Research Statement

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Summary

Algebraic topology tries to solve topological or geometric problems by projecting them into a simpler (discrete) algebraic context where calculations become possible. The main invariants are homotopy groups and homology groups, which do not change under continuous deformations (called homotopies). While the field is most strongly represented at the handful of leading research institutions, the footprint that it leaves on contemporary mathematics is vast and growing. Its methods are used in a wide variety of fields from geometry and number theory, to algebra and analysis. For instance, students in a first course in algebraic topology learn quickly how to compute the homology groups of an n-sphere S^n, which are then used to give a simple and conceptual proof of the fundamental theorem of algebra and Brouwer's fixed point theorem. The deeper one goes into algebraic topology, the more elaborate the invariants become, but also the bigger the potential payoffs. For instance, in their landmark 2003 Annals paper, Hesselholt-Madsen use sophisticated methods of algebraic topology to calculate the algebraic K-theory groups of local fields.

In recent years the methods of algebraic topology and homotopy theory have started to become important in other disciplines as well, including applied mathematics (e.g., qualitative analysis of large data sets) and mathematical physics (e.g., two dimensional quantum field theory and string theory). Often the topological and geometric problems have naturally occurring sophisticated algebraic structure that reveals more about the problem (e.g., homotopy theoretic ring structures that naturally arise in algebraic K-theory and derived algebraic geometry). Topological Quillen (TQ) homology is an invariant of such enriched algebraic-topological structure, in the same way that ordinary homology is an invariant of topological spaces. An important part of the current "enriched algebra" program in homotopy theory is the development of standard tools of the homotopy theory of spaces in this new algebraic-topological context (which can intuitively be thought of as "homotopy theoretic algebra" as we explain below).

The main focus of my recent work has been an intensive development of these new tools with a special emphasis on recovering algebraic and topological structures from TQ-homology. Together with my co-authors, I have shown in [6, 7] that certain generalized "homotopy theoretic algebras" can be completely recovered (up to homotopy) from a natural sequence of approximations built entirely from TQ-homology and its co-operations, via a sort of "algebraic completion" process. This new result means that TQ-homology tells us a lot more about these sophisticated algebraic-topological objects than one might think. Together with my co-authors, I have also proved a finiteness result for TQ-homology [18] (that can be thought of as an algebraic-topological analog of Serre's finiteness theorem for spaces), together with absolute and relative Hurewicz theorems for TQ-homology [18]. These new results mean that TQ-homology detects important finiteness and connectivity properties of these generalized "homotopy theoretic algebras". I also have relevant earlier results [15, 16, 17], but here I am only going to focus on recent work.

Introduction to Spectra and "Homotopy Theoretic Algebra"

Spectra (described below) naturally arise as the "linear approximations" to topological spaces, in a sense made precise by Goodwillie's "Taylor tower" expansions of homotopy functors [13, 14]; these approximations can be thought of as algebraic-topological analogs of the Taylor series expansions of functions studied in calculus and analysis. One of the most important themes in current work in homotopy theory is the investigation and exploitation of enriched algebraic structures on spectra that naturally arise, for instance, in algebraic topology, algebraic K-theory, and derived algebraic geometry (e.g., [11, 19, 22, 28, 29]). Such "homotopy theoretic rings" (e.g. [10, 20, 24, 30]) are sophisticated algebraic-topological generalizations of the notion of ring from algebra and algebraic geometry.

A spectrum X is a sequence of based spaces $X_0, X_1, X_2, \ldots$ with maps between them $\Sigma X_n \longrightarrow X_{n+1}$ (here, the symbol ΣX_n denotes the suspension of X_n; e.g., $\Sigma S^1 \cong S^2$). It can be thought of as the geometric analog of a chain complex of abelian groups, and as will be explained shortly, the theory of spectra properly contains a lot of algebra and homological algebra as a special case. To tell the story, we start with Eilenberg-MacLane spaces $K(R,n)$ (with R a commutative ring) which are homotopically very simple in the sense that their homotopy groups are concentrated in degree n with value R (i.e., $\pi_* K(R,n) = 0$ for $* \neq n$, and $\pi_n K(R,n) = R$). It is a basic fact of algebraic topology that Eilenberg-MacLane spaces $K(R,n)$ represent ordinary cohomology of spaces with coefficients in R (sometimes called ordinary R-cohomology, for short). This means that there are natural isomorphisms of abelian groups

$$\tilde{H}^n(Y,R) \cong [Y, K(R,n)]$$

(where Y is a based space and [,] denotes homotopy classes of maps), which amounts to the fact that there are maps of based spaces $\Sigma K(R,n) \longrightarrow K(R,n+1)$. This gives a spectrum $H(R)$, with $H(R)_n = K(R,n)$, called the Eilenberg-MacLane spectrum associated to R.

