Algebraic Topology (topics course) **Spring 2010**

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Series 2

Exercise 1. Prove Proposition 1.

Proposition 1. *Let* C *be a model category.*

(a) *The cofibrations in* C *are the maps which have the left lifting property with respect to acyclic fibrations.*
(b) *The acyclic cofibrations in* C *are the maps which have the left lifting property with respect to fibrations.*
(c) *The fibrations in* C *are the maps which have the right lifting property with respect to acyclic cofibrations.*
(d) *The acyclic fibrations in* C *are the maps which have the right lifting property with respect to cofibrations.*

Exercise 2. Prove Proposition 2.

Proposition 2. *Let* C *be a model category.*

(a) *The class of cofibrations in* C *is closed under pushouts.*
(b) *The class of acyclic cofibrations in* C *is closed under pushouts.*
(c) *The class of fibrations in* C *is closed under pullbacks.*
(d) *The class of acyclic fibrations in* C *is closed under pullbacks.*

If C is a category and D is a small category, denote by C^{D} the *diagram category* with objects the diagrams $X\colon \mathsf{D}\longrightarrow\mathsf{C}$ and morphisms their natural transformations. If the colimit $\mathrm{colim}_{\mathsf{D}} X$ exists for every diagram $X\in\mathsf{C}^{\mathsf{D}}$, then it follows easily—from the universal property of colimits—that the objects $\mathrm{colim}_{\mathsf{D}} X$ give a well-defined functor $\mathrm{colim}\colon \mathsf{C}^{\mathsf{D}}\longrightarrow\mathsf{C}$.

Exercise 3. Prove Proposition 3.

Let C be a category and let $\mathsf{D}=\{a\leftarrow b\rightarrow c\}$. Then a morphism $f\colon X\longrightarrow Y$ in C^{D} is a collection of maps f_a, f_b, f_c which makes the left-hand diagram

(1)
$$\begin{array}{ccccc} X_a & \longleftarrow & X_b & \longrightarrow & X_c \\ \downarrow{\scriptstyle f_a} & & \downarrow{\scriptstyle f_b} & & \downarrow{\scriptstyle f_c} \\ Y_a & \longleftarrow & Y_b & \longrightarrow & Y_c \end{array} \qquad\qquad \begin{array}{c} \mathrm{colim}\,X \\ {\scriptstyle \mathrm{colim}\,f}\downarrow \\ \mathrm{colim}\,Y \end{array}$$

in C commute, and $\mathrm{colim}\,f$ is the induced map $X_a\amalg_{X_b}X_c\longrightarrow Y_a\amalg_{Y_b}Y_c$ on pushouts. The following proposition explores some homotopical properties of the colimit functor $\mathrm{colim}\colon \mathsf{C}^{\mathsf{D}}\longrightarrow\mathsf{C}$. We will establish additional homotopical properties after developing certain results on left (and right) homotopic maps.

Proposition 3. *Let* C *be a model category and let* $f\colon X\longrightarrow Y$ *be a map in* C^{D}.

(a) *If* f_a, f_b *are acyclic cofibrations,* f_c *is a weak equivalence, and the induced map* $Y_b\amalg_{X_b}X_c\longrightarrow Y_c$ *is a cofibration, then* $\mathrm{colim}\,f$ *is an acyclic cofibration.*
(b) *If* f_a *and the induced map* $Y_b\amalg_{X_b}X_c\longrightarrow Y_c$ *are cofibrations, then* $\mathrm{colim}\,f$ *is a cofibration.*

Exercise 4. Use duality in model categories to obtain a corresponding proposition for the limit functor $\lim\colon \mathsf{C}^{\mathsf{D}^{\mathrm{op}}}\longrightarrow\mathsf{C}$. Note that D^{op} is the category $\{a\rightarrow b\leftarrow c\}$.

Exercise 5. Prove Proposition 4.

Let C be a category and let D be the category $\{0 \to 1 \to 2 \to 3 \to \cdots\}$ with objects the non-negative integers and a single morphism $i \to j$ for each $i \leq j$. Then a morphism $f\colon X \longrightarrow Y$ in C^{D} is a collection of maps $f_0, f_1, f_2, f_3, \ldots$ which makes the left-hand diagram

$$(2) \qquad \begin{array}{ccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \cdots \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \cdots \end{array} \qquad\qquad \begin{array}{c} \operatorname{colim} X \\ {\scriptstyle \operatorname{colim} f}\downarrow \\ \operatorname{colim} Y \end{array}$$

in C commute, and $\operatorname{colim} f$ is the induced map $\operatorname{colim} X \longrightarrow \operatorname{colim} Y$ on colimits. The following proposition explores some homotopical properties of the colimit functor $\operatorname{colim}\colon \mathsf{C}^{\mathsf{D}} \longrightarrow \mathsf{C}$. We will establish additional homotopical properties after developing certain results on left (and right) homotopic maps.

