is called the *pullback* (or base change) of j along h. It follows that any diagram isomorphic to a pullback diagram is a pullback diagram.

Exercise 8. (a) Prove Proposition 5. (b) Prove Proposition 6.

Proposition 5. *Let C be a category with all pullbacks and finite products. Let $g, h\colon X \longrightarrow Y$ be a pair of maps in C. Consider any pullback diagram of the form*

$$\begin{array}{ccc} E & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id},h)} \\ X & \xrightarrow[(\mathrm{id},g)]{} & X \times Y \end{array}$$

in C. Then the equalizer of the pair g, h exists and is isomorphic to E.

Proposition 6. *Let C be a category with all pullbacks and small (resp. finite) products. Then C has all small (resp. finite) limits.*

Exercise 9. Let $\mathsf{D} = \{a \leftleftarrows b\}$. Then a diagram $X\colon \mathsf{D} \longrightarrow \mathsf{Top}$ has the form

$$X(a) \overset{g}{\underset{h}{\leftleftarrows}} X(b)$$

and the colimit $\operatorname{colim}_{\mathsf{D}} X$ is a space with the following mapping properties: (1) there is a map i_a

$$\operatorname{colim}_{\mathsf{D}} X \xleftarrow{i_a} X(a) \overset{g}{\underset{h}{\leftleftarrows}} X(b) \qquad\qquad \begin{array}{ccccc} \operatorname{colim}_{\mathsf{D}} X & \xleftarrow{i_a} & X(a) & \overset{g}{\underset{h}{\leftleftarrows}} & X(b) \\ {\scriptstyle \exists!}\ \vdots\ {\scriptstyle \overline{f}} & \swarrow{\scriptstyle f_a} & & & \\ A & & & & \end{array}$$

such that $i_a g = i_a h$, (2) (universal property) for any space A and map f_a such that $f_a g = f_a h$, there exists a unique map $\overline{f}$ which makes the diagram commute. Prove that $\operatorname{colim}_{\mathsf{D}} X$ is isomorphic to the quotient space

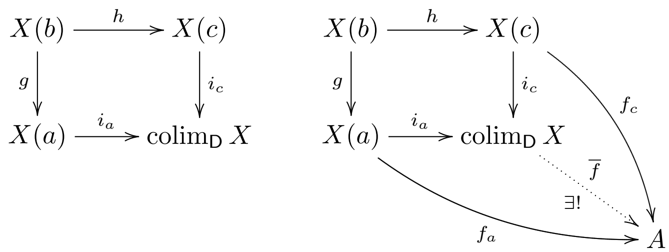
$$\operatorname{colim}_{\mathsf{D}} X \cong X(a)/\sim$$

of $X(a)$ with respect to the equivalence relation $\sim$ generated by $g(x) \sim h(x)$, $x \in X(b)$. In this case, the colimit $\operatorname{colim}_{\mathsf{D}} X$ is called the *coequalizer* of the pair of maps g, h.

Exercise 10. Let $\mathsf{D} = \{a \leftarrow b \rightarrow c\}$. Then a diagram $X\colon \mathsf{D} \longrightarrow \mathsf{Top}$ has the form

$$X(a) \xleftarrow{g} X(b) \xrightarrow{h} X(c) \tag{3}$$

and $\mathrm{colim}_{\mathsf{D}}\, X$ is a space with the following mapping properties: (1) there are maps i_a, i_c which make the left-hand diagram



commute, (2) (universal property) for any space A and maps f_a, f_c which make the right-hand outer diagram commute, there exists a unique map $\overline{f}$ which makes the diagram commute. Prove that $\mathrm{colim}_{\mathsf{D}}\, X$ is isomorphic to the quotient space

$$\mathrm{colim}_{\mathsf{D}}\, X \cong X(a) \amalg X(c)/\sim$$

of the disjoint union $X(a) \amalg X(c)$ with respect to the equivalence relation $\sim$ generated by $g(x) \sim h(x)$, $x \in X(b)$. In this case, the colimit $\mathrm{colim}_{\mathsf{D}}\, X$ is called the *pushout* of the diagram (3) and is usually denoted by $X(a) \amalg_{X(b)} X(c)$.

Exercise 11. Let D be the empty category. Then there is a unique diagram $X\colon \mathsf{D} \longrightarrow \mathsf{Top}$ (the empty diagram). Prove that $\mathrm{colim}_{\mathsf{D}}\, X \cong \emptyset$. Here, $\emptyset$ denotes the empty space.

