Algebraic Topology (topics course) **Spring 2010**

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Series 3

In the exercises and propositions below, we are working in a fixed model category C. References for the following include [1, Sections 5-6] and [2, Chapter 7].

Exercise 1. Prove Proposition 1.

Proposition 1. *Given a map $f\colon X \longrightarrow Y$ in C, there exists a map Qf which makes the diagram*

$$\begin{array}{ccc} QX & \xrightarrow{Qf} & QY \\ {\scriptstyle p_X}\downarrow\sim & & \sim\downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commute. Furthermore:

(a) *The map Qf is a weak equivalence if and only if the map f is a weak equivalence.*
(b) *If a map $\hat{f}\colon QX \longrightarrow QY$ also makes the diagram commute (i.e., $p_Y\hat{f} = fp_X$), then $\hat{f} \overset{l}{\sim} Qf$ and $\hat{f} \overset{r}{\sim} Qf$.*
(c) *If Y is fibrant and $[f'] = [f]$ in $\pi^l(X,Y)$ for some map $f'\colon X \longrightarrow Y$, then $Qf' \overset{l}{\sim} Qf$ and $Qf' \overset{r}{\sim} Qf$.*

Exercise 2. Use duality in model categories to obtain a corresponding proposition involving existence of a map Rf which makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow\sim & & \sim\downarrow{\scriptstyle i_Y} \\ RX & \xrightarrow[Rf]{} & RY \end{array}$$

commute.

Exercise 3. Prove Proposition 2.

Proposition 2. *The restriction of the functor $Q\colon \mathsf{C} \longrightarrow \pi\mathsf{C}_c$ to C_f induces a functor $Q'\colon \pi\mathsf{C}_f \longrightarrow \pi\mathsf{C}_{cf}$. The restriction of the functor $R\colon \mathsf{C} \longrightarrow \pi\mathsf{C}_f$ to C_c induces a functor $R'\colon \pi\mathsf{C}_c \longrightarrow \pi\mathsf{C}_{cf}$.*

Exercise 4. Prove Proposition 3.

Proposition 3. *If $F, G\colon \mathsf{Ho}(\mathsf{C}) \longrightarrow \mathsf{D}$ is a pair of functors and $t\colon F\gamma \longrightarrow G\gamma$ is a natural transformation, then t also gives a natural transformation from F to G.*

Exercise 5. Prove Proposition 4.

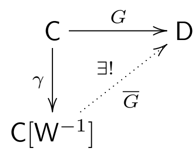
Proposition 4. *Let C be a model category and $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ a functor which sends weak equivalences in C to isomorphisms in D. If $f \overset{l}{\sim} g\colon A \longrightarrow X$ or $f \overset{r}{\sim} g\colon A \longrightarrow X$, then $F(f) = F(g)$ in D.*

Exercise 6. Prove Proposition 5.

Proposition 5. *Suppose that A is a cofibrant object of C and X is a fibrant object of C. Then the map $\gamma\colon \hom_{\mathsf{C}}(A,X) \longrightarrow \hom_{\mathsf{Ho(C)}}(A,X)$ is surjective, and induces a bijection $\gamma\colon \pi(A,X) \xrightarrow{\cong} \hom_{\mathsf{Ho(C)}}(A,X)$.*

Exercise 7. Prove Theorem 7.

Definition 6. Let C be a category and $\mathsf{W} \subset \mathsf{C}$ a class of morphisms. A *localization* of C with respect to W is a category $\mathsf{C}[\mathsf{W}^{-1}]$ with the following mapping properties: (1) there is a functor $\gamma\colon \mathsf{C} \longrightarrow \mathsf{C}[\mathsf{W}^{-1}]$ such that $\gamma(f)$ is an isomorphism for each $f \in \mathsf{W}$, (2) (universal property) the functor γ is initial with respect to all such functors; i.e., for any category D and functor $G\colon \mathsf{C} \longrightarrow \mathsf{D}$ such that $G(f)$ is an isomorphism for each $f \in \mathsf{W}$, then there exists a unique functor $\overline{G}$ which makes the diagram



commute.

Theorem 7. *Let C be a model category and $\mathsf{W} \subset \mathsf{C}$ the class of weak equivalences. Then the functor $\gamma\colon \mathsf{C} \longrightarrow \mathsf{Ho(C)}$ is a localization of C with respect to W.*

References for the following include [1, Section 9] and [2, Chapter 7].

Exercise 8. Prove Proposition 8.

Proposition 8. *Let C be a model category and $F\colon \mathsf{C}_c \longrightarrow \mathsf{D}$ a functor such that $F(f)$ is an isomorphism whenever f is an acyclic cofibration between objects of C_c. Suppose that $f, g\colon A \longrightarrow B$ are maps in C_c such that f is right homotopic to g in C. Then $F(f) = F(g)$.*

Exercise 9. Use duality in model categories to obtain a corresponding proposition involving acyclic fibrations between objects of C_f.

Exercise 10. Prove Proposition 9

Proposition 9 (K. Brown's lemma)**.** *Let $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ be a functor between model categories. If F sends acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects.*

Exercise 11. Use duality in model categories to obtain a corresponding proposition involving acyclic fibrations between fibrant objects.

Exercise 12. Please read [1, Sections 5-6].

References

[1] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[2] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.