Algebraic Topology (topics course) **Spring 2010**
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Series 7

References for the following include [1, Sections 9-10], [2, Section 2] and [4, 5].

Exercise 1. Prove Proposition 1.

Proposition 1. *Let* C, D *be model categories and consider any adjunction*

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$$

with left adjoint on top. Then

(a) *F preserves cofibrations if and only if G preserves acyclic fibrations, and*
(b) *F preserves acyclic cofibrations if and only if G preserves fibrations.*

Exercise 2. Prove Theorem 2.

Theorem 2. *Let* C, D *be model categories and consider any adjunction*

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$$

with left adjoint on top. Suppose that

(i) *F preserves cofibrations and G preserves fibrations.*

Then the total derived functors $\mathsf{L}F$ and $\mathsf{R}G$ exist and fit into an adjunction

$$\mathsf{Ho}(\mathsf{C}) \underset{\mathsf{R}G}{\overset{\mathsf{L}F}{\rightleftarrows}} \mathsf{Ho}(\mathsf{D}) \tag{1}$$

with left adjoint on top. If in addition we have

(ii) *for each cofibrant object $A \in \mathsf{C}$ and fibrant object $X \in \mathsf{D}$, a map $f\colon A \longrightarrow G(X)$ is a weak equivalence in C if and only if its adjoint $F(A) \longrightarrow X$ is a weak equivalence in D,*

then the adjunction (1) is an equivalence of categories; i.e., the natural maps

$$A \xrightarrow[\cong]{\eta_A} \mathsf{R}G(\mathsf{L}F(A)) \qquad \mathsf{L}F(\mathsf{R}G(X)) \xrightarrow[\cong]{\varepsilon_X} X$$

of the adjunction (1) are isomorphisms for each $A \in \mathsf{C}$ and $X \in \mathsf{D}$.

Definition 3. Let C, D be model categories and consider any adjunction

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D} \tag{2}$$

with left adjoint on top. If the conditions in Theorem 2(i) are satisfied, then F is a *left Quillen functor*, G is a *right Quillen functor*, and the adjunction (2) is a *Quillen adjunction*. If in addition the conditions in Theorem 2(ii) are satisfied, then the adjunction (2) is a *Quillen equivalence*.

Let R be a ring and denote by Ch_R the category of unbounded chain complexes over R.

Exercise 3. Use a 5-lemma argument to prove Proposition 4.

Proposition 4. *Consider any commutative diagram of the form*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow[\alpha']{} & B' & \xrightarrow[\beta']{} & C' & \longrightarrow & 0 \end{array}$$

in Ch_R *with exact rows. If any two of the three vertical maps is a homology isomorphism, then so is the third.*

Recall the following right-exactness property of the tensor product functors.

Proposition 5. *If* X *is a right* R*-module and* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is an exact sequence of left* R*-modules, then*

$$X \otimes_R A \xrightarrow{\mathrm{id} \otimes_R \alpha} X \otimes_R B \xrightarrow{\mathrm{id} \otimes_R \beta} X \otimes_R C \to 0$$

is an exact sequence of abelian groups.

Definition 6. A right R-module X is *flat* if the functor $X \otimes_R -$ preserves monomorphisms; i.e., if for any monomorphism $\alpha\colon A \longrightarrow B$ of left R-modules, the induced map $\mathrm{id} \otimes_R i\colon X \otimes_R A \longrightarrow X \otimes_R B$ of abelian groups is a monomorphism.

Exercise 4. Prove Proposition 7.

Proposition 7. *The following properties of a left* R*-module* X *are equivalent.*

(a) *X is flat.*
(b) *The functor* $X \otimes_R -\colon \mathsf{Mod}_R \longrightarrow \mathsf{Mod}_{\mathbb{Z}}$ *preserves short exact sequences.*

Exercise 5. Prove Proposition 8.

Proposition 8.

(a) *The free right* R*-module* R *is flat.*
(b) *Every free right* R*-module is flat.*
(c) *Every projective right* R*-module is flat.*

Exercise 6. Prove Proposition 9.

The following proposition indicates how the $\mathrm{Tor}_n^R(X, -)$ functors provide a measure of the inexactitude of $X \otimes_R -$.