A category C *has all small* (resp. *finite*) *colimits* if $\mathrm{colim}_{\mathsf{D}}\, X$ exists for each small (resp. finite) diagram $X\colon \mathsf{D} \longrightarrow \mathsf{C}$. Recall from lecture the following proposition.

Proposition 7. *Let C be a category with all coequalizers and small (resp. finite) coproducts. If $X\colon \mathsf{D} \longrightarrow \mathsf{C}$ is a small (resp. finite) diagram, then the colimit $\mathrm{colim}_{\mathsf{D}}\, X$ exists and is isomorphic to a coequalizer of the form*

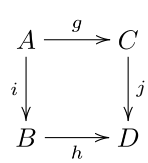
$$\mathrm{colim}_{\mathsf{D}}\, X \cong \mathrm{colim}\Bigg(\coprod_{d \in \mathsf{D}} X(d) \leftleftarrows \coprod_{(\alpha\colon d \to d') \in \mathsf{D}} X(d) \Bigg)$$

in C. In particular, the category C has all small (resp. finite) colimits.

An object $\emptyset$ of a category C is an *initial object* if for each object $X \in \mathsf{C}$ there exists a unique map $\emptyset \longrightarrow X$. It follows that the colimit of the empty diagram in C, if it exists, is an initial object of C. Since the empty category is discrete, if C has all small (resp. finite) coproducts, then C has an initial object $\emptyset$.

Let C be a category with all pushouts. Given a commutative diagram of the form

(4)



in C, the induced map $B \amalg_A C \longrightarrow D$ is often called the *pushout corner map*; if this map is an isomorphism, then (4) is called a *pushout diagram* and j is called the *pushout* (or cobase change) of i along g. It follows that any diagram isomorphic to a pushout diagram is a pushout diagram.

Exercise 12. (a) Prove Proposition 8. (b) Prove Proposition 9.

Proposition 8. *Let* C *be a category with all pushouts and finite coproducts. Let* $g, h\colon Y \longrightarrow X$ *be a pair of maps in* C*. Consider any pushout diagram of the form*

$$\begin{array}{ccc} X \amalg Y & \xrightarrow{(\mathrm{id},h)} & X \\ {\scriptstyle (\mathrm{id},g)}\downarrow & & \downarrow \\ X & \longrightarrow & C \end{array}$$

in C*. Then the coequalizer of the pair* g, h *exists and is isomorphic to* C*.*

Proposition 9. *Let* C *be a category with all pushouts and small (resp. finite) coproducts. Then* C *has all small (resp. finite) colimits.*

Exercise 13. Prove Proposition 10.

Proposition 10. *Let* $F\colon \mathsf{C} \longrightarrow \mathsf{C}'$ *be a functor.*

(a) *If* F *preserves all equalizers and small (resp. finite) products, then* F *preserves all small (resp. finite) limits.*
(b) *If* F *preserves all pullbacks and small (resp. finite) products, then* F *preserves all small (resp. finite) limits.*
(c) *If* F *preserves all coequalizers and small (resp. finite) coproducts, then* F *preserves all small (resp. finite) colimits.*
(d) *If* F *preserves all pushouts and small (resp. finite) coproducts, then* F *preserves all small (resp. finite) colimits.*

Exercise 14. Prove Proposition 11.

Proposition 11. *Let* C *be a category with all small limits and colimits. If* A, B, C *are objects in* C *and* $X\colon \mathsf{D} \longrightarrow \mathsf{C}$ *is a small diagram, then there are natural isomorphisms of sets:*

(a) $\hom(A, B \times C) \cong \hom(A, B) \times \hom(A, C)$
(b) $\hom(A, \lim_{\mathsf{D}} X) \cong \lim_{\mathsf{D}} \hom(A, X)$
(c) $\hom(A \amalg B, C) \cong \hom(A, B) \times \hom(A, C)$
(d) $\hom(\mathrm{colim}_{\mathsf{D}} X, B) \cong \lim_{\mathsf{D}^{\mathrm{op}}} \hom(X, B)$

Exercise 15. Please read [1, Sections 1-2] and review the notions of *category*, *subcategory*, *functor*, and *natural transformation* [2, I], [3, 2.1-2.3, 2.6]; adjunctions will be introduced in lecture.

References

[1] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.
[2] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
[3] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. Available at: `http://www.math.uchicago.edu/~may/` .