Often the spectra that arise in applications (e.g., in algebraic topology, algebraic K-theory, and derived algebraic geometry) are naturally equipped with an enriched algebraic-topological structure. For instance, the multiplication map $R \otimes R \longrightarrow R$ associated to the ring structure on R induces a pairing $H(R) \wedge H(R) \longrightarrow H(R)$ on the associated Eilenberg-MacLane spectrum $H(R)$; its meaning can be understood as encoding the cup product structure $\tilde{H}^*(Y,R) \otimes \tilde{H}^*(Y,R) \longrightarrow \tilde{H}^*(Y,R)$ on the ordinary R-cohomology of spaces. The symbol $\wedge$ denotes the "smash product" of spectra, which sometimes behaves like the tensor product of abelian groups and sometimes like the smash product of based spaces. Another fundamental example of a spectrum is the sphere spectrum S. It consists of the sequence of spheres $S^0, S^1, S^2, \ldots$ together with the isomorphisms $\Sigma S^n \longrightarrow S^{n+1}$ of based spaces. The sphere spectrum plays the role of the "unit" with respect to smash product (in the sense that $S \wedge X \cong X \cong X \wedge S$) in the same way that the group of integers $\mathbb{Z}$ plays the role of the "unit" with respect to tensor product $\otimes$ of abelian groups. It is another example of a spectrum that has enriched algebraic-topological structure; here, the isomorphisms $S^m \wedge S^n \longrightarrow S^{m+n}$ of based spaces induce a pairing $S \wedge S \longrightarrow S$ on the level of spectra.

As remarked above, spectra can be thought of as the geometric analogs of chain complexes of abelian groups (instead of having a sequence of abelian groups together with boundary maps between them, in a spectrum we have sequences of based spaces together with maps between them). Analogous to the homology groups of a chain complex, the main invariants of a spectrum X are the homotopy groups $\pi_*(X)$ defined by

$$\pi_i(X) := \operatorname{colim} \pi_{i+n} X_n,$$

which are built out of directed limits (also called colimits) of the homotopy groups of the spaces X_n. Eilenberg-MacLane spectra $H(R)$ are homotopically very simple in the sense that their homotopy groups $\pi_* H(R)$ are concentrated in degree 0 with value R (i.e., $\pi_* H(R) = 0$

for $* \neq 0$, and $\pi_0 H(R) = R$). Since the pairing $H(R) \wedge H(R) \longrightarrow H(R)$ induces a pairing on graded abelian groups $\pi_* H(R) \otimes \pi_* H(R) \longrightarrow \pi_* H(R)$ that is concentrated in degree 0 with value the ring multiplication $R \otimes R \longrightarrow R$, it follows that the Eilenberg-MacLane spectrum $H(R)$ is a type of "homotopy theoretic commutative ring" model of R itself.

We are now in a good position to explain how the theory of spectra contains a lot of algebra and homological algebra. It is a fundamental result of recent advances in algebraic topology that if one works with spectra that are further equipped with symmetric group actions (such spectra are called symmetric spectra), then the corresponding smash product $\wedge$ is sufficiently well-behaved to allow one to "do algebra" in this sophisticated algebraic-topological context. By analogy with Eilenberg-MacLane spectra $H(R)$ above, a commutative ring spectrum is a symmetric spectrum X together with two maps

$$\begin{aligned} X \wedge X &\longrightarrow X, &\quad& \text{"multiplication"} \\ S &\longrightarrow X, && \text{"unit"} \end{aligned}$$

