Algebraic Topology (topics course) John E. Harper

Spring 2010

Series 4

Exercise 1. Prove Proposition 1.

Proposition 1. Let $\{j_{\alpha}: A_{\alpha} \longrightarrow B_{\alpha}\}$ be a collection of maps in Top and consider the induced map $\amalg j_{\alpha}: \amalg A_{\alpha} \longrightarrow \amalg B_{\alpha}$.

- (a) If each map j_{α} is injective, then the induced map $\coprod j_{\alpha}$ is injective.
- (b) If each map j_{α} is closed injective, then the induced map $\coprod j_{\alpha}$ is closed injective.
- (c) If each map j_α is closed injective with image j_α(A_α) a strong deformation retract of B_α, then the image of the induced map IIj_α is a strong deformation retract of IIB_α.

Exercise 2. Prove Proposition 2.

Proposition 2. Consider any pushout diagram of the form



in Top.

- (a) If *i* is injective, then *j* is injective.
- (b) If *i* is closed injective, then *j* is closed injective.
- (c) If i is closed injective with image i(A) a strong deformation retract of B, then j(C) is a strong deformation retract of D.

Exercise 3. Prove Proposition 3.

Proposition 3. Let X be a non-empty space and consider the diagram of the form

constructed in the proof of MC5(ii) (factorization axiom) for a map $p: X \longrightarrow Y$ in Top.

- (a) The map i_k is a weak equivalence for each $k \ge 1$.
- (b) The induced map

$$\pi_n(X, x) = [S^n, X]_* \xrightarrow{(i_\infty)_*} [S^n, \operatorname{colim}_k G^k]_* = \pi_n(\operatorname{colim}_k G^k, i_\infty(x))$$

is surjective for each $n \ge 0$ and $x \in X$.

(c) The induced map

$$\pi_n(X, x) = [S^n, X]_* \xrightarrow{(i_\infty)_*} [S^n, \operatorname{colim}_k G^k]_* = \pi_n(\operatorname{colim}_k G^k, i_\infty(x))$$

is injective for each $n \ge 0$ and $x \in X$.

(d) The map i_{∞} is a weak equivalence.

Exercise 4. Prove Proposition 4.

Proposition 4. Consider any pair of functors of the form

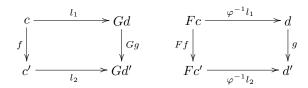
(1)
$$C \xrightarrow{F}_{G} D$$

such that there are isomorphisms

$$\varphi \colon \hom(Fc, d) \xrightarrow{\cong} \hom(c, Gd)$$

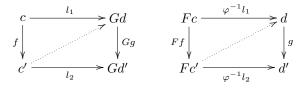
natural in c, d; i.e., such that (1) is an adjunction with left adjoint on top.

(a) The left-hand diagram



in C commutes if and only if the right-hand diagram in D commutes.

(b) The left-hand solid commutative diagram

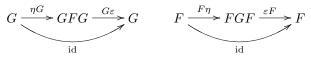


in C has a lift if and only if the right-hand solid commutative diagram in D has a lift.

- (c) The left adjoint F preserves colimits.
- (d) The right adjoint G preserves limits.
- (e) There exist natural transformations

$$\eta: \operatorname{id} \longrightarrow GF, \qquad \varepsilon: FG \longrightarrow \operatorname{id}$$

such that the diagrams



commute.

Exercise 5. Prove Proposition 5.

Let G be a finite group. Consider the adjunction

$$\operatorname{Top} \xrightarrow{G \times -}_{U} \operatorname{Top}^{G}$$

with left adjoint on top and U the forgetful functor.

Proposition 5. Let G be a finite group. Define a map $f: X \longrightarrow Y$ in Top^G to be (a) a weak equivalence if the underlying map Uf is a weak equivalence in Top ,

- (b) a fibration if the underlying map Uf is a fibration in Top,
- (c) a cofibration if f has the left lifting property with respect to all acyclic fibrations.

These three classes of maps give Top^G the structure of a model category.

Exercise 6. Prove Proposition 6.

Let G be a finite group and $H \subset G$ a subgroup. Consider the adjunction

$$\operatorname{Top} \xrightarrow[(-)]{G/H \times -} \operatorname{Top}^{G}$$

with left adjoint on top.

Proposition 6. Let G be a finite group. Define a map $f: X \longrightarrow Y$ in Top^G to be

- (a) a weak equivalence if the map f^H: X^H→Y^H is a weak equivalence in Top for each subgroup H ⊂ G,
 (b) a fibration if the map f^H: X^H→Y^H is a fibration in Top for each sub-
- (b) a fibration if the map $f^H \colon X^H \longrightarrow Y^H$ is a fibration in Top for each subgroup $H \subset G$,
- (c) a cofibration if it has the left lifting property with respect to all acyclic fibrations.

These three classes of maps give Top^G the structure of a model category.

Exercise 7. Prove Proposition 7.

Proposition 7. The factorizations constructed by the small object argument in the proofs of MC5(i) and MC5(i) for Top are functorial factorizations.