Math 35300: Section 002. Linear algebra II **Fall 2012**

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Homework 10

Exercises 1–10 should be regarded as warm-up exercises. They are intended to test your understanding of some of the definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction.

Exercise 1. An endomorphism $f\colon V \longrightarrow V$ of a Euclidean vector space V is said to be *self-adjoint* (or *symmetric*) if, for all $v, w \in V$, we have

(a) $\langle f(v), f(w)\rangle = \langle v, w\rangle$.
(b) $\langle v, f(w)\rangle = \langle f(v), w\rangle$
(c) $\langle f(v), w\rangle = \langle w, f(v)\rangle$

Exercise 2. If $c_1, \dots, c_r$ are eigenvalues of a self-adjoint endomorphism, $c_i \neq c_j$ for $i \neq j$, and v_i is an eigenvector for c_i, $i = 1, \dots r$, then for $i \neq j$

(a) $c_i \perp c_j$
(b) $v_i \perp v_j$
(c) $E_{c_i} \perp E_{c_j}$

Exercise 3. Let V be a finite-dimensional Euclidean vector space. The assertion that for each f-invariant subspace $U \subset V$ also $U^\perp \subset V$ is invariant under f holds

(a) for each self-adjoint endomorphism $f\colon V \longrightarrow V$
(b) for each orthogonal endomorphism $f\colon V \longrightarrow V$
(c) for each endomorphism $f\colon V \longrightarrow V$

Exercise 4. Let A be a real $n \times n$ matrix and $z \in \mathbb{C}^n$ a complex eigenvector, $z = x + iy$ with $x, y \in \mathbb{R}^n$, for the real eigenvalue c. Suppose that $y \neq 0$. Then

(a) $y \in \mathbb{R}^n$ is an eigenvector of A for the eigenvalue c.
(b) $y \in \mathbb{R}^n$ is an eigenvector of A for the eigenvalue ic.
(c) if $x \neq 0$, then $y \in \mathbb{R}^n$ *cannot* be an eigenvector of A.

Exercise 5. Let V be an n-dimensional Euclidean vector space and $U \subset V$ be a k-dimensional subspace. When is the orthogonal projection $P\colon V \longrightarrow U \subset V$ self-adjoint?

(a) always
(b) only for $0 < k \leq n$
(c) only for $0 \leq k < n$

Exercise 6. Does there exist an inner product on $\mathbb{R}^2$ for which the shear is self-adjoint?

(a) No, because $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.
(b) Yes, let $\langle x, y\rangle := x_1y_1 + x_1y_2 + x_2y_2$.
(c) Yes, because the standard inner product already has this property.

Exercise 7. Let $f\colon V \longrightarrow V$ be a self-adjoint endomorphism and let $(v_1, \dots, v_n)$ be a basis of eigenvectors with $\|v_i\| = 1$ for $i = 1, \dots, n$. Is $(v_1, \dots, v_n)$ then already an orthonormal basis?

(a) Yes, by definition of an orthonormal basis.
(b) Yes, because the eigenvectors of a self-adjoint endomorphism are orthogonal to each other.
(c) No, because the eigenspaces do not need to be one-dimensional.

Exercise 8. If a symmetric real $n \times n$ matrix A has only one eigenvalue c, then

(a) A is already diagonal.
(b) $a_{ij} = c$ for all $i, j = 1, \dots, n$.
(c) $n = 1$.

Exercise 9. How does the Jordan normal form of $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ look?

(a) $\begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & & \\ & 2 & 1 \\ & & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}$

Exercise 10. Is the Jordan normal form of a real symmetric $n \times n$ matrix always diagonal?

(a) Yes, because it can be brought into diagonal form by means of the Spectral Theorem.
(b) No, because a symmetric matrix can also have fewer than n distinct eigenvalues. The argument in favor of the answer "Yes" is unsound, since $O(n) \neq GL(n, \mathbb{C})$.
(c) The question has no meaning and therefore does not deserve an answer, since the Jordan normal form theorem is stated not for real but for *complex* $n \times n$ matrices.