Proposition 9. *Let* X *be a right* R*-module and* $\varepsilon\colon P \longrightarrow X$ *in* $\mathsf{Ch}^+_{R^{\mathrm{op}}}$ *a projective resolution of* X*. Consider any short exact sequence* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *of left* R*-modules. Then there is a short exact sequence*

$$0 \to P \otimes_R A \xrightarrow{\mathrm{id} \otimes_R \alpha} P \otimes_R B \xrightarrow{\mathrm{id} \otimes_R \beta} P \otimes_R C \to 0$$

in $\mathsf{Ch}^+_{\mathbb{Z}}$*, and hence a natural corresponding long exact sequence*

$$\begin{aligned} \cdots \to \mathrm{Tor}_{n+1}^R(X, C) \xrightarrow{\partial} \mathrm{Tor}_n^R(X, A) \to \mathrm{Tor}_n^R(X, B) \to \mathrm{Tor}_n^R(X, C) \\ \cdots \to \mathrm{Tor}_1^R(X, A) \to \mathrm{Tor}_1^R(X, B) \to \mathrm{Tor}_1^R(X, C) \\ \xrightarrow{\partial} X \otimes_R A \to X \otimes_R B \to X \otimes_R C \to 0 \end{aligned}$$

of abelian groups.

Exercise 7. Prove Propositions 10 and 11.

Proposition 10. *Let X be a right R-module. A map $\varepsilon\colon P \longrightarrow X$ in $\mathsf{Ch}^+_{R^{\mathrm{op}}}$ is a projective resolution of X if and only if it is a cofibrant replacement of X in $\mathsf{Ch}^+_{R^{\mathrm{op}}}$.*

Proposition 11. *Let Y be a left R-module. Then the total left derived functor*

$$\begin{array}{ccccc} \mathsf{Ch}^+_{R^{\mathrm{op}}} & \xrightarrow{-\otimes_R Y} & \mathsf{Ch}^+_{\mathbb{Z}} & \longrightarrow & \mathsf{Ho}(\mathsf{Ch}^+_{\mathbb{Z}}) \\ \downarrow & & & & \\ \mathsf{Ho}(\mathsf{Ch}^+_{R^{\mathrm{op}}}) & & \xrightarrow[\text{total left derived functor}]{-\otimes^{\mathsf{L}}_R Y} & & \mathsf{Ho}(\mathsf{Ch}^+_{\mathbb{Z}}) \end{array}$$

of the tensor product functor $\mathsf{Ch}^+_{R^{\mathrm{op}}} \longrightarrow \mathsf{Ch}^+_{\mathbb{Z}}$ exists, and there are natural isomorphisms

$$H_n(X \otimes^{\mathsf{L}}_R Y) \cong \operatorname{Tor}^R_n(X, Y) \qquad (n \in \mathbb{Z})$$

of abelian groups for each $X \in \mathsf{Mod}_{R^{\mathrm{op}}} \subset \mathsf{Ch}^+_{R^{\mathrm{op}}}$.

Exercise 8. Prove Proposition 12.

Proposition 12. *The total left derived functor*

$$\begin{array}{ccccc} \mathsf{Ch}^+_{R^{\mathrm{op}}} \times \mathsf{Ch}^+_R & \xrightarrow{-\otimes_R -} & \mathsf{Ch}^+_{\mathbb{Z}} & \longrightarrow & \mathsf{Ho}(\mathsf{Ch}^+_{\mathbb{Z}}) \\ \downarrow & & & & \\ \mathsf{Ho}(\mathsf{Ch}^+_{R^{\mathrm{op}}}) \times \mathsf{Ho}(\mathsf{Ch}^+_R) & & \xrightarrow[\text{total left derived functor}]{-\otimes^{\mathsf{L}}_R -} & & \mathsf{Ho}(\mathsf{Ch}^+_{\mathbb{Z}}) \end{array}$$

of the tensor product functor $\mathsf{Ch}^+_{R^{\mathrm{op}}} \times \mathsf{Ch}^+_R \longrightarrow \mathsf{Ch}^+_{\mathbb{Z}}$ exists, and there are natural isomorphisms

$$H_n(X \otimes^{\mathsf{L}}_R Y) \cong \operatorname{Tor}^R_n(X, Y) \qquad (n \in \mathbb{Z})$$

of abelian groups for each $X \in \mathsf{Mod}_{R^{\mathrm{op}}} \subset \mathsf{Ch}^+_{R^{\mathrm{op}}}$ and $Y \in \mathsf{Mod}_R \subset \mathsf{Ch}^+_R$.

Recall the following left-exactness property of the hom object functors.

