Algebraic Topology (topics course) **Spring 2010**

John E. Harper

Series 5

Exercise 1. Prove Proposition 1.

Proposition 1. *Let G be a finite group, $H \subset G$ a subgroup, and $\{X_\alpha\}$ a set of left G-spaces. There are isomorphisms*

$$(\amalg X_\alpha)^H \cong \amalg\,(X_\alpha^H)$$

in Top natural in X_α; i.e., the H-fixed points functor $(-)^H\colon \mathsf{Top}^G \longrightarrow \mathsf{Top}$ commutes with all small coproducts.

Exercise 2. Prove Proposition 2.

Proposition 2. *Let G be a finite group and $H \subset G$ a subgroup. If $f\colon A \longrightarrow B$ is a closed injective map in Top^G, then the induced map $f^H\colon A^H \longrightarrow B^H$ is a closed injective map in Top; i.e., the H-fixed points functor $(-)^H\colon \mathsf{Top}^G \longrightarrow \mathsf{Top}$ preserves closed injective maps.*

Recall the following proposition.

Proposition 3. *Let X, Y, Z be spaces. If Y is Hausdorff and locally compact, then there are isomorphisms*

$$\hom_{\mathsf{Top}}(X \times Y, Z) \cong \hom_{\mathsf{Top}}(X, \mathrm{Map}(Y, Z))$$

natural in such X, Y, Z.

Exercise 3. Prove Proposition 4.

Proposition 4. *Let G be a finite group and $X, Y, Z \in \mathsf{Top}^G$. If Y is Hausdorff and locally compact, then there are isomorphisms*

$$\hom_{\mathsf{Top}^G}(X \times Y, Z) \cong \hom_{\mathsf{Top}^G}(X, \mathrm{Map}(Y, Z))$$

natural in such X, Y, Z. In particular, the functor $- \times I\colon \mathsf{Top}^G \longrightarrow \mathsf{Top}^G$ is a left adjoint and hence preserves colimits. Here, $I := [0, 1]$ with trivial left G-action.

Exercise 4. Prove Proposition 5.

Proposition 5. *Let G be a finite group, $H \subset G$ a subgroup, and $\{j_\alpha\colon A_\alpha \longrightarrow B_\alpha\}$ a set of maps in Top^G. Consider the induced map $\amalg j_\alpha\colon \amalg A_\alpha \longrightarrow \amalg B_\alpha$ in Top^G and the induced map $\amalg(j_\alpha^H)\colon \amalg(A_\alpha^H) \longrightarrow \amalg(B_\alpha^H)$ in Top.*

(a) *If each map j_α is closed injective with image $j_\alpha(A_\alpha)$ a strong deformation retract of B_α in Top^G, then the image of the induced map $\amalg j_\alpha$ is a strong deformation retract of $\amalg B_\alpha$ in Top^G.*

(b) *If each map j_α is closed injective with image $j_\alpha(A_\alpha)$ a strong deformation retract of B_α in Top^G, then the image of the induced map $\amalg(j_\alpha^H)$ is a strong deformation retract of $\amalg(B_\alpha^H)$ in Top.*

Recall the following proposition.

Proposition 6. *Let X be a space. If X is Hausdorff, then the diagonal $X \subset X \times X$ is a closed subspace of $X \times X$.*

Exercise 5. Prove Proposition 7.

Proposition 7. *Let G be a finite group, $H \subset G$ a subgroup, and X a left G-space. If X is Hausdorff, then the H-fixed points $X^H \subset X$ is a closed subspace of X.*

Exercise 6. Prove Proposition 8.

Proposition 8. *Let G be a finite group and $H \subset G$ a subgroup. Consider any left-hand pushout diagram of the form*

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xrightarrow[g]{} & D \end{array} \qquad \qquad \begin{array}{ccc} A^H & \xrightarrow{f^H} & C^H \\ {\scriptstyle i^H}\downarrow & & \downarrow{\scriptstyle j^H} \\ B^H & \xrightarrow[g^H]{} & D^H \end{array}$$

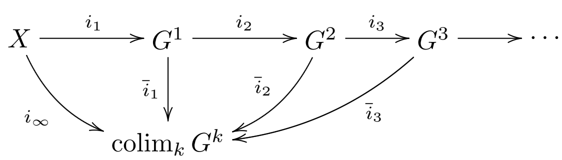
in Top^G and the corresponding right-hand diagram in Top. If A, B are Hausdorff and i is a closed injective map, then the right-hand diagram is a pushout diagram in Top.