Exercise 11. Consider the symmetric matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Find an orthogonal matrix $P \in O(3)$ so that PAP^t is diagonal.

Exercise 12. Prove the following proposition from lecture.

Proposition 1. *Let $(V, \langle \ , \ \rangle)$ be a Euclidean vector space.*

(a) *If $(v_1, \dots, v_n)$ is an orthonormal basis of V, then the corresponding matrix A of an endomorphism $f \colon V \longrightarrow V$ is given by*

$$a_{ij} = \langle f(v_j), v_i \rangle.$$

(b) *If $(v_1, \dots, v_n)$ is an orthonormal basis of V, then an endomorphism $f \colon V \longrightarrow V$ is self-adjoint if and only if the corresponding matrix A is symmetric.*

Exercise 13. Prove the following proposition. (Hint: this may be argued similar to the proof of the Spectral Theorem given in lecture).

Proposition 2.

(a) *(Vector space form): Let $f \colon V \longrightarrow V$ be an endomorphism of a finite-dimensional complex vector space V. Then there is a basis $\mathbf{B}$ of V such that the corresponding matrix A of f is upper triangular (i.e., all entries below the diagonal are zero).*

(b) *(Matrix form): Let A be a complex $n \times n$ matrix. Then there is a matrix $P \in GL(n, \mathbb{C})$ such that PAP^{-1} is upper triangular.*

Exercise 14. Use the Spectral Theorem proved in lecture to give a short proof of the following proposition. (Hint: Even a one line proof can be given).

Proposition 3. *Let $f\colon V \longrightarrow V$ be a self-adjoint endomorphism on an n-dimensional ($n \geq 1$) Euclidean vector space. Then there exists an orthogonal map*

$$P\colon \mathbb{R}^n \xrightarrow{\cong} V$$

so that the matrix of f with respect to P has the form

$$\begin{bmatrix} c_1 & & & & & & \\ & \ddots & & & & & \\ & & c_1 & & & & \\ & & & \ddots & & & \\ & & & & c_r & & \\ & & & & & \ddots & \\ & & & & & & c_r \end{bmatrix}$$

of the indicated diagonal matrix. Here $c_1, \ldots, c_r$ are the distinct eigenvalues of f, the number of each appearing on the diagonal being equal to the geometric multiplicity.

Exercise 15. Let $f\colon V \longrightarrow V$ be a self-adjoint endomorphism on a Euclidean vector space V. Prove the following: If v, w are eigenvectors of f corresponding to distinct eigenvalues $c \neq d$, then $v \perp w$.

Exercise 16. Use the Spectral Theorem proved in lecture to give a proof of the following proposition. (Hint: Exercise 15 should also be helpful).

Proposition 4. *Let $f\colon V \longrightarrow V$ be a self-adjoint endomorphism of a finite-dimensional Euclidean vector space, $c_1, \ldots, c_r$ its distinct eigenvalues, and $P_k\colon V \longrightarrow E_{c_k} \subset V$ the orthogonal projection onto the eigenspace E_{c_k}. Then*

$$f = c_1 P_1 + \cdots + c_r P_r.$$

Exercise 17. Let V be a finite-dimensional real vector space. Show that an endomorphism $f\colon V \longrightarrow V$ is diagonalizable if and only if there exists an inner product $\langle\ ,\ \rangle$ on V for which f is self-adjoint.

Exercise 18. Let V be a finite-dimensional Euclidean vector space and $U \subset V$ a subspace. Show that the orthogonal projection $P\colon V \longrightarrow U \subset V$ is self-adjoint, and determine its eigenvalues and eigenspaces.

Exercise 19. Let V be a finite-dimensional Euclidean vector space. Show that two self-adjoint endomorphisms $f, g\colon V \longrightarrow V$ can be diagonalized by the same orthogonal map $P\colon \mathbb{R}^n \xrightarrow{\cong} V$ if and only if they *commute* (i.e., $f \circ g = g \circ f$).