Algebraic Topology (topics course) **Spring 2010**
John E. Harper

Series 4

Exercise 1. Prove Proposition 1.

Proposition 1. *Let* $\{j_\alpha\colon A_\alpha \longrightarrow B_\alpha\}$ *be a collection of maps in* Top *and consider the induced map* $\amalg j_\alpha\colon \amalg A_\alpha \longrightarrow \amalg B_\alpha$.

(a) *If each map* j_α *is injective, then the induced map* $\amalg j_\alpha$ *is injective.*
(b) *If each map* j_α *is closed injective, then the induced map* $\amalg j_\alpha$ *is closed injective.*
(c) *If each map* j_α *is closed injective with image* $j_\alpha(A_\alpha)$ *a strong deformation retract of* B_α*, then the image of the induced map* $\amalg j_\alpha$ *is a strong deformation retract of* $\amalg B_\alpha$.

Exercise 2. Prove Proposition 2.

Proposition 2. *Consider any pushout diagram of the form*

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xrightarrow[g]{} & D \end{array}$$

in Top.

(a) *If* i *is injective, then* j *is injective.*
(b) *If* i *is closed injective, then* j *is closed injective.*
(c) *If* i *is closed injective with image* $i(A)$ *a strong deformation retract of* B*, then* $j(C)$ *is a strong deformation retract of* D.

Exercise 3. Prove Proposition 3.

Proposition 3. *Let* X *be a non-empty space and consider the diagram of the form*

$$\begin{array}{ccccccccc} X & \xrightarrow{i_1} & G^1 & \xrightarrow{i_2} & G^2 & \xrightarrow{i_3} & G^3 & \longrightarrow & \cdots \\ & {\scriptstyle i_\infty}\searrow & \downarrow{\scriptstyle \bar{i}_1} & \swarrow{\scriptstyle \bar{i}_2} & & \swarrow{\scriptstyle \bar{i}_3} & & & \\ & & \operatorname{colim}_k G^k & & & & & & \end{array}$$

constructed in the proof of MC5(ii) (factorization axiom) for a map $p\colon X \longrightarrow Y$ *in* Top.

(a) *The map* i_k *is a weak equivalence for each* $k \geq 1$.
(b) *The induced map*

$$\pi_n(X,x) = [S^n, X]_* \xrightarrow{(i_\infty)_*} [S^n, \operatorname{colim}_k G^k]_* = \pi_n(\operatorname{colim}_k G^k, i_\infty(x))$$

is surjective for each $n \geq 0$ *and* $x \in X$.

(c) *The induced map*

$$\pi_n(X,x) = [S^n, X]_* \xrightarrow{(i_\infty)_*} [S^n, \operatorname{colim}_k G^k]_* = \pi_n(\operatorname{colim}_k G^k, i_\infty(x))$$

is injective for each $n \geq 0$ *and* $x \in X$.

(d) *The map i_∞ is a weak equivalence.*

Exercise 4. Prove Proposition 4.

Proposition 4. *Consider any pair of functors of the form*

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D} \tag{1}$$

such that there are isomorphisms

$$\varphi\colon\ \hom(Fc,d) \xrightarrow{\cong} \hom(c,Gd)$$

natural in c, d; i.e., such that (1) is an adjunction with left adjoint on top.

(a) *The left-hand diagram*

$$\begin{array}{ccc} c & \xrightarrow{l_1} & Gd \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Gg} \\ c' & \xrightarrow[l_2]{} & Gd' \end{array} \qquad\qquad \begin{array}{ccc} Fc & \xrightarrow{\varphi^{-1}l_1} & d \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle g} \\ Fc' & \xrightarrow[\varphi^{-1}l_2]{} & d' \end{array}$$

in C commutes if and only if the right-hand diagram in D commutes.

(b) *The left-hand solid commutative diagram*

$$\begin{array}{ccc} c & \xrightarrow{l_1} & Gd \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle Gg} \\ c' & \xrightarrow[l_2]{} & Gd' \end{array} \qquad\qquad \begin{array}{ccc} Fc & \xrightarrow{\varphi^{-1}l_1} & d \\ {\scriptstyle Ff}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ Fc' & \xrightarrow[\varphi^{-1}l_2]{} & d' \end{array}$$

in C has a lift if and only if the right-hand solid commutative diagram in D has a lift.

(c) *The left adjoint F preserves colimits.*

(d) *The right adjoint G preserves limits.*

(e) *There exist natural transformations*

$$\eta\colon\ \mathrm{id} \longrightarrow GF, \qquad \varepsilon\colon\ FG \longrightarrow \mathrm{id}$$

such that the diagrams

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G, \quad (G\varepsilon)(\eta G) = \mathrm{id} \qquad\qquad F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F, \quad (\varepsilon F)(F\eta) = \mathrm{id}$$

commute.

Exercise 5. Prove Proposition 5.

Let G be a finite group. Consider the adjunction

$$\mathsf{Top} \underset{U}{\overset{G\times -}{\rightleftarrows}} \mathsf{Top}^G$$

with left adjoint on top and U the forgetful functor.

Proposition 5. *Let G be a finite group. Define a map $f\colon\ X \longrightarrow Y$ in Top^G to be*

(a) *a* weak equivalence *if the underlying map Uf is a weak equivalence in* Top,

(b) *a* fibration *if the underlying map Uf is a fibration in* Top*,*
(c) *a* cofibration *if f has the left lifting property with respect to all acyclic fibrations.*

These three classes of maps give Top^G the structure of a model category.

Exercise 6. Prove Proposition 6.

Let G be a finite group and $H \subset G$ a subgroup. Consider the adjunction

$$\mathsf{Top} \underset{(-)^H}{\overset{G/H\times -}{\rightleftarrows}} \mathsf{Top}^G$$

with left adjoint on top.

Proposition 6. *Let G be a finite group. Define a map $f\colon X \longrightarrow Y$ in Top^G to be*
(a) *a* weak equivalence *if the map $f^H\colon X^H \longrightarrow Y^H$ is a weak equivalence in* Top *for each subgroup $H \subset G$,*
(b) *a* fibration *if the map $f^H\colon X^H \longrightarrow Y^H$ is a fibration in* Top *for each subgroup $H \subset G$,*
(c) *a* cofibration *if it has the left lifting property with respect to all acyclic fibrations.*

These three classes of maps give Top^G the structure of a model category.

Exercise 7. Prove Proposition 7.

Proposition 7. *The factorizations constructed by the small object argument in the proofs of MC5(ii) and MC5(i) for* Top *are functorial factorizations.*