which satisfy the usual associativity, two-sided unit, and commutativity relations—this is precisely analogous to the conditions in classical algebra satisfied by a commutative ring X, where the smash product $\wedge$ would be replaced by the tensor product $\otimes$ of abelian groups, and the sphere spectrum S would be replaced by the abelian group of integers $\mathbb{Z}$. Examples of commutative ring spectra (which can be thought of as "homotopy theoretic commutative rings") include the sphere spectrum S, the Eilenberg-MacLane spectrum $H(R)$, and the algebraic K-theory spectrum $K(R)$ (discussed further below). The upshot is that one can "do algebra" in this enriched algebraic-topological context (e.g., as in derived algebraic geometry). For instance, this leads to the notion of $H(R)$-module spectra and $H(R)$-algebra spectra, which have the following direct connection with classical algebra:

$$\begin{aligned} H(R)\text{-module spectra} &\quad \simeq_h \quad \text{dg } R\text{-modules}, \\ H(R)\text{-algebra spectra} &\quad \simeq_h \quad \text{dg } R\text{-algebras}, \\ \text{commutative } H(R)\text{-algebra spectra} &\quad \simeq_h \quad E_\infty \text{ dg } R\text{-algebras} \\ (\text{if} \quad R \supset \mathbb{Q}) &\quad \simeq_h \quad \text{commutative dg } R\text{-algebras} \end{aligned}$$

Here, the notation $\simeq_h$ means that they have equivalent homotopy theories, and in particular, equivalent derived categories (or homotopy categories). Hence as far as their derived categories are concerned, dg R-algebras are the same as $H(R)$-algebra spectra. There are other types of homotopy theoretic algebraic structures that naturally arise; for instance, E_n algebra spectra are types of "homotopy theoretic algebras" that interpolate between the two extremes of commutative ring spectra (equivalent to E_∞ algebra spectra) and ring spectra (equivalent to E_1 algebra spectra). We will refer to such (generalized) "homotopy theoretic algebras" as *structured ring spectra*.

Some Applications of "Homotopy Theoretic Algebra"

While there is a close relationship between certain multiplicative cohomology theories of spaces and structured ring spectra, it is important to note that one can do other things with spectra besides cohomology theories. One of the initial motivations for studying structured ring spectra goes back to Waldhausen's work in the 1970's and 1980's relating the geometric topology of a high dimensional manifold M to the algebraic K-theory of the (spectral) group algebra $S[\Omega M]$, where ΩM is the loop group of M. Structured ring spectra also have applications in computations of invariants of interest in classical algebra, such as the algebraic K-theory of rings. If R is a commutative ring, then Waldhausen's construction builds an associated commutative ring spectrum whose homotopy groups are essentially Quillen's algebraic K-theory of the ring R. The upshot is that the collection of algebraic K-theory invariants has now been refined into a more sophisticated algebraic-topological structure that reveals more about the problem. For instance, in their 2003 Annals paper, Hesselholt-Madsen [19] calculate the algebraic K-theory of local fields using techniques that rely heavily on structured ring spectra and their associated constructions and invariants; this was a decisive result and it followed from Waldhausen's program. The theory of structured ring spectra also contains a lot of classical algebra by working over Eilenberg-Mac Lane ring spectra $H(R)$. For instance, as remarked above, the categories of $H(R)$-module spectra and $H(R)$-algebra spectra are homotopically equivalent to the categories of dg R-modules and dg R-algebras, respectively (see, [31, 1.1]). There is a similar homotopical equivalence for the category of commutative $H(R)$-algebras (see, [23, 7.2, 7.22] and [31, 1.2]). Sometimes this algebraic-topological enrichment of constructions and phenomena in algebra can be turned around. For instance, Dwyer-Greenlees-Iyengar [9] use constructions of structured ring spectra to obtain new results in algebra that otherwise appear inaccessible from classical algebraic techniques alone. The upshot is that working in the sophisticated algebraic-topological framework of structured ring spectra allows one to give more relationships between things.

TQ-homology of "Homotopy Theoretic Algebras"

