**Math 35300: Section 002. Linear algebra II** **Fall 2012**
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**Homework 5**

Exercises 1–10 should be regarded as warm-up exercises. They are intended to test your understanding of some of the definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction.

**Exercise 1.** Let $A \in \mathrm{M}(2 \times 3, \mathbb{F})$, $B \in \mathrm{M}(2 \times 3, \mathbb{F})$. Then

(a) $A + B \in \mathrm{M}(2 \times 3, \mathbb{F})$.
(b) $A + B \in \mathrm{M}(4 \times 6, \mathbb{F})$.
(c) $A + B \in \mathrm{M}(4 \times 9, \mathbb{F})$.

**Exercise 2.** For which of the following $3 \times 3$ matrices $A$ do we have $AB = BA = B$ for all $B \in \mathrm{M}(3 \times 3, \mathbb{F})$?

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

**Exercise 3.** For $A \in \mathrm{M}(m \times n, \mathbb{F})$, we have

(a) $A$ has $m$ rows and $n$ columns.
(b) $A$ has $n$ rows and $m$ columns.
(c) The rows of $A$ have length $m$ and the columns of $A$ have length $n$.

**Exercise 4.** Which of the following matrix products is zero?

$$\text{(a)} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

**Exercise 5.** Which of the following properties does matrix multiplication lack?

(a) associativity (b) commutativity (c) distributivity

**Exercise 6.** For $A \in \mathrm{M}(n \times n, \mathbb{F})$ we have:

(a) $\operatorname{rk} A = n \Rightarrow A$ is invertible, but there exist invertible matrices with $\operatorname{rk} A \neq n$.
(b) $A$ is invertible $\Rightarrow \operatorname{rk} A = n$, but there exist matrices $A$ with $\operatorname{rk} A = n$, which are not invertible.
(c) $\operatorname{rk} A = n \Leftrightarrow A$ is invertible.

**Exercise 7.** Let $A \in \mathrm{M}(m \times n, \mathbb{F})$, $B \in \mathrm{M}(n \times m, \mathbb{F})$, so that we have

$$\mathbb{F}^n \xrightarrow{A} \mathbb{F}^m \xrightarrow{B} \mathbb{F}^n.$$

Let $BA = I$ (which equals $\mathrm{Id}_{\mathbb{F}^n}$ as a linear map). Then

(a) $m \geq n$, $A$ injective, $B$ surjective.
(b) $m \leq n$, $A$ surjective, $B$ injective.
(c) $m = n$, $A$ and $B$ invertible (bijective).

**Exercise 8.** For $A \in \mathrm{M}(m \times n, \mathbb{F})$ with $m \leq n$, we always have

(a) $\operatorname{rk} A \leq m$ (b) $m \leq \operatorname{rk} A \leq n$ (c) $n \leq \operatorname{rk} A$

**Exercise 9.** Which of the following properties of an equivalence relation is (are) not fulfilled for the relation on $\mathbb{R}$ defined by $x \leq y$?

(a) reflexivity (b) symmetry (c) transitivity

**Exercise 10.** We define an equivalence relation on $\mathbb{Z}$ by

$$n \sim m \Leftrightarrow n - m \text{ is even.}$$

How many elements are there in $\mathbb{Z}/\!\sim$ (the set of equivalence classes)?

**Exercise 11.** Show that if $A, B \in \mathrm{M}(n \times n, \mathbb{F})$ then

$$\operatorname{rk} A + \operatorname{rk} B - n \leq \operatorname{rk} AB \leq \min(\operatorname{rk} A, \operatorname{rk} B).$$

Hint: use the dimension formula for linear maps.

**Exercise 12.** Let $V$ be a finite-dimensional vector space over $\mathbb{F}$ and $f\colon V \longrightarrow V$ an endomorphism. Show that if with respect to all bases $f$ is represented by the same matrix $A$; i.e., $A = \Phi^{-1} f \Phi$ for all isomorphisms $\Phi\colon \mathbb{F}^n \longrightarrow V$, then there exists some $c \in \mathbb{F}$ with $f = c(\mathrm{Id})$.

