

Jordan Normal Form

The classification of $n \times n$ matrices up to similarity is not so simple.

We will merely state (rather than prove) the result, and then only for the case $F := \mathbb{C}$. That is, only for complex $n \times n$ matrices.

(The theorem will at least give you a feel for the nature of complex endomorphisms.)

Definition: Let c be a complex number and $m \geq 1$. The $m \times m$ matrix

$$J_m(c) := \begin{bmatrix} c & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & c \end{bmatrix}$$

is called the Jordan block of degree m for the eigenvalue c .

Def: as an endomorphism of \mathbb{C}^m

$$\mathbb{C}^m \xrightarrow{J_m(c)} \mathbb{C}^m$$

The Jordan block $J_m(c)$ has only one eigenvalue c , and the dimension of the eigenspace is the smallest that an eigenspace can have

$$(i.e., \dim E_c = 1).$$

For this reason, $J_m(c)$ is regarded as being as non-diagonalizable as a complex $n \times n$ matrix can be.

Theorem : (Jordan Normal Form).

If A is a complex $n \times n$ matrix, and if $c_1, \dots, c_r \in \mathbb{C}$ are its distinct eigenvalues, then for each $k=1, \dots, r$ there exist uniquely determined positive natural numbers n_k and

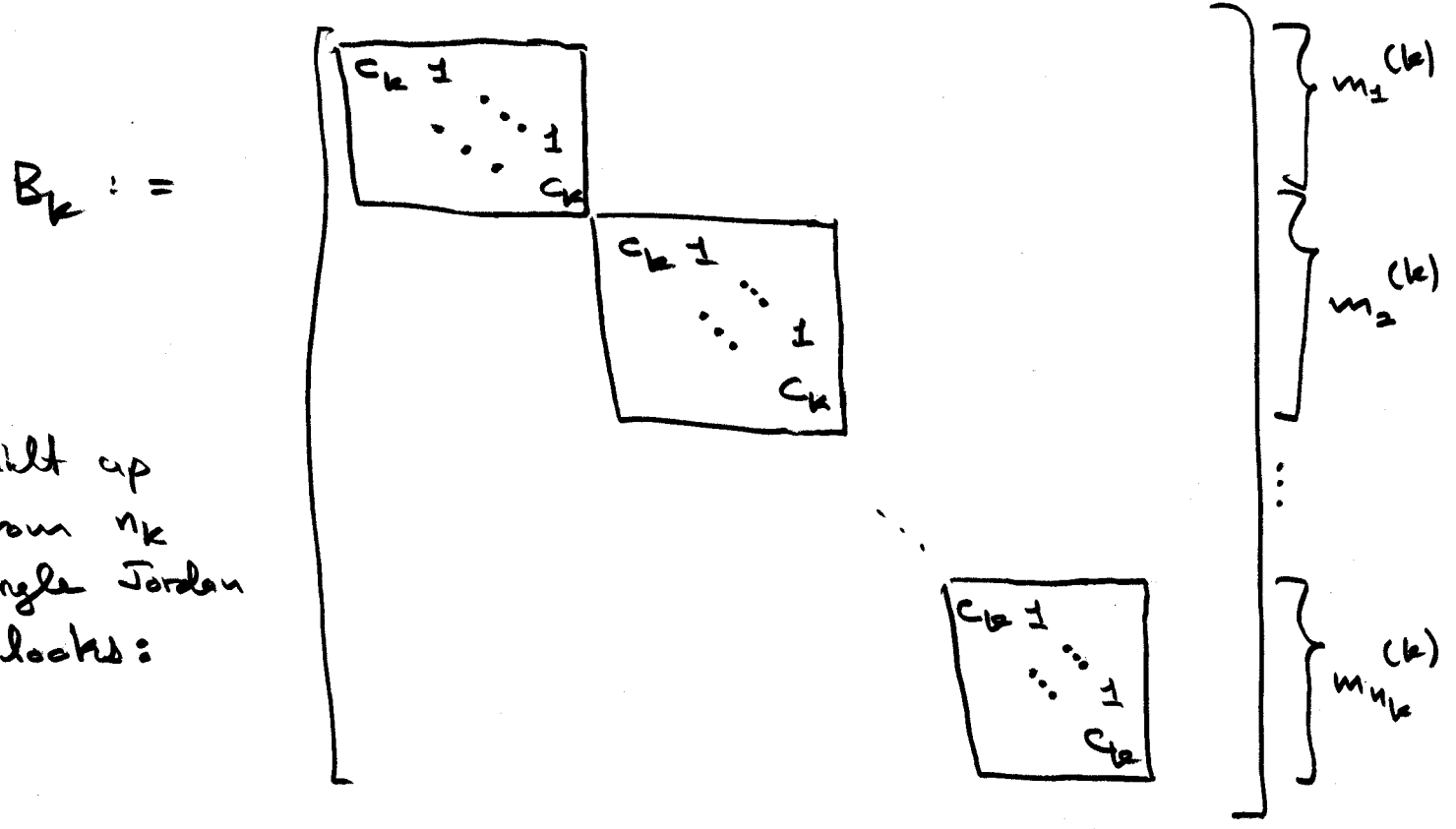
$$m_1^{(k)} \leq m_2^{(k)} \leq \dots \leq m_{n_k}^{(k)}$$

with the property that there exists an invertible complex non matrix P such that $P^{-1}AP$ is the "block matrix" obtained by adjunction of the Jordan blocks

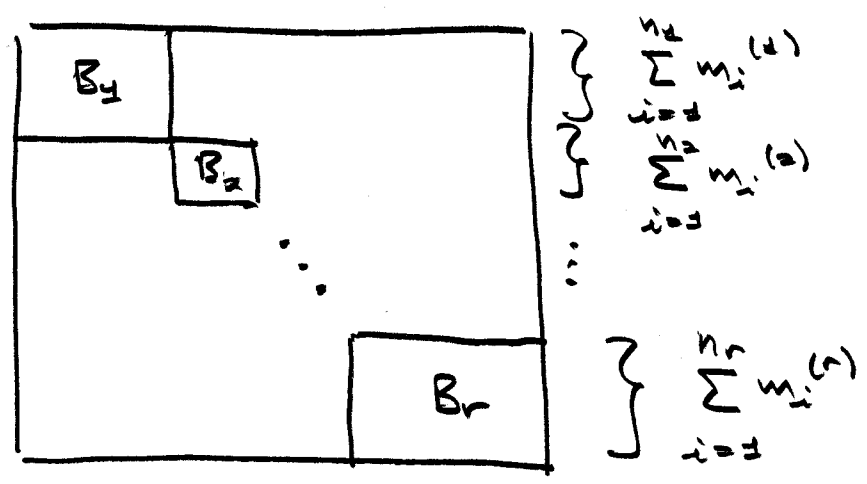
$$J_{m_1(c_1)}(c_1), \dots, J_{m_{n_1}(c_1)}(c_1), \dots, J_{m_1(c_r)}(c_r), \dots, J_{m_{n_r}(c_r)}(c_r)$$

along the diagonal.

Thus the k -th eigenvalue c_k contributes a smaller block matrix B_k



The whole Jordan normal form of A is then :



Note: Only if each Jordan block has dimension 1 is A diagonalizable.