Some of the most well-developed and powerful tools for studying spaces are those that relate their two primary invariants—homotopy groups and homology groups—and an important part of the current "enriched algebra" program in homotopy theory is the development of standard tools of the homotopy theory of spaces in this new algebraic-topological context of structured ring spectra. The main focus of my recent and current work is to give an intensive development of these tools with a special emphasis on recovering algebraic and topological structures from associated homology objects. I have a particular interest in generalized algebras in spectra (e.g., associative, commutative, or more generally, E_n ring spectra), for which the associated homology object is topological Quillen (TQ) homology (a type of derived abelianization of the object). In this new context, TQ-homology is the precise analog for structured ring spectra of ordinary homology of spaces. Its theory was begun by Basterra [2] for commutative ring spectra, and has been developed in certain directions by Basterra-Mandell [3, 4], Goerss-Hopkins [12], Kuhn [21], Mandell [23], McCarthy-Minasian [25], and Rognes [28, 29], among others. TQ-homology specializes to topological André-Quillen homology in the context of commutative ring spectra (equivalently E_∞ ring spectra), and to topological Hochschild homology (essentially, up to a dimension shift) in the context of ring spectra (equivalently A_∞ ring spectra or E_1 ring spectra), both of which, for instance, play a key role in the Ausoni-Rognes [1] program for analyzing the layers of the K-theory chromatic tower in terms of Galois theory of commutative ring spectra [28, 29]. A notable result of Basterra-Mandell [3] is that every "axiomatic" homology theory on a category of E_∞ ring spectra is a form of TQ homology with appropriate coefficients. An important aim of my current research is to go further with this work.

Together with my co-authors, I have proved a Serre-type finiteness result for TQ [18] that can be thought of as a "spectral algebra" analog of Serre's finiteness theorem for spaces (i.e., that certain finiteness properties of the TQ-homology groups imply corresponding finiteness properties of the homotopy groups), together with absolute and relative Hurewicz theorems for TQ-homology [18] (e.g., that the first non-trivial TQ-homology group is naturally isomorphic to the first non-trivial homotopy group), at least for connected non-unital ring spectra (either associative, commutative, or more generally E_n ring spectra). It is important to note that non-unital structured ring spectra naturally arise whenever one is studying augmented structured ring spectra (e.g., [3, Section 6]). Even though the proof techniques in [18] are totally different from the proofs that work in the context of spaces, these results provide evidence suggesting that TQ-homology behaves very much like the ordinary homology of spaces, and hence hints, for instance, at the tantalizing possibility that perhaps the structured ring spectra themselves can be recovered from TQ-homology and their co-operations. This is suggested by the analogous situation in the homotopy theory of spaces using Sullivan [32, 33] and Bousfield-Kan [5] completions, where the general idea is to use ordinary homology to give a sequence of approximations to the space, that under some conditions converges to the space itself. Progress in this direction has been made in current work with my co-authors. For instance, it is shown in [6] that a connected non-unital ring spectrum (either associative, commutative, or more generally an E_n ring spectrum) can be recovered (up to homotopy) from the inverse limit of a tower of fibrations that by construction encodes all co-operations on TQ-homology. It is also shown in [6] that one can relax the connectivity condition to a nilpotency condition if TQ-homology is suitably adjusted. These results are elaborated in more detail below, together with some discussion on the background and motivation.

TQ-Completion Tower of "Homotopy Theoretic Algebras"

A major part of the homotopy theory of spaces is the unstable Adams spectral sequence, which roughly speaking goes from the ordinary homology $\tilde{H}_*(X;R)$ of a space (with coefficients in a commutative ring R), and gets at the homotopy groups π_*X of the space itself, at least under some conditions. For instance, in his 1984 Annals paper [26], H.R. Miller proves the Sullivan conjecture (on maps from classifying spaces) using the unstable Adams spectral sequence as an important tool; he also uses the algebraic version of TQ-homology for augmented commutative algebras as a key ingredient. Since the homology groups $\tilde{H}_*(X;R)$ of a space are usually easier to understand and compute than its homotopy groups $\pi_*(X)$, this important tool provides a method for taking simple kinds of objects and combining them together to get everything. In a little more detail, the unstable Adams spectral sequence takes information about the homology $\tilde{H}_*(X;R)$ of a space and its naturally occurring co-operations, and combines them together to get information about the homotopy groups π_*X, at least under some conditions.

