Math 35300: Section 002. Linear algebra II **Fall 2012**
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Homework 3

Exercises 1–10 should be regarded as warm-up exercises. They are intended to test your understanding of some of the definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction.

Exercise 1. For which of the following objects does the description "linearly dependent" or "linearly independent" make sense?

(a) An n-tuple $(v_1, \ldots, v_n)$ of elements of a vector space
(b) An n-tuple $(v_1, \ldots, v_n)$ of real vector spaces
(c) A linear combination $c_1v_1 + \cdots + c_nv_n$

Exercise 2. Let V be a vector space over a field $\mathbb{F}$. Let $v_1, \ldots, v_n \in V$. What does $\mathrm{Span}(v_1, \ldots, v_n) = V$ mean?

(a) Each linear combination $c_1v_1 + \cdots + c_nv_n$ is an element of V.
(b) Each element of V is a linear combination $c_1v_1 + \cdots + c_nv_n$.
(c) The dimension of V is n.

Exercise 3. Let V be a vector space over a field $\mathbb{F}$. For linearly independent triples (v_1, v_2, v_3) of vectors in V,

(a) (v_1, v_2) is always linearly dependent.
(b) (v_1, v_2) may or may not be linearly dependent, depending on the choice of (v_1, v_2, v_3).
(c) (v_1, v_2) is always linearly independent.

Exercise 4. Let $\mathbb{F}$ be a field. The i-th vector of the canonical basis of $\mathbb{F}^n$ is defined by

(a) $e_i = (0, \cdots, i, \cdots, 0)$.
(b) $e_i = (0, \cdots, 1, \cdots, 0)$.
(c) $e_i = (1, \cdots, 1, \cdots, 0)$.

Exercise 5. Let V be a vector space over a field $\mathbb{F}$. Which of the following statements implies the linear independence of the n-tuple $(v_1, \ldots, v_n)$ of elements of V?

(a) $c_1v_1 + \cdots c_nv_n = 0$ only if $c_1 = c_2 = \cdots = c_n = 0$.
(b) If $c_1 = \cdots = c_n = 0$, then $c_1v_1 + \cdots + c_nv_n = 0$.
(c) $c_1v_1 + \cdots + c_nv_n = 0$ for all $(v_1, \ldots, v_n) \in \mathbb{F}^n$.

Recall from lecture the following proposition.

Proposition 1. *Let V be a finite dimensional vector space over a field $\mathbb{F}$. Any linearly independent finite ordered set $L \subset V$ can be extended by adding elements, to get a basis.*

Exercise 6. What does Proposition 1 and its proof imply in the case that $L = \emptyset$?

(a) If $(w_1, \cdots, w_s)$ is an s-tuple of vectors in V and $\mathrm{Span}(w_1, \cdots, w_s) = V$, then one can extend $w_1, \cdots, w_s$ to a basis.

(b) If $(w_1, \cdots, w_s)$ is a linearly independent s-tuple of vectors in V, then there exists a basis consisting of vectors from $(w_1, \cdots, w_s)$.
(c) If $(w_1, \cdots, w_s)$ is an s-tuple of vectors in V and $\mathrm{Span}(w_1, \cdots, w_s) = V$, then there exists a basis consisting of vectors from $(w_1, \cdots, w_s)$.

Exercise 7. The vector space $V = \{0\}$ over a field $\mathbb{F}$ consisting of the zero element

(a) has the basis (0).
(b) has the basis $\emptyset$.
(c) has no basis.

Exercise 8. Let V be a vector space over a field $\mathbb{F}$. If one were to define

$$U_1 - U_2 := \{x - y \mid x \in U_1, y \in U_2\}$$

for subspaces U_1, U_2 of V, then one would have

(a) $U_1 - U_1 = \{0\}$.
(b) $(U_1 - U_2) + U_2 = U_1$.
(c) $U_1 - U_2 = U_1 + U_2$.

Exercise 9. Let V be a vector space over a field $\mathbb{F}$. For subspaces U_1, U_2 of V, one always has

(a) $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$.
(b) $U_1 \cap (U_2 + U_3) = (U_1 \cap U_2) + (U_1 \cap U_3)$.
(c) $U_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap (U_1 + U_3)$.

Exercise 10. Let V be a vector space over a field $\mathbb{F}$. Subspaces U_1, U_2 of V are said to be *transverse* (to each other) if $U_1 + U_2 = V$. One calls $\mathrm{codim}\, U := \dim V - \dim U$ the *codimension* of U in V. For transverse U_1, U_2, one has

(a) $\dim U_1 + \dim U_2 = \dim U_1 \cap U_2$.
(b) $\dim U_1 + \dim U_2 = \mathrm{codim}\, U_1 \cap U_2$.
(c) $\mathrm{codim}\, U_1 + \mathrm{codim}\, U_2 = \mathrm{codim}\, U_1 \cap U_2$.

Definition 2. Let V be a vector space over a field $\mathbb{F}$ and suppose $S \subset V$ is a subset. A subspace U of V is called the *smallest subspace of V containing S* if (i) $U \supset S$ and (ii) if W is a subspace of V and $W \supset S$, then $W \supset U$.

