Math 35300: Section 002. Linear algebra II **Fall 2012**
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Handout 1

Recall from lecture that instead of calculating in $\mathbb{R}$ we may want to calculate in the field of complex numbers $\mathbb{C}$. This leads to the notion of a "complex vector space" which is defined analogously to a real vector space: one has only to replace $\mathbb{R}$ by $\mathbb{C}$ and "real" by "complex" in all instances.

Definition 1. A triple $(V, +, \cdot)$ consisting of a set V and two maps

$$V \times V \xrightarrow{+} V, \qquad (x, y) \longmapsto x + y \qquad (\text{"addition"})$$
$$\mathbb{C} \times V \xrightarrow{\cdot} V, \qquad (a, x) \longmapsto ax \qquad (\text{"scalar multiplication"})$$

is called a *complex vector space* if the following eight axioms hold for the maps $+$ and $\cdot$:

(1) $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$.
(2) $x + y = y + x$ for all $x, y \in V$.
(3) There exists an element $0 \in V$ with $x + 0 = x$ for all $x \in V$.
(4) For each $x \in V$ there exists an element $-x \in V$ with $x + (-x) = 0$.
(5) $a(bx) = (ab)x$ for all $a, b \in \mathbb{C}$, $x \in V$.
(6) $1x = x$ for all $x \in V$.
(7) $a(x + y) = ax + ay$ for all $a \in \mathbb{C}$, $x, y \in V$.
(8) $(a + b)x = ax + bx$ for all $a, b \in \mathbb{C}$, $x \in V$.

Notation. Instead of "complex vector space" one also says "vector space over $\mathbb{C}$".

Instead of calculating with only $\mathbb{R}$ or $\mathbb{C}$, one can work with any subfield of the complex numbers. A *subfield* of $\mathbb{C}$ is any subset which is closed under the four operations addition, subtraction, multiplication, and division, and which contains 1. In other words, a subset $\mathbb{F} \subset \mathbb{C}$ is a subfield of $\mathbb{C}$ if the following properties hold:

(a) $a + b \in \mathbb{F}$ for all $a, b \in \mathbb{F}$.
(b) If $a \in \mathbb{F}$, then $-a \in \mathbb{F}$.
(c) $ab \in \mathbb{F}$ for all $a, b \in \mathbb{F}$.
(d) If $a \in \mathbb{F}$ and $a \neq 0$, then $a^{-1} \in \mathbb{F}$.
(e) $1 \in \mathbb{F}$.

The following are examples of subfields of $\mathbb{C}$.

(i) $\mathbb{F} = \mathbb{R}$ the field of real numbers.
(ii) $\mathbb{F} = \mathbb{Q}$ the field of rational numbers.
(iii) $\mathbb{F} = \mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

Instead of working only with subfields of $\mathbb{C}$, one can list the properties of the "scalars" that are needed axiomatically. This leads to the abstract notion of a "field", which contains important new classes of fields including the finite fields. Axioms (1)–(9) below are modeled on calculation with real or complex numbers. In other words, they are essentially designed so that one can calculate in a field "exactly as" one would calculate in $\mathbb{R}$ or $\mathbb{C}$.

Definition 2. A *field* is a triple $(\mathbb{F}, +, \cdot)$ consisting of a set $\mathbb{F}$ and two operations

$$\mathbb{F} \times \mathbb{F} \xrightarrow{+} \mathbb{F}, \qquad (x,y) \longmapsto x+y \qquad (\text{“addition”})$$

$$\mathbb{F} \times \mathbb{F} \xrightarrow{\cdot} \mathbb{F}, \qquad (x,y) \longmapsto xy \qquad (\text{“multiplication”})$$

such that the following nine axioms hold:

(1) $(a+b)+c = a+(b+c)$ for all $a,b,c \in \mathbb{F}$.
(2) $a+b = b+a$ for all $a,b \in \mathbb{F}$.
(3) There exists an element $0 \in \mathbb{F}$ with $a+0 = a$ for all $a \in \mathbb{F}$.
(4) For each $a \in \mathbb{F}$ there exists an element $-a \in \mathbb{F}$ with $a+(-a) = 0$.
(5) $a(bc) = (ab)c$ for all $a,b,c \in \mathbb{F}$.
(6) $ab = ba$ for all $a,b \in \mathbb{F}$.
(7) There exists an element $1 \in \mathbb{F}$, $1 \neq 0$, such that $1a = a$ for all $a \in \mathbb{F}$.
(8) For all $a \in \mathbb{F}$ with $a \neq 0$ there exists an element $a^{-1} \in \mathbb{F}$ with $a^{-1}a = 1$.
(9) $a(b+c) = ab+ac$ for all $a,b,c \in \mathbb{F}$.

If $\mathbb{F}$ is any field, one defines the concept of a *vector space over* $\mathbb{F}$ analogously to the notion of a complex vector space above — replace $\mathbb{C}$ by $\mathbb{F}$ everywhere. It is probably better to once again write out the whole definition.

Definition 3. A triple $(V, +, \cdot)$ consisting of a set V and two maps

$$V \times V \xrightarrow{+} V, \qquad (x,y) \longmapsto x+y \qquad (\text{“addition”})$$

$$\mathbb{F} \times V \xrightarrow{\cdot} V, \qquad (a,x) \longmapsto ax \qquad (\text{“scalar multiplication”})$$

is called a *vector space over* $\mathbb{F}$ if the following eight axioms hold for the maps $+$ and $\cdot$:

(1) $(x+y)+z = x+(y+z)$ for all $x,y,z \in V$.
(2) $x+y = y+x$ for all $x,y \in V$.
(3) There exists an element $0 \in V$ with $x+0 = x$ for all $x \in V$.
(4) For each $x \in V$ there exists an element $-x \in V$ with $x+(-x) = 0$.
(5) $a(bx) = (ab)x$ for all $a,b \in \mathbb{F}$, $x \in V$.
(6) $1x = x$ for all $x \in V$.
(7) $a(x+y) = ax+ay$ for all $a \in \mathbb{F}$, $x,y \in V$.
(8) $(a+b)x = ax+bx$ for all $a,b \in \mathbb{F}$, $x \in V$.

Similarly, one defines the concept of a *vector subspace* of a vector space over $\mathbb{F}$ analogously to the notion of a vector subspace of a real vector space as we did in lecture — replace $\mathbb{R}$ by $\mathbb{F}$ everywhere. It is probably better to once again write out the whole definition.

Definition 4. Let V be a vector space over a field $\mathbb{F}$. A subset $U \subset V$ is called a *vector subspace* (or just *subspace* for short) of V if the following are satisfied:

(i) $U \neq \emptyset$.
(ii) For all $x,y \in U$ we have $x+y \in U$.
(iii) For all $a \in \mathbb{F}$ and $x \in U$ we have $ax \in U$.