Math 35300: Section 002. Linear algebra II **Fall 2012**
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Homework 4

Exercises 1–10 should be regarded as warm-up exercises. They are intended to test your understanding of some of the definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction.

Exercise 1. A map $f\colon V \longrightarrow W$ between vector spaces V and W over a field $\mathbb{F}$ is linear, if

(a) $f(ax+by) = af(x) + bf(y)$ for all $x, y \in V$, $a, b \in \mathbb{F}$.
(b) f satisfies the eight axioms for a vector space.
(c) $f\colon V \longrightarrow W$ is bijective.

Exercise 2. By the kernel of a linear map $f\colon V \longrightarrow W$ one understands

(a) $\{w \in W \mid f(0) = w\}$
(b) $\{f(v) \mid v = 0\}$
(c) $\{v \in V \mid f(v) = 0\}$

Exercise 3. Which of the following statements are correct? If $f\colon V \longrightarrow W$ is a linear map, we have

(a) $f(0) = 0$.
(b) $f(-x) = -x$ for all $x \in V$.
(c) $f(av) = f(a) + f(v)$ for all $a \in \mathbb{F}$, $v \in V$.

Exercise 4. A linear map $f\colon V \longrightarrow W$ is called an isomorphism if

(a) there exists a linear map $g\colon W \longrightarrow V$ with $fg = \mathrm{Id}_W$ and $gf = \mathrm{Id}_V$.
(b) V and W are isomorphic.
(c) for each n-tuple $(v_1, \ldots, v_n)$ of vectors in V, the n-tuple $(f(v_1), \ldots, f(v_n))$ is a basis of W.

Exercise 5. By the rank $\mathrm{rk}(f)$ of a linear map $f\colon V \longrightarrow W$, one understands

(a) $\dim \operatorname{Ker} f$ (b) $\dim \operatorname{Im} f$ (c) $\dim W$

Exercise 6. $\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} =$

(a) $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Exercise 7. The map $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x, y) \longmapsto (x+y, x-y)$, is given by the following matrix ("The columns are the ..."):

(a) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Exercise 8. Let V and W be vector spaces over a field $\mathbb{F}$ with bases (v_1, v_2, v_3) and (w_1, w_2, w_3), respectively, and let $f\colon V \longrightarrow W$ be the linear map with $f(v_i) = w_i$. Then the "associated" matrix is

$$\text{(a)} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \text{(b)} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(c)} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise 9. A linear map $f\colon V \longrightarrow W$ is injective if and only if

(a) f is surjective. (b) $\dim \operatorname{Ker} f = 0$. (c) $\operatorname{rk} f = 0$.

Exercise 10. Let $f\colon V \longrightarrow W$ be a surjective linear map and $\dim V = 5$, $\dim W = 3$. Then

(a) $\dim \operatorname{Ker} f \geq 3$.
(b) $\dim \operatorname{Ker} f$ is 0, 1, or 2 and each of these cases can arise.
(c) $\dim \operatorname{Ker} f = 2$.

Exercise 11. Let V and W be vector spaces over a field $\mathbb{F}$, let $(v_1, \dots, v_n)$ be a basis of V, and let $f\colon V \longrightarrow W$ be a linear map. Show that f is injective if and only if $(f(v_1), \dots, f(v_n))$ is linearly independent.

We can define the notion of a *polynomial* with coefficients in a field $\mathbb{F}$ to mean a linear combination of powers of the variable (or indeterminate):

$$f(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_1 t + a_0, \tag{1}$$

where $a_i \in \mathbb{F}$. Such expressions are sometimes called *formal polynomials*, to distinguish them from polynomial functions. Every formal polynomial with coefficients in $\mathbb{F}$ determines a polynomial function on $\mathbb{F}$. The variable appearing in (1) is an arbitrary symbol, and the monomials t^i are considered linearly independent. This means that if

$$g(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0$$

is a polynomial with coefficients in $\mathbb{F}$, then $f(t)$ and $g(t)$ are equal if and only if $a_i = b_i$ for all $i = 0, 1, 2, \dots$. Sometimes it is useful to write a polynomial in the standard form

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots , \tag{2}$$

where the coefficients a_i are all in the field $\mathbb{F}$ and *only finitely many of the coefficients are different from zero*. Formally, the polynomial (2) is determined by its sequence of coefficients a_i:

$$a = (a_0, a_1, a_2, \dots),$$

where $a_i \in \mathbb{F}$ and all but a finite number of a_i are zero. Every such sequence corresponds to a polynomial.