Proposition 4. *Let C be a model category and let $f\colon X \longrightarrow Y$ be a map in C^{D}. Assume that C has all small colimits.*

(a) *If f_0 and each of the induced maps $Y_i \amalg_{X_i} X_{i+1} \longrightarrow Y_{i+1}$ $(i \geq 0)$ is an acyclic cofibration, then* $\operatorname{colim} f$ *is an acyclic cofibration.*
(b) *If f_0 and each of the induced maps $Y_i \amalg_{X_i} X_{i+1} \longrightarrow Y_{i+1}$ $(i \geq 0)$ is a cofibration, then* $\operatorname{colim} f$ *is a cofibration.*

Exercise 6. Use duality in model categories to obtain a corresponding proposition for the limit functor $\lim\colon \mathsf{C}^{\mathsf{D}^{\mathrm{op}}} \longrightarrow \mathsf{C}$. Note that $\mathsf{D}^{\mathrm{op}} = \{0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots\}$ is the category with objects the non-negative integers and a single morphism $i \leftarrow j$ for each $i \leq j$.

Exercise 7. Prove Proposition 5

Proposition 5. *Let C be a model category.*

(a) *The class of cofibrations contains all isomorphisms.*
(b) *The class of fibrations contains all isomorphisms.*
(c) *The class of weak equivalences contains all isomorphisms.*

Exercise 8. Prove Proposition 6.

Proposition 6. *Let C be a model category. Assume that C has all small coproducts and products.*

(a) *The class of cofibrations in C is closed under coproducts.*
(b) *The class of acyclic cofibrations in C is closed under coproducts.*
(c) *The class of fibrations in C is closed under products.*
(d) *The class of acyclic fibrations in C is closed under products.*

In the exercises and propositions which follow, we are working in a fixed model category C; references for this material include [1, Section 4] and [2, Chapter 7]. Recall the following proposition.

Proposition 7. *If A is cofibrant and $A \wedge I$ is a good cylinder object for A, then the maps $i_0, i_1\colon A \longrightarrow A \wedge I$ are acyclic cofibrations.*

Exercise 9. Use duality in model categories to obtain a corresponding proposition involving path objects.

Exercise 10. Prove Proposition 8.

Proposition 8. *. If $f \overset{l}{\sim} g\colon A \longrightarrow X$, then there exists a good left homotopy from f to g. If in addition X is fibrant, then there exists a very good left homotopy from f to g.*

Exercise 11. Use duality in model categories to obtain a corresponding proposition involving right homotopies.

Exercise 12. Prove Proposition 9.

Proposition 9. *If A is cofibrant, then $\overset{l}{\sim}$ is an equivalence relation on $\hom_{\mathsf{C}}(A, X)$.*

Exercise 13. Use duality in model categories to obtain a corresponding proposition involving right homotopy.

Exercise 14. Prove Proposition 10.

Proposition 10. *If A is cofibrant and $p\colon Y \longrightarrow X$ is an acyclic fibration, then composition with p induces a bijection:*

$$p_* : \pi^l(A, Y) \xrightarrow{\cong} \pi^l(A, X), \qquad [A \xrightarrow{f} Y] \longmapsto [A \xrightarrow{f} Y \xrightarrow{p} X].$$

Exercise 15. Use duality in model categories to obtain a corresponding proposition involving right homotopy classes of maps.

Exercise 16. Prove Proposition 11

Proposition 11. *Suppose that X is fibrant, that $f \overset{l}{\sim} g\colon\ A \longrightarrow X$, and that $h\colon\ A' \longrightarrow A$ is a map. Then $fh \overset{l}{\sim} gh$.*

Exercise 17. Use duality in model categories to obtain a corresponding proposition involving right homotopy.

Exercise 18. Prove Proposition 12.

Proposition 12. *If X is fibrant, then the composition in C induces a map:*

$$\pi^l(A', A) \times \pi^l(A, X) \longrightarrow \pi^l(A', X), \qquad ([A' \xrightarrow{h} A], [A \xrightarrow{f} X]) \longmapsto [A' \xrightarrow{h} A \xrightarrow{f} X].$$

Exercise 19. Use duality in model categories to obtain a corresponding proposition involving right homotopy classes of maps.

Exercise 20. Prove Proposition 13.

Proposition 13. *Let $f, g\colon\ A \longrightarrow X$ be maps.*

(a) *If A is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$.*
(b) *If X is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.*

Exercise 21. Prove Proposition 14

The following proposition is a key result of this section.

Proposition 14. *Suppose that $f\colon\ A \longrightarrow X$ is a map between objects A and X which are both fibrant and cofibrant. Then f is a weak equivalence if and only if f has a homotopy inverse; i.e., if and only if there exists a map $g\colon X \longrightarrow A$ such that $gf \sim \mathrm{id}$ and $fg \sim \mathrm{id}$.*

Exercise 22. Please read [1, Sections 3-4].

References

[1] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[2] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.