Algebraic Topology (topics course) **Spring 2010**

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Series 6

Let R be a ring and denote by Ch_R^+ (resp. Mod_R) the category of non-negative chain complexes over R (resp. the category of left R-modules). Define a map $f\colon M\longrightarrow N$ in Ch_R^+ to be

(i) a *weak equivalence* if it is a homology isomorphism,
(ii) a *fibration* if the map $f_k\colon M_k\longrightarrow N_k$ is an epimorphism for each $k\geq 1$,
(iii) a *cofibration* if it has the left lifting property with respect to all acyclic fibrations.

The purpose of this series is to give a proof of the following proposition.

Proposition 1. *These three classes of maps give Ch_R^+ the structure of a model category.*

Exercise 1. Prove Proposition 2.

Proposition 2. *The left-hand solid commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[h]{} & Y \end{array} \qquad \begin{array}{ccc} A_k & \xrightarrow{g_k} & X_k \\ {\scriptstyle i_k}\downarrow & \overset{f_k}{\nearrow} & \downarrow{\scriptstyle p_k} \\ B_k & \xrightarrow[h_k]{} & Y_k \end{array} \qquad \begin{array}{ccc} B_{k-1} & \xleftarrow{\partial} & B_k \\ {\scriptstyle f_{k-1}}\downarrow & & \downarrow{\scriptstyle f_k} \\ X_{k-1} & \xleftarrow[\partial]{} & X_k \end{array} \qquad (k\geq 0)$$

in Ch_R^+ has a lift if and only if the right-hand sequence of lifting problems in Mod_R has a solution, if and only if the sequence of lifting problems

$$\begin{array}{ccc} A_k & \xrightarrow{\qquad g_k \qquad} & X_k \\ {\scriptstyle i_k}\downarrow & \overset{f_k}{\nearrow} & \downarrow \\ B_k & \longrightarrow & \mathrm{Cy}_{k-1}(X)\times_{\mathrm{Cy}_{k-1}(Y)} Y_k \end{array} \qquad (k\geq 0)$$

in Mod_R has a solution.

Exercise 2. Prove Proposition 3.

Proposition 3. *Let $p\colon X\longrightarrow Y$ be a map in Ch_R^+.*

(a) *The map p is an acyclic fibration if and only if the induced map*
$$X_k\longrightarrow \mathrm{Cy}_{k-1}(X)\times_{\mathrm{Cy}_{k-1}(Y)} Y_k$$
is an epimorphism for each $k\geq 0$.

(b) *If p is an acyclic fibration, then the induced map*
$$(p_k)_*\colon\ \mathrm{Cy}_k(X)\longrightarrow \mathrm{Cy}_k(Y)$$
is an epimorphism for each $k\geq 0$.

(c) *If the induced map $X_k\longrightarrow \mathrm{Cy}_{k-1}(X)\times_{\mathrm{Cy}_{k-1}(Y)} Y_k$ is an epimorphism for each $k\geq 0$, then the induced map $(p_k)_*\colon\ \mathrm{Cy}_k(X)\longrightarrow \mathrm{Cy}_k(Y)$ is an epimorphism for each $k\geq 0$.*

Exercise 3. Prove Proposition 4.

Proposition 4. *Let $i\colon A \longrightarrow B$ be a map in Ch_R^+. If the map $i_k\colon A_k \longrightarrow B_k$ is a monomorphism with* $\operatorname{coker}(i_k)$ *a projective R-module for each $k \geq 0$, then i is a cofibration.*

Definition 5. Let A be a left R-module and $n \geq 1$. The chain complex $D_n(A)$ in Ch_R^+ has the form

$$D_n(A): \qquad \cdots \leftarrow 0 \leftarrow 0 \leftarrow A \xleftarrow{\text{id}} A \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

and is defined degreewise by

$$D_n(A)_k := \begin{cases} A, & \text{for } k = n, n-1, \\ 0, & \text{otherwise.} \end{cases}$$

The *n-disk* chain complex D^n in Ch_R^+ is defined by $D^n := D_n(R)$.

Note that the map $0 \longrightarrow D^n$ is a weak equivalence for each $n \geq 1$; i.e., the n-disk chain complex D^n is acyclic.

Exercise 4. Prove Proposition 6.

Proposition 6. *Let $n \geq 1$. There is an adjunction*

$$\mathsf{Mod}_R \underset{\text{Ev}_n}{\overset{D_n}{\rightleftarrows}} \mathsf{Ch}_R^+$$

with left adjoint on top and Ev_n *the evaluation functor defined objectwise by* $\text{Ev}_n(B) := B_n$*; i.e., there are isomorphisms*

$$\hom_{\mathsf{Ch}_R^+}(D_n(A), B) \cong \hom_{\mathsf{Mod}_R}(A, \text{Ev}_n(B))$$

natural in A, B.

Exercise 5. Prove Proposition 7.

Proposition 7. *Let $n \geq 1$. A solid commutative diagram of the form*

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ D^n & \longrightarrow & Y \end{array}$$

in Ch_R^+ is equivalent to an element $y \in Y_n$. A lift in such a solid commutative diagram is equivalent to an element $x \in X_n$ such that $p_n x = y$.

Exercise 6. Prove Proposition 8.

Proposition 8. *A map $p\colon X \longrightarrow Y$ in Ch_R^+ is a fibration if and only if it has the right lifting property with respect to the set of maps*

$$j_n\colon 0 \longrightarrow D^n, \qquad n \geq 1.$$

Definition 9. Let A be a left R-module and $n \geq 0$. The chain complex $S_n(A)$ in Ch_R^+ has the form

$$S_n(A): \qquad \cdots \leftarrow 0 \leftarrow 0 \leftarrow A \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

and is defined degreewise by

$$S_n(A)_k := \begin{cases} A, & \text{for } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

The *n-sphere* chain complex S^n in Ch_R^+ is defined by $S^n := S_n(R)$. For notational convenience, define the chain complexes $S^{-1} := 0$, $D^0 := S_0(R)$, and denote by $j_n\colon\ S^{n-1}\longrightarrow D^n$ the natural inclusion map in Ch_R^+.