Addition and multiplication of polynomials mimic the familiar operations on polynomial functions. Let $f(t)$ be as in (2), and let

$$g(t) = b_0 + b_1 t + b_2 t^2 + \cdots , \tag{3}$$

be a polynomial with coefficients in the same field $\mathbb{F}$, determined by the sequence $b = (b_0, b_1, b_2, \dots)$. The *sum* of f and g is

$$\begin{aligned} f(t) + g(t) :=& (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots \\ =& \sum_k (a_k + b_k)t^k, \end{aligned}$$

which corresponds to addition of sequences: $a + b = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots)$. The *product* of f and g is computed by multiplying term by term and collecting coefficients of the same degree in t. If we expand the product using the distributive law, but without collecting terms, we obtain

$$f(t)g(t) = \sum_{i,j} a_i b_j t^{i+j}.$$

Note that there are only finitely many nonzero coefficients $a_i b_j$. The right-hand side is not in standard form since the same monomial t^n appears many times—once for each pair (i, j) of indices such that $i + j = n$. Putting the right-hand side back into standard form (by collecting terms) leads to the definition

$$f(t)g(t) := p_0 + p_1 t + p_2 t^2 + \cdots ,$$

where

$$p_k := a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0 = \sum_{i+j=k} a_i b_j.$$

Exercise 12. Let $\mathbb{F}$ be a field and $\mathcal{P}_n = \{a_0 + a_1 t + \cdots + a_n t^n \mid a_i \in \mathbb{F}\}$ be the vector space of polynomials in the indeterminate t of degree $\leq n$ with coefficients in $\mathbb{F}$. If $f(t) \in \mathcal{P}_m$ and $g(t) \in \mathcal{P}_n$, the product $f(t)g(t) \in \mathcal{P}_{m+n}$ is defined as above. We call $(1, t, \dots, t^m)$ the canonical basis of $\mathcal{P}_m$. Determine the matrix of the linear map

$$\mathcal{P}_3 \longrightarrow \mathcal{P}_4, \qquad f(t) \longmapsto (2 - t)f(t)$$

relative to the canonical bases.

Exercise 13. By a *finite chain complex* C of vector spaces over a field $\mathbb{F}$ one understands a sequence of homomorphisms

$$0 \xrightarrow{f_{n+1}} V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \xrightarrow{f_0} 0$$

of vector spaces over $\mathbb{F}$ with the property that $f_i f_{i+1} = 0$ for each i; i.e., such that $\operatorname{Ker} f_i \supset \operatorname{Im} f_{i+1}$. The quotient vector space $H_i(C) := \operatorname{Ker} f_i / \operatorname{Im} f_{i+1}$ is called the i-th *homology group* of the complex. Show that if all the V_i are finite-dimensional, then

$$\sum_{i=0}^{n} (-1)^i \dim V_i = \sum_{i=0}^{n} (-1)^i \dim H_i(C).$$

Exercise 14. Consider the following commutative diagram of homomorphisms of vector spaces over a field $\mathbb{F}$.

$$\begin{tikzcd} V_4 \arrow[r, "f_4"] \arrow[d, "\varphi_4", "\text{epi.}"'] & V_3 \arrow[r, "f_3"] \arrow[d, "\varphi_3", "\cong"'] & V_2 \arrow[r, "f_2"] \arrow[d, "\varphi_2"] & V_1 \arrow[r, "f_1"] \arrow[d, "\varphi_1", "\cong"'] & V_0 \arrow[d, "\varphi_0", "\text{mono.}"'] \\ W_4 \arrow[r, "g_4"'] & W_3 \arrow[r, "g_3"'] & W_2 \arrow[r, "g_2"'] & W_1 \arrow[r, "g_1"'] & W_0 \end{tikzcd}$$

Assume that the rows are *exact*, i.e., $\operatorname{Ker} f_i = \operatorname{Im} f_{i+1}$ and $\operatorname{Ker} g_i = \operatorname{Im} g_{i+1}$ for $i = 1, 2, 3$, and suppose furthermore that the vertical homomorphisms have the indicated properties; i.e., φ_4 is an epimorphism, φ_3 and φ_1 are isomorphisms, and φ_0 is a monomorphism. Show that under these conditions φ_2 is an isomorphism.