In the 1970's Bousfield-Kan [5] used cosimplicial methods to describe what can be thought of as a more precise space-level version of the unstable Adams spectral sequence. The general idea is to use ordinary homology with coefficients in R to give a natural sequence of approximations to the space X, that under some conditions converges to the space itself. This sequence of approximations is organized in the form of a tower $\{R_nX\}$ of fibrations, today called the Bousfield-Kan R-completion tower of X, that by construction encodes all space-level co-operations on ordinary homology with coefficients in R. The homotopy limit of this tower is the Bousfield-Kan R-completion $X^\wedge_R$ of X. An important property of this tower is that its associated homotopy spectral sequence agrees with the unstable Adams spectral sequence, and it is in this sense that the tower $\{R_nX\}$ provides a space-level refinement of this important tool. Most topologists are familiar with the particular Bousfield-Kan R-completions obtained by taking $R=\mathbb{F}_p$ and $R=\mathbb{Q}$, where the corresponding constructions are usually called the p-completion $X^\wedge_p := X^\wedge_{\mathbb{F}_p}$ of X and the $\mathbb{Q}$-localization $X^\wedge_{\mathbb{Q}}$ of X, respectively, since under appropriate conditions they agree (up to homotopy) with Sullivan's [32, 33] p-completion and $\mathbb{Q}$-localization of spaces constructions. Bousfield-Kan [5] prove that for simply connected (or more generally, nilpotent) spaces X, the sequence of approximations $\{R_nX\}$ in the Bousfield-Kan tower (for $R=\mathbb{Z}$) converges to the space X itself, in the sense that the natural coaugmentation $X\longrightarrow X^\wedge_{\mathbb{Z}}$ of spaces induces an isomorphism

$$\pi_*X \cong \pi_*X^\wedge_{\mathbb{Z}}$$

on homotopy groups. It follows immediately that the natural coaugmentation $X\longrightarrow X^\wedge_{\mathbb{Z}}$ induces an isomorphism on ordinary integral homology groups $\tilde{H}_*(X;\mathbb{Z}) \cong \tilde{H}_*(X^\wedge_{\mathbb{Z}},\mathbb{Z})$. In summary, Bousfield-Kan [5] prove that the space X is homotopy equivalent to its $\mathbb{Z}$-completion

$X^\wedge_{\mathbb{Z}}$, and hence it can be recovered (up to homotopy) from ordinary integral homology and co-operations. A corollary of their proof is that the associated unstable Adams spectral sequence converges to $\pi_* X$.

Since the Bousfield-Kan completion tower of a space and its associated unstable Adams spectral sequence are important tools in the homotopy theory of spaces, it is natural to work on developing an analogous tool in the new algebraic-topological context of structured ring spectra. As noted above, I have a particular interest in operadic algebras in spectra (e.g., associative, commutative, or more generally, E_n ring spectra), for which the associated homology object is topological Quillen (TQ) homology. Together with my co-authors, so far I have constructed the precise analog of the Bousfield-Kan completion tower in this new context, called the TQ-completion tower of X, at least for non-unital ring spectra X (either associative, commutative, or more generally E_n ring spectra). As noted above, non-unital structured ring spectra naturally arise whenever one is studying augmented structured ring spectra (e.g., [3, Section 6]). Unlike the situation for Bousfield-Kan, making the TQ-completion construction work required the solution of a homotopy rigidification problem that was recently solved in [18]. By construction [18] the tower encodes all co-operations on TQ-homology, and gives a natural sequence of approximations to the structured ring spectrum X. The homotopy limit of this tower is the TQ-completion $X^\wedge_{\mathrm{TQ}}$ of X. Together with my co-authors, in current work I have proved that under some conditions this sequence of approximations converges to X itself. The following convergence result is proved in [6] with my co-authors.

Theorem. *Suppose X is a non-unital ring spectrum (either associative, commutative, or more generally an E_n ring spectrum). If X is connected, then the natural map $X \longrightarrow X^\wedge_{\mathrm{TQ}}$ induces an isomorphism on homotopy groups*

$$\pi_* X \cong \pi_* X^\wedge_{\mathrm{TQ}}.$$

It follows immediately that the natural coaugmentation $X \longrightarrow X^\wedge_{\mathrm{TQ}}$ induces an isomorphism on TQ-homology groups $\pi_* \mathrm{TQ}(X) \cong \pi_* \mathrm{TQ}(X^\wedge_{\mathrm{TQ}})$. The proof of this theorem also implies strong convergence (in the pro-constant sense) of the associated TQ-completion spectral sequence [6], which is the precise analog in this new context of the unstable Adams spectral sequence for spaces.

In current work, together with my co-authors, I also prove the following structured ring spectra analog of Dror's [8] generalization of the Whitehead theorem to nilpotent spaces. The following result is proved in [7] with my co-authors.