Proposition 13. *If X is a right R-module and $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is an exact sequence of right R-modules, then*

$$0 \to \operatorname{Hom}_R(X, A) \xrightarrow{\alpha_*} \operatorname{Hom}_R(X, B) \xrightarrow{\beta_*} \operatorname{Hom}_R(X, C)$$

is an exact sequence of abelian groups.

Exercise 9. Prove Proposition 14.

Proposition 14. *The following properties of a right R-module X are equivalent.*

(a) *X is projective.*
(b) *The functor $\operatorname{Hom}_R(X, -)\colon \mathsf{Mod}_{R^{\mathrm{op}}} \longrightarrow \mathsf{Mod}_{\mathbb{Z}}$ preserves short exact sequences.*

Exercise 10. Prove Proposition 15.

The following proposition indicates how the $\operatorname{Ext}^n_R(X, -)$ functors provide a measure of the inexactitude of $\operatorname{Hom}_R(X, -)$.

Proposition 15. *Let X be a right R-module and $\varepsilon\colon P \longrightarrow X$ in $\mathsf{Ch}^+_{R^{\mathrm{op}}}$ a projective resolution of X. Consider any short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of right R-modules. Then there is a short exact sequence*

$$0 \to \mathrm{Hom}_R(P, A) \xrightarrow{\alpha_*} \mathrm{Hom}_R(P, B) \xrightarrow{\beta_*} \mathrm{Hom}_R(P, C) \to 0$$

of cochain complexes, and hence a natural corresponding long exact sequence

$$\begin{aligned} &0 \to \mathrm{Hom}_R(X, A) \to \mathrm{Hom}_R(X, B) \to \mathrm{Hom}_R(X, C) \\ &\xrightarrow{\delta} \mathrm{Ext}^1_R(X, A) \to \mathrm{Ext}^1_R(X, B) \to \mathrm{Ext}^1_R(X, C) \\ \cdots \to \ &\mathrm{Ext}^n_R(X, A) \to \mathrm{Ext}^n_R(X, B) \to \mathrm{Ext}^n_R(X, C) \xrightarrow{\delta} \mathrm{Ext}^{n+1}_R(X, A) \to \cdots \end{aligned}$$

of abelian groups.

Define a map $f\colon M \longrightarrow N$ in Ch_R to be

(i) a *weak equivalence* if it is a homology isomorphism,
(ii) a *fibration* if the map $f_k\colon M_k \longrightarrow N_k$ is an epimorphism for each $k \in \mathbb{Z}$,
(iii) a *cofibration* if it has the left lifting property with respect to all acyclic fibrations.

The following proposition can be proved using similar arguments as in the case of non-negative chain complexes Ch^+_R.

Proposition 16. *These three classes of maps give Ch_R the structure of a model category.*

(a) *A map $i\colon A \longrightarrow B$ in Ch_R between bounded below chain complexes is a cofibration if and only if the map $i_k\colon A_k \longrightarrow B_k$ is a monomorphism with* $\mathrm{coker}(i_k)$ *a projective R-module for each $k \in \mathbb{Z}$.*
(b) *A bounded below chain complex $B \in \mathsf{Ch}_R$ is cofibrant if and only if B_k is a projective R-module for each $k \geq 0$.*
(c) *Every chain complex $B \in \mathsf{Ch}_R$ is fibrant.*

Proof. A proof is given in [3, 2.3]. □

Exercise 11. Prove Proposition 17.

Proposition 17. *Let Y be a right R-module. Then the total right derived functor*

$$\begin{array}{ccccc} (\mathsf{Ch}^+_{R^{\mathrm{op}}})^{\mathrm{op}} & \xrightarrow{\mathrm{Hom}_R(-,Y)} & \mathsf{Ch}_{\mathbb{Z}} & \longrightarrow & \mathsf{Ho}(\mathsf{Ch}_{\mathbb{Z}}) \\ \downarrow & & & & \\ \mathsf{Ho}(\mathsf{Ch}^+_{R^{\mathrm{op}}})^{\mathrm{op}} & & \xrightarrow[\text{total right derived functor}]{\mathsf{R}\,\mathrm{Hom}_R(-,Y)} & & \mathsf{Ho}(\mathsf{Ch}_{\mathbb{Z}}) \end{array}$$

of the hom object functor $(\mathsf{Ch}^+_{R^{\mathrm{op}}})^{\mathrm{op}} \longrightarrow \mathsf{Ch}_{\mathbb{Z}}$ exists, and there are natural isomorphisms

$$H^n(\mathsf{R}\,\mathrm{Hom}_R(X, Y)) \cong \mathrm{Ext}^n_R(X, Y) \qquad (n \in \mathbb{Z})$$

of abelian groups for each $X \in \mathsf{Mod}_{R^{\mathrm{op}}} \subset \mathsf{Ch}^+_{R^{\mathrm{op}}}$.