Definition 9. Let G be a finite group. The *G-equivariant model structure* on Top^G is defined by the following three classes of maps: a map $f\colon X \longrightarrow Y$ in Top^G is

(i) a *weak equivalence* if the map $f^H\colon X^H \longrightarrow Y^H$ is a weak equivalence in Top for each subgroup $H \subset G$,
(ii) a *fibration* if the map $f^H\colon X^H \longrightarrow Y^H$ is a fibration in Top for each subgroup $H \subset G$,
(iii) a *cofibration* if it has the left lifting property with respect to all acyclic fibrations.

Exercise 7. Prove Proposition 10.

Proposition 10. *Let G be a finite group and $H \subset G$ a subgroup. Let X be a non-empty left G-space and consider the diagram of the form*



constructed in the proof of MC5(ii) (factorization axiom) for a map $p\colon X \longrightarrow Y$ in Top^G with the G-equivariant model structure.

(a) *The map $(i_k)^H\colon (G^{k-1})^H \longrightarrow (G^k)^H$ is a weak equivalence in Top for each $k \geq 1$.*
(b) *The map $(i_\infty)^H\colon X^H \longrightarrow (\operatorname{colim}_k G^k)^H$ is a weak equivalence in Top.*

Definition 11. Let G be a finite group. A map $f\colon A \longrightarrow X$ in Top^G is a *relative G-CW inclusion* if there is a sequence of maps of the form

$$A = X^{-1} \xrightarrow{i_0} X^0 \xrightarrow{i_1} X^1 \xrightarrow{i_2} X^2 \rightarrow \cdots$$

in Top^G such that

(i) $X \cong \operatorname{colim}_n X^n$ under A in Top^G, and

(ii) each map i_n fits into a pushout diagram of the form

$$\begin{array}{ccc} \coprod_H \coprod_{\mathcal{A}_H} G/H \times S^{n-1} & \longrightarrow & X^{n-1} \\ \big\downarrow{\scriptstyle \amalg\amalg j_n} & & \big\downarrow{\scriptstyle i_n} \\ \coprod_H \coprod_{\mathcal{A}_H} G/H \times D^n & \longrightarrow & X^n \end{array}$$

in Top^G; in other words, X^n is obtained from X^{n-1} by attaching G-cells $G/H \times D^n$. Here, the outer coproduct is indexed over all subgroups $H \subset G$, and $\mathcal{A}_H$ denotes an indexing set (possibly empty).

A left G-space X is a *G-CW complex* if the map $\emptyset \longrightarrow X$ is a relative G-CW inclusion.

Exercise 8. Prove Proposition 12.

Proposition 12. *Let G be a finite group and consider Top^G with the G-*equivariant model structure.

(a) *If $f\colon A \longrightarrow X$ in Top^G is a relative G-CW inclusion, then f is a cofibration in Top^G.*
(b) *If X is a G-CW complex, then X is cofibrant in Top^G.*

Definition 13. Let G be a finite group. A map $f\colon A \longrightarrow X$ in Top^G is a *generalized relative G-CW inclusion* if there is a sequence of maps of the form

$$A = G^0 \xrightarrow{i_1} G^1 \xrightarrow{i_2} G^2 \xrightarrow{i_3} G^3 \rightarrow \cdots$$

in Top^G such that

(i) $X \cong \operatorname{colim}_k G^k$ under A in Top^G, and
(ii) each map i_k fits into a pushout diagram of the form

$$\begin{array}{ccc} \coprod_{n,H} \coprod_{\mathcal{A}_{n,H}} G/H \times S^{n-1} & \longrightarrow & G^{k-1} \\ \big\downarrow{\scriptstyle \amalg\amalg j_n} & & \big\downarrow{\scriptstyle i_k} \\ \coprod_{n,H} \coprod_{\mathcal{A}_{n,H}} G/H \times D^n & \longrightarrow & G^k \end{array}$$

in Top^G; in other words, G^k is obtained from G^{k-1} by attaching G-cells $G/H \times D^n$. Here, the outer coproduct is indexed over all $n \geq 0$ and subgroups $H \subset G$, and $\mathcal{A}_{n,H}$ denotes an indexing set (possibly empty).