**Exercise 13.** Let $S$ be a nonempty set. The purpose of this exercise is to verify that the notions of a partition on $S$ and an equivalence relation on $S$ are logically equivalent. Given a partition $P$ on $S$, we can define an equivalence relation $R$ by the rule $a \sim b$ if $a$ and $b$ lie in the same subset of the partition.

(a) Show that $\sim$ satisfies the axioms of an equivalence relation on $S$.

Conversely, given an equivalence relation $R$ on $S$, we can define a partition $P$ as follows: The subset containing $a$ is the set of all elements $b$ such that $a \sim b$. This subset is the *equivalence class* of $a$. We want to show that $S$ is partitioned into equivalence classes. Denote by $C_a$ the equivalence class of $a$; i.e.,

$$C_a := \{b \in S \mid a \sim b\}.$$

It suffices to show that $S$ is the disjoint union of these equivalence classes $C_a$, and that each equivalence class $C_a$ is nonempty. (Careful: the partition consists of the subsets, not of the notations.)

(b) Show that the classes $C_a$ are nonempty.
(c) Prove the following: If $C_a$ and $C_b$ have an element $d$ in common, then $C_a = C_b$. Conclude that the equivalence classes do not overlap.

**Exercise 14.** Let $S$ be a nonempty set. The purpose of this exercise is to verify that any map of sets $\varphi\colon S \longrightarrow T$ defines an equivalence relation on the domain $S$, namely the relation given by the rule $a \sim b$ if $\varphi(a) = \varphi(b)$.

(a) Show that $\sim$ satisfies the axioms of an equivalence relation on $S$.

The corresponding partition is made up of the nonempty inverse images of the elements of $T$. In more detail, if $t \in T$, then the *fiber* of $t$ is

$$\varphi^{-1}(t) := \{s \in S \mid \varphi(s) = t\}.$$

the inverse image of $t$.

(b) Prove that the nonempty fibers of a map $\varphi\colon S\longrightarrow T$ form a partition of the domain. Denote by $\overline{S}$ the set of equivalence classes, which is the set of nonempty fibers of the map $\varphi$.
(c) Show that there is a bijective map

$$\overline{\varphi}\colon \overline{S}\longrightarrow\varphi(S),$$

which sends an element $\overline{s}$ of $\overline{S}$ to $\varphi(s)$.

Hence there is a bijective correspondence between the set of equivalence classes and the image set of $\varphi$. In other words, the image set $\varphi(S)$ can be identified with the set of equivalence classes.

**Exercise 15.** Let $S$ be a nonempty set. Is the intersection $R\cap R'$ of two equivalence relations $R, R' \subset S\times S$ an equivalence relation? Is the union?

**Exercise 16.** Let $R$ be an equivalence relation on the set $\mathbb{R}$ of real numbers. We may view $R$ as a subset of the $(x, y)$-plane. Explain the geometric meaning of the reflexive and symmetric properties.

**Exercise 17.** With each of the following subsets $R$ of the $(x, y)$-plane, determine which of the axioms for an equivalence relation are satisfied and whether or not $R$ is an equivalence relation on the set $\mathbb{R}$ of real numbers.
(a) $R = \{(s, s) \mid s \in \mathbb{R}\}$.
(b) $R = \emptyset$.
(c) $R = \{(x, y) \mid y = 0\}$.
(d) $R = \{(x, y) \mid xy + 1 = 0\}$.
(e) $R = \{(x, y) \mid x^2y - xy^2 - x + y = 0\}$.
(f) $R = \{(x, y) \mid x^2 - xy + 2x - 2y = 0\}$.

**Exercise 18.** Describe the smallest equivalence relation on the set $\mathbb{R}$ of real numbers which contains the line $x - y = 1$ in the $(x, y)$-plane, and sketch it.

**Basic Assumption.** From now on in this section, assume that $V, W$ are vector spaces over a field $\mathbb{F}$, unless otherwise specified.

Recall from lecture that if $U \subset V$ is a subspace, then a *coset* of $U$ is a subset of the form $x + U := \{x + u \mid u \in U\}$. The following exercise motivates this definition.