Exercise 7. Prove Proposition 10.

Proposition 10. *Let $n \geq 0$. There is an adjunction*

$$\mathsf{Mod}_R \underset{\mathrm{Cy}_n}{\overset{S_n}{\rightleftarrows}} \mathsf{Ch}_R^+$$

with left adjoint on top and Cy_n the "n-dimensional cycles" functor defined objectwise by $\mathrm{Cy}_n(B) := \ker(\partial\colon\ B_n\longrightarrow B_{n-1})$; i.e., there are isomorphisms

$$\hom_{\mathsf{Ch}_R^+}(S_n(A), B) \cong \hom_{\mathsf{Mod}_R}(A, \mathrm{Cy}_n(B))$$

natural in A, B.

Exercise 8. Prove Proposition 11.

Proposition 11. *Let $n \geq 1$. A solid commutative diagram of the form*

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & X \\ {\scriptstyle j_n}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ D^n & \longrightarrow & Y \end{array}$$

in Ch_R^+ is equivalent to an element $(y, z) \in Y_n \oplus \mathrm{Cy}_{n-1}(X)$ such that $p_{n-1}z = \partial y$. A lift in such a solid commutative diagram is equivalent to an element $x \in X_n$ such that $p_n x = y$ and $\partial x = z$.

Exercise 9. Prove Proposition 12.

Proposition 12. *A map $p\colon\ X\longrightarrow Y$ in Ch_R^+ is an acyclic fibration if and only if it has the right lifting property with respect to the set of maps*

$$j_n\colon\ S^{n-1}\longrightarrow D^n, \qquad n \geq 0.$$

Recall the following proposition which is a special case of the property that homology commutes with filtered colimits.

Proposition 13. *Let $n \geq 0$ and consider any diagram of the form*

$$G^0 \xrightarrow{i_1} G^1 \xrightarrow{i_2} G^2 \xrightarrow{i_3} G^3 \rightarrow \cdots$$

in Ch_R^+. Then the natural map $\operatorname{colim}_k H_n(G^k) \xrightarrow{\cong} H_n(\operatorname{colim}_k G^k)$ in Mod_R is an isomorphism.

Exercise 10. Use Proposition 8 together with a small object argument to prove Proposition 14.

Proposition 14. *Let $p\colon\ X\longrightarrow Y$ be a map in Ch_R^+. Then p has a factorization*

$$X \xrightarrow{j} X' \xrightarrow{q} Y$$

in Ch_R^+ as an acyclic cofibration j followed by a fibration q; i.e., MC5(ii) *is satisfied.*

Exercise 11. Prove Proposition 15 using the factorizations constructed in the proof of Proposition 14.

Proposition 15. *Consider any solid commutative diagram of the form*

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[h]{} & Y \end{array}$$

in Ch_R^+*. If* i *is an acyclic cofibration and* p *is a fibration, then the diagram has a lift; i.e.,* MC4(ii) *is satisfied.*

Exercise 12. Use Proposition 12 together with a small object argument to prove Proposition 16.

Proposition 16. *Let* $p\colon X \longrightarrow Y$ *be a map in* Ch_R^+*. Then* p *has a factorization*

$$X \xrightarrow{j} Y' \xrightarrow{q} Y$$

in Ch_R^+ *as a cofibration* j *followed by an acyclic fibration* q*; i.e.,* MC5(i) *is satisfied.*

Exercise 13. Prove Proposition 17.

Proposition 17. *Every identity map in* Ch_R^+ *is a fibration, cofibration, and weak equivalence.*

Exercise 14. Prove Proposition 18.

Proposition 18. *The three classes of maps in* Ch_R^+*—weak equivalences, fibrations, and cofibrations—are each closed under composition.*

Exercise 15. Prove Proposition 19.

Proposition 19. *The category* Ch_R^+ *has all small limits and colimits, and they are calculated degreewise. In particular,* MC1 *is satisfied.*

Exercise 16. Prove Proposition 20.

Proposition 20. *The class of weak equivalences in* Ch_R^+ *satisfies the "two out of three axiom"* MC2*.*

Exercise 17. Prove Proposition 21.

Proposition 21. *The three classes of maps in* Ch_R^+*—weak equivalences, fibrations, and cofibrations—are each closed under retracts; i.e.,* MC3 *is satisfied.*

Exercise 18. Prove Proposition 1.

Exercise 19. Use the factorizations constructed in the proof of Proposition 16 together with Proposition 4 to prove Proposition 22.

Proposition 22.

(a) *A map* $i\colon A \longrightarrow B$ *in* Ch_R^+ *is a cofibration if and only if the map* $i_k\colon A_k \longrightarrow B_k$ *is a monomorphism with* $\operatorname{coker}(i_k)$ *a projective* R*-module for each* $k \geq 0$*.*
(b) *A chain complex* $B \in \mathsf{Ch}_R^+$ *is cofibrant if and only if* B_k *is a projective* R*-module for each* $k \geq 0$*.*
(c) *Every chain complex* $B \in \mathsf{Ch}_R^+$ *is fibrant.*

Exercise 20. Please read [1, Sections 7-8] and [2, 1.1-1.4]; see also [3, 2.3].

References

[1] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[2] P. G. Goerss and K. Schemmerhorn. Model categories and simplicial methods. In *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pages 3–49. Amer. Math. Soc., Providence, RI, 2007.

[3] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.