Theorem. *Suppose $f\colon X \longrightarrow Y$ is a map of non-unital ring spectra (either associative, commutative, or more generally E_n ring spectra) and $N \geq 2$. If X, Y are nilpotent of order N, then f induces an isomorphism on homotopy groups $\pi_* X \cong \pi_* Y$ if and only if f induces an isomorphism on* TQ*-homology groups $\pi_* \mathrm{TQ}(X) \cong \pi_* \mathrm{TQ}(Y)$.*

Homotopy Completion Tower of "Homotopy Theoretic Algebras"

Prior work on the homotopy completion tower. Associated to each non-unital commutative ring X is the completion tower arising in commutative ring theory

$$X/X^2 \leftarrow X/X^3 \leftarrow \cdots \leftarrow X/X^n \leftarrow X/X^{n+1} \leftarrow \cdots$$

of non-unital commutative rings. The limit of this tower is the completion $X^\wedge$ of X, which is sometimes also called the X-adic completion of X. Here, X/X^n denotes the quotient of X in the underlying category by the image of the multiplication map $X^{\otimes n} \longrightarrow X$.

Since completion towers are important tools in commutative algebra, it is natural to work on developing an analogous tool in the new algebraic-topological context of structured ring spectra. As noted above, I have a particular interest in operadic algebras in spectra (e.g., associative, commutative, or more generally, E_n ring spectra), for which the associated homology object is topological Quillen (TQ) homology. Together with my co-authors, so far I have constructed in [18] the precise analog of the completion tower in this new context, called the homotopy completion tower of X, at least for non-unital ring spectra X (either associative, commutative, or more generally E_n ring spectra). As noted above, non-unital structured ring spectra naturally arise whenever one is studying augmented structured ring spectra (e.g., [3, Section 6]). By construction [18] the tower gives a natural sequence of approximations to the structured ring spectrum X. The homotopy limit of this tower is the homotopy completion $X^{\mathsf{h}\wedge}$ of X. Together with my co-authors, I have proved in previous work that under some conditions this sequence of approximations converges to X itself. The following convergence result is proved in [18] with my co-authors.

Theorem. *If X is a connected non-unital ring spectrum (either associative, commutative, or more generally an E_n ring spectrum), then the natural map $X \longrightarrow X^{\mathsf{h}\wedge}$ induces an isomorphism on homotopy groups $\pi_* X \cong \pi_* X^{\mathsf{h}\wedge}$.*

This generalizes earlier results of Kuhn [21], Minasian [27], and McCarthy-Minasian [25] for the special case of connected non-unital commutative ring spectra to more general operadic algebras in spectra, and plays a key role in the proofs of the Serre-type finiteness theorem [18] for TQ-homology and the absolute and relative Hurewicz theorems [18] for TQ-homology described above for structured ring spectra.

Algebraic K-theory and TQ-Homology

In current work, I have been working on applying recent results on TQ-homology to the algebraic K-theory of a commutative ring R. The following is a speculative application of TQ-homology in this direction. Suppose one wants to calculate the algebraic K-theory groups of R by comparing the associated commutative ring spectrum, call it $K(R)$, to another (-1)-connected commutative ring spectrum X, via a map of ring spectra $K(R) \longrightarrow X$. Then one could look at the natural augmentations $K(R) \longrightarrow H(\pi_0 K(R))$ and $X \longrightarrow H(\pi_0 X)$ and consider the induced map $\tilde{K}(R) \longrightarrow \tilde{X}$ on homotopy fibers of non-unital commutative ring spectra. The following is an immediate consequence of the results in [18] with my co-authors.

Theorem. *The map $K(R) \longrightarrow X$ induces an isomorphism $\pi_* K(R) \cong \pi_* X$ on homotopy groups if and only if it induces an isomorphism $\pi_0 K(R) \cong \pi_0 X$ and the induced map on TQ-homology groups $\pi_* \mathsf{TQ}(\tilde{K}(R)) \cong \pi_* \mathsf{TQ}(\tilde{X})$ is an isomorphism.*

There is an analogous statement for verifying that $K(R) \longrightarrow X$ is n-connected, using the relative Hurewicz theorem for TQ-homology proved in [18]. One objective of my current work is to explore such speculative applications of TQ-homology to the algebraic K-theory of rings.

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