A left G-space X is a *generalized G-CW complex* if the map $\emptyset \longrightarrow X$ is a generalized relative G-CW inclusion.

Exercise 9. Prove Proposition 14.

Proposition 14. *Let G be a finite group and consider Top^G with the G-*equivariant model structure.

(a) *Every left G-space is fibrant in Top^G.*
(b) *A map $f\colon A \longrightarrow X$ in Top^G is a cofibration if and only if f is a retract of a generalized relative G-CW inclusion in Top^G.*
(c) *A left G-space X is cofibrant in Top^G if and only if X is a retract of a generalized G-CW complex in Top^G.*

Exercise 10. Prove Proposition 15.

Proposition 15. *Let* C *be a category with all small limits and colimits. Let* D *be a small category. There are adjunctions*

$$\mathsf{C} \underset{\lim_{\mathsf{D}}}{\overset{\Delta}{\rightleftarrows}} \mathsf{C}^{\mathsf{D}} \underset{\Delta}{\overset{\operatorname{colim}_{\mathsf{D}}}{\rightleftarrows}} \mathsf{C}$$

with left adjoints on top. Here, Δ *is the "diagonal" functor with* $\Delta(X) \in \mathsf{C}^{\mathsf{D}}$ *the constant diagram with value* X*.*

Exercise 11. Prove Proposition 16.

Proposition 16. *Let* C *be a category with all small limits and colimits. Let* I, J *small categories.*

(a) *The diagram category* C^{J} *has all small limits and colimits, and they are calculated objectwise.*

(b) *There are natural isomorphisms of diagram categories*

$$(\mathsf{C}^{\mathsf{I}})^{\mathsf{J}} \cong \mathsf{C}^{\mathsf{I}\times\mathsf{J}} \cong (\mathsf{C}^{\mathsf{J}})^{\mathsf{I}}.$$

(c) *If* $X \in \mathsf{C}^{\mathsf{I}\times\mathsf{J}}$*, there are natural isomorphisms*

$$\operatorname{colim}_{\mathsf{J}}(\operatorname{colim}_{\mathsf{I}} X) \cong \operatorname{colim}_{\mathsf{I}\times\mathsf{J}} X \cong \operatorname{colim}_{\mathsf{I}}(\operatorname{colim}_{\mathsf{J}} X).$$

(d) *If* $X \in \mathsf{C}^{\mathsf{I}\times\mathsf{J}}$*, there are natural isomorphisms*

$$\lim_{\mathsf{J}}(\lim_{\mathsf{I}} X) \cong \lim_{\mathsf{I}\times\mathsf{J}} X \cong \lim_{\mathsf{I}}(\lim_{\mathsf{J}} X).$$

Exercise 12. Prove Proposition 17.

Proposition 17. *Let* J *be a small category. Every left-hand adjunction*

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D} \qquad \mathsf{C}^{\mathsf{J}} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}^{\mathsf{J}}$$

with left adjoint on top, induces the right-hand adjunction on diagram categories with left adjoint on top. Here, the induced functors F *and* G *are defined objectwise; i.e.,* $F(X)(j) := F(X(j))$ *and* $G(Y)(j) := G(Y(j))$ *for each* $X \in \mathsf{C}^{\mathsf{J}}$ *and* $Y \in \mathsf{D}^{\mathsf{J}}$*.*

Exercise 13. Prove Proposition 18.

Proposition 18. *Consider any left-hand pair of adjunctions of the form*

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathsf{E} \qquad \mathsf{C} \underset{GG'}{\overset{F'F}{\rightleftarrows}} \mathsf{E}$$

with left adjoints on top. Then the right-hand pair of composite functors is an adjunction with left adjoint on top.

Exercise 14. Prove Proposition 19.

Proposition 19. *Consider any adjunctions of the form*

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D} \qquad \mathsf{C} \underset{G}{\overset{F'}{\rightleftarrows}} \mathsf{D} \qquad \mathsf{C} \underset{G'}{\overset{F}{\rightleftarrows}} \mathsf{D}$$

with left adjoints on top. Then there are isomorphisms of functors $F' \cong F$ *and* $G' \cong G$*; in other words, left adjoints (resp. right adjoints) are unique up to isomorphism.*