Definition 18. Let $l \in \mathbb{Z}$. Denote by $\mathsf{Ch}^{\geq l}_R \subset \mathsf{Ch}_R$ the full subcategory of chain complexes M such that $M_k = 0$ for each $k < l$; for instance, $\mathsf{Ch}^{\geq 0}_R = \mathsf{Ch}^+_R$.

Exercise 12. Prove Propositions 19 and 20.

Proposition 19. *Let $l \in \mathbb{Z}$. There is an isomorphism of categories $\mathsf{Ch}_R^{\geq l} \cong \mathsf{Ch}_R^{+}$.*

For each $l \in \mathbb{Z}$, define a map $f\colon M \longrightarrow N$ in $\mathsf{Ch}_R^{\geq l}$ to be

(i) a *weak equivalence* if it is a homology isomorphism,
(ii) a *fibration* if the map $f_k\colon M_k \longrightarrow N_k$ is an epimorphism for each $k \geq l+1$,
(iii) a *cofibration* if it has the left lifting property with respect to all acyclic fibrations.

Proposition 20. *These three classes of maps give $\mathsf{Ch}_R^{\geq l}$ the structure of a model category.*

(a) *A map $i\colon A \longrightarrow B$ in $\mathsf{Ch}_R^{\geq l}$ is a cofibration if and only if the map $i_k\colon A_k \longrightarrow B_k$ is a monomorphism with* $\operatorname{coker}(i_k)$ *a projective R-module for each $k \geq l$.*
(b) *A chain complex $B \in \mathsf{Ch}_R^{\geq l}$ is cofibrant if and only if B_k is a projective R-module for each $k \geq l$.*
(c) *Every chain complex $B \in \mathsf{Ch}_R^{\geq l}$ is fibrant.*

Definition 21. Let $n \in \mathbb{Z}$. A chain complex M in Ch_R is *n-connected* if $H_k(M) = 0$ for each $k \leq n$, and is *connective* if it is -1-connected.

Definition 22. Let $l \in \mathbb{Z}$ and $X \in \mathsf{Ch}_R$. The chain complex $\tau_{\geq l}(X)$ in $\mathsf{Ch}_R^{\geq l}$, called a *good truncation* of X, has the form

$$\tau_{\geq l}(X): \qquad \cdots \leftarrow 0 \leftarrow 0 \leftarrow \ker \partial_l \leftarrow X_{l+1} \leftarrow X_{l+2} \leftarrow X_{l+3} \leftarrow \cdots$$

and is defined degreewise by

$$\tau_{\geq l}(X)_k := \begin{cases} X_k, & \text{for } k \geq l+1, \\ \ker \partial_l, & \text{for } k = l, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 13. Prove Propositions 23 and 24.

Proposition 23. *Let $l \in \mathbb{Z}$. The good truncation functor $\tau_{\geq l}$ fits into an adjunction*

$$\mathsf{Ch}_R^{\geq l} \underset{\tau_{\geq l}}{\overset{i}{\rightleftarrows}} \mathsf{Ch}_R$$

with left adjoint on top and i the inclusion functor. The natural inclusion of chain complexes $j\colon \tau_{\geq l}(X) \longrightarrow X$ in Ch_R induces isomorphisms

$$H_k(\tau_{\geq l}(X)) \xrightarrow[\cong]{j_*} H_k(X) \qquad (k \geq l).$$

Proposition 24. *Let $l, m, n \in \mathbb{Z}$ and $X, Y \in \mathsf{Ch}_R^{\geq l}$. The total left derived functor*

$$\begin{array}{ccccc} \mathsf{Ch}_{R^{\mathrm{op}}}^{\geq l} \times \mathsf{Ch}_R^{\geq l} & \xrightarrow{\quad -\otimes_R - \quad} & \mathsf{Ch}_{\mathbb{Z}} & \longrightarrow & \mathsf{Ho}(\mathsf{Ch}_{\mathbb{Z}}) \\ \downarrow & & & & \\ \mathsf{Ho}(\mathsf{Ch}_{R^{\mathrm{op}}}^{\geq l}) \times \mathsf{Ho}(\mathsf{Ch}_R^{\geq l}) & & \xrightarrow[\text{total left derived functor}]{-\otimes^{\mathsf{L}}_R -} & & \mathsf{Ho}(\mathsf{Ch}_{\mathbb{Z}}) \end{array}$$

of the tensor product functor $\mathsf{Ch}_{R^{\mathrm{op}}}^{\geq l} \times \mathsf{Ch}_R^{\geq l} \longrightarrow \mathsf{Ch}_{\mathbb{Z}}$ exists.