**Exercise 19.** Let $\varphi\colon V\longrightarrow W$ be a linear map. Let $U := \operatorname{Ker}\varphi$ and let $x, y \in V$.
(a) Prove that $\varphi(x) = \varphi(y)$ if and only if $y = x + u$ for some element $u \in U$, or equivalently, if and only if $y - x \in U$.
(b) Conclude that the cosets of $U$ partition $V$.

**Exercise 20.** Let $U \subset V$ be a subspace.
(a) Prove that the cosets of $U$ are equivalence classes for the relation

$$x \sim y \quad \text{if} \quad y = x + u, \quad \text{for some } u \in U$$

(b) Conclude that the cosets of $U$ partition $V$.

Recall from lecture the following definition.

**Definition 1.** Let $U \subset V$ be a subspace. The *quotient space* $V/U$ of $V$ modulo $U$ is the set

$$V/U := \{x + U \mid x \in V\}$$

of all cosets of $U$, with addition and scalar multiplication defined by

$$(x+U)+(y+U) := (x+y)+U \qquad \text{“addition”}$$
$$a(x+U) := ax+U \qquad \text{“scalar multiplication”}$$

for every $x, y \in V$, $a \in \mathbb{F}$. The *projection* map is defined by

$$V \longrightarrow V/U =: \overline{V}, \qquad x \longmapsto x+U := \overline{x}.$$

**Exercise 21.** Let $U \subset V$ be a subspace.

(a) Prove that the operations “addition” and “scalar multiplication” in Definition 1 determine well-defined maps

$$V/U \times V/U \xrightarrow{+} V/U \qquad \text{“addition”}$$
$$\mathbb{F} \times V/U \xrightarrow{\cdot} V/U \qquad \text{“scalar multiplication”}$$

(b) Prove that $(V/U, +, \cdot)$ is a vector space over $\mathbb{F}$.

**Exercise 22.** Let $U \subset V$ be a subspace. Prove that if $V$ is finite-dimensional, then

$$\dim V/U = \dim V - \dim U.$$

**Exercise 23.** Consider the subspaces $V \subset V$ and $0 \subset V$. Prove that $V/V = 0$ and $V/0 \cong V$.

**Exercise 24.** Prove Proposition 2.

**Proposition 2.** *Let $f\colon V \longrightarrow V'$ be an epimorphism, and let $U := \operatorname{Ker} f$.*

(a) *Then the induced map*

$$V/U \xrightarrow{\overline{f}} V',$$
$$\overline{x} = x+U \longmapsto \overline{f}(\overline{x}) = f(x)$$

*is an isomorphism.*

(b) *The set of subspaces $A' \subset V'$ is in bijective correspondence with the set of subspaces $A \subset V$ which contain $U$, the correspondence being defined by the maps $A \longmapsto f(A)$ and $A' \longmapsto f^{-1}(A')$.*

**Exercise 25.** Prove the following.

(a) If $A, B \subset V$ are subspaces, then there is an isomorphism of the form

$$A/(A \cap B) \cong (A+B)/B.$$

(b) If $A \subset A' \subset B' \subset B \subset V$ are subspaces, then there is an isomorphism of the form

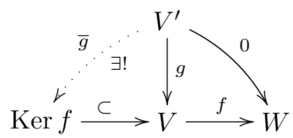
$$B'/A' \cong (B'/A)/(A'/A).$$

Hint: For part (a), consider the inclusion map $A \longrightarrow A+B$. For part (b), note that $A'/A \subset B'/A$ is a subspace and consider the projection map $B' \longrightarrow B'/A$.

**Exercise 26.** Prove Proposition 3.

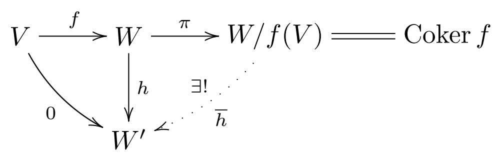
**Proposition 3.** *Let $f\colon V \longrightarrow W$ be a linear map.*

(a) *If $g$ is a linear map which makes the solid diagram*




*commute, then there exists a unique linear map $\overline{g}$ which makes the diagram commute.*

(b) *If $h$ is a linear map which makes the solid diagram*




*commute, then there exists a unique linear map $\overline{h}$ which makes the diagram commute.*