Jordan Normal Form

The classification of $n \times n$ matrices up to similarity is not so simple.

We will merely state (rather than prove) the result, and then only for the case $\mathbb{F} := \mathbb{C}$. That is, only for complex $n \times n$ matrices.

(The theorem will at least give you a feel for the nature of complex endomorphisms.)

Definition: Let c be a complex number and $m \geq 1$. The $m \times m$ matrix

$$J_m(c) := \begin{bmatrix} c & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & c \end{bmatrix}$$

is called the <u>Jordan block</u> of degree m for the eigenvalue c.

<u>Note</u>: as an endomorphism of $\mathbb{C}^m$

$$\mathbb{C}^m \xrightarrow{J_m(c)} \mathbb{C}^m$$

the Jordan block $J_m(c)$ has only one eigenvalue c, and the dimension of the eigenspace is the <u>smallest</u> that an eigenspace can have

(i.e., $\dim E_c = 1$).

(For this reason, $J_m(c)$ is regarded as being as <u>nondiagonalizable</u> as a complex $m \times m$ matrix can be.)

<u>Theorem</u>: (Jordan Normal Form).

If A is a complex $n \times n$ matrix, and if $c_1, \ldots, c_r \in \mathbb{C}$ are its distinct eigenvalues, then for each $k = 1, \ldots, r$ there exist uniquely determined positive natural numbers n_k and

$$m_1^{(k)} \leq m_2^{(k)} \leq \cdots \leq m_{n_k}^{(k)}$$

with the property that there exists an invertible complex $n \times n$ matrix $\underline{P}$ such that $\underline{P}^{-1} A \underline{P}$ is the "block matrix" obtained by adjunction of the Jordan blocks

$$J_{m_1^{(1)}}(c_1), \dots, J_{m_{n_1}^{(1)}}(c_1), \dots, J_{m_1^{(r)}}(c_r), \dots, J_{m_{n_r}^{(r)}}(c_r)$$

along the diagonal.

Thus the k-th eigenvalue c_k contributes a smaller block matrix B_k.

built up from n_k single Jordan blocks:

$$B_k := \begin{bmatrix} \boxed{\begin{matrix} c_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & c_k \end{matrix}} & & & \\ & \boxed{\begin{matrix} c_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & c_k \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix} c_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & c_k \end{matrix}} \end{bmatrix} \begin{matrix} \}\, m_1^{(k)} \\ \\ \}\, m_2^{(k)} \\ \vdots \\ \}\, m_{n_k}^{(k)} \end{matrix}$$

The whole Jordan normal form of A is then:

$$\begin{array}{cl} \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{pmatrix} & \begin{array}{l} \} \sum_{i=1}^{n_1} m_i^{(1)} \\ \} \sum_{i=1}^{n_2} m_i^{(2)} \\ \vdots \\ \} \sum_{i=1}^{n_r} m_i^{(r)} \end{array} \end{array}$$

(Note: Only if each Jordan block has dimension 1 is A diagonalizable.)