Condition (i) is read as "U contains S" and condition (ii) is read as "if W is a subspace of V and W contains S, then W contains U."

Exercise 11. Let V be a vector space over a field $\mathbb{F}$. Prove the following.

(a) If $v_1, \ldots, v_r$ are elements of V, then $\mathrm{Span}(v_1, \ldots, v_r)$ is the smallest subspace of V containing $v_1, \ldots, v_r$. (See Definition 2.)
(b) The span of $v_1, \ldots, v_r$ is the same as the span of any reordering of $v_1, \ldots, v_r$.

Exercise 12. Let V be a vector space over a field $\mathbb{F}$. Prove the following.

(a) Any reordering of a linearly independent r-tuple of vectors $(v_1, \ldots, v_r)$ is linearly independent.
(b) An r-tuple of vectors $(v_1, \ldots, v_r)$ is linearly independent if and only if none of these vectors is a linear combination of the others.

Exercise 13. Let V be a finite dimensional vector space over a field $\mathbb{F}$.

(a) Show that any subset of a linearly independent set is linearly independent.
(b) Show that any reordering of a basis is also a basis.

Exercise 14. Prove Proposition 3. (This proposition is a more precise—but perhaps less attractive—reformulation of Proposition 1 and its proof. A good starting point for this exercise would be to study the proof of Proposition 1 given in lecture.)

Proposition 3. *Let V be a vector space over a field $\mathbb{F}$ and let $v_1, \ldots, v_r$, $w_1, \ldots, w_s$ be vectors of V. If $(v_1, \ldots, v_r)$ is linearly independent and $\operatorname{Span}(v_1, \ldots, v_r, w_1, \ldots, w_s) = V$, then by suitably chosen vectors from $(w_1, \ldots, w_s)$ one can extend $(v_1, \ldots, v_r)$ to a basis of V.*

Exercise 15. Let V be a vector space of dimension n over a field $\mathbb{F}$, and let $0 \leq r \leq n$. Prove that V contains a subspace of dimension r.

Exercise 16. Prove Proposition 4.

Proposition 4. *Let V be a vector space over a field $\mathbb{F}$. If $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ are bases of V, then for each v_i there exists some w_j, so that on replacing v_i by w_j in $(v_1, \ldots, v_n)$ we still have a basis.*

Exercise 17. Let V be a real vector space and $a, b, c, d \in V$. Suppose that

$$\begin{aligned}
v_1 &= a + b + c + d \\
v_2 &= 2a + 2b + c - d \\
v_3 &= a + b + 3c - d \\
v_4 &= a - c + d \\
v_5 &= -b + c - d
\end{aligned}$$

Show that $(v_1, \ldots, v_5)$ is linearly dependent. (Remark: One can solve this exercise by expressing one of the v_i as a linear combination of the other four. But there is a proof in which one does not need to do any calculations.)

Definition 5. Let V be a vector space over a field $\mathbb{F}$ and let U_1, U_2 be subspaces of V. We say that U_1 and U_2 are *complementary subspaces* if $U_1 + U_2 = V$ and $U_1 \cap U_2 = \{0\}$.

For instance, consider the real vector space $V = \mathbb{R}^3$. It is easy to check that (i) $U_1 = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$ and $U_2 = \{x \in \mathbb{R}^3 \mid x_1 = 0, x_2 = 0\}$ are complementary to each other and (ii) $U_1 = V$ and $U_2 = \{0\}$ are complementary to each other. The following exercise shows that there are many other examples.

Exercise 18. Let V be a vector space of dimension n over a field $\mathbb{F}$. Show that if U_1 is a subspace of dimension p, then there exists a subspace U_2 complementary to U_1, and each such subspace U_2 has dimension $n - p$.

Given a complex vector space V one can make a real vector space from it by simply restricting the scalar multiplication $\mathbb{C} \times V \longrightarrow V$ to $\mathbb{R} \times V \longrightarrow V$. Since on restriction the concepts "span" and "dimension" take on a new meaning, we sometimes write $\operatorname{Span}_{\mathbb{C}}$ and $\dim_{\mathbb{C}}$ (resp. $\operatorname{Span}_{\mathbb{R}}$ and $\dim_{\mathbb{R}}$), when regarding V as a complex (resp. real) vector space.

Exercise 19. For each $n \geq 0$ determine for which pairs (r, s) of numbers there exists a complex vector space and vectors $(v_1, \ldots, v_n)$ in it, such that

$$\begin{aligned}
r &= \dim_{\mathbb{R}} \operatorname{Span}_{\mathbb{C}}(v_1, \ldots, v_n), \\
s &= \dim_{\mathbb{R}} \operatorname{Span}_{\mathbb{R}}(v_1, \ldots, v_n).
\end{aligned}$$

Exercise 20. Let V be a finite dimensional vector space over a field $\mathbb{F}$, and let U_1, U_2 be subspaces of V. The formula $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$ is analogous to the formula $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$, which holds for sets. If three sets are given, then

$$\begin{aligned} |S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| \\ - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|. \end{aligned}$$

Does the corresponding formula for dimensions of subspaces hold?