(a) *If X is m-connected and Y is n-connected, then the derived tensor product $X \otimes^{\mathsf{L}}_R Y$ is $(m+n+1)$-connected.*
(b) *If X is m-connected and cofibrant and Y is n-connected, then $X \otimes_R Y$ is $(m+n+1)$-connected.*

Let C be a model category and let $\mathsf{D} = \{a \leftarrow b \rightarrow c\}$. Then a morphism $f\colon X \longrightarrow Y$ in C^{D} is a collection of maps f_a, f_b, f_c which makes the diagram

$$\begin{array}{ccccc} X_a & \longleftarrow & X_b & \longrightarrow & X_c \\ {\scriptstyle f_a}\downarrow & & {\scriptstyle f_b}\downarrow & & {\scriptstyle f_c}\downarrow \\ Y_a & \longleftarrow & Y_b & \longrightarrow & Y_c \end{array}$$

in C commute. Define a map $f\colon X \longrightarrow Y$ in C^{D} to be

(i) a *weak equivalence* if it is an objectwise weak equivalence; i.e., if the maps f_a, f_b, f_c are weak equivalences in C,
(ii) a *fibration* if it is an objectwise fibration; i.e., if the maps f_a, f_b, f_c are fibrations in C,
(iii) a *cofibration* if the induced maps

$$X_a \amalg_{X_b} Y_b \longrightarrow Y_a, \qquad X_b \longrightarrow Y_b, \qquad X_c \amalg_{X_b} Y_b \longrightarrow Y_c$$

are cofibrations in C.

Exercise 14. Prove Proposition 25.

Proposition 25. *These three classes of maps give C^{D} the structure of a model category.*

(a) *The total left derived functor*

$$\begin{array}{ccccc} \mathsf{C}^{\mathsf{D}} & \xrightarrow{\mathrm{colim}_{\mathsf{D}}} & \mathsf{C} & \longrightarrow & \mathsf{Ho}(\mathsf{C}) \\ \downarrow & & & & \\ \mathsf{Ho}(\mathsf{C}^{\mathsf{D}}) & & \xrightarrow[\textit{total left derived functor}]{\mathrm{hocolim}_{\mathsf{D}}} & & \mathsf{Ho}(\mathsf{C}) \end{array}$$

of the colimit functor $\mathsf{C}^{\mathsf{D}} \longrightarrow \mathsf{D}$ exists.
(b) *A diagram $Y \in \mathsf{C}^{\mathsf{D}}$ is cofibrant if and only if the maps*

$$Y_b \longrightarrow Y_a, \qquad \emptyset \longrightarrow Y_b, \qquad Y_b \longrightarrow Y_c$$

are cofibrations in C.
(c) *If $Y \in \mathsf{C}^{\mathsf{D}}$ is a diagram and $\emptyset \to Y^c \to Y$ is a cofibration followed by a weak equivalence in C^{D}, then* $\mathrm{hocolim}_{\mathsf{D}}(Y) \simeq \mathrm{colim}_{\mathsf{D}}(Y^c)$.
(d) *If $f\colon X \longrightarrow Y$ is a weak equivalence between cofibrant diagrams, then the induced map* $\mathrm{colim}_{\mathsf{D}} X \longrightarrow \mathrm{colim}_{\mathsf{D}} Y$ *is a weak equivalence.*

Sometimes $\mathrm{hocolim}_{\mathsf{D}}(X)$ is called the *homotopy pushout* of the diagram X.

Exercise 15. Use duality in model categories to obtain a corresponding proposition involving the total right derived functor of the limit functor $\lim_{\mathsf{D}^{\mathrm{op}}}\colon \mathsf{C}^{\mathsf{D}^{\mathrm{op}}} \longrightarrow \mathsf{C}$; note that D^{op} is the category $\{a \rightarrow b \leftarrow c\}$. Describe the corresponding model structure on $\mathsf{C}^{\mathsf{D}^{\mathrm{op}}}$.

Exercise 16. Please read [1, Sections 9-10] and [2, Section 2].

References

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