Math 35300: Section 002. Linear algebra II **Fall 2012**
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Homework 9

Exercises 1–10 should be regarded as warm-up exercises. They are intended to test your understanding of some of the definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction.

Exercise 1. In order to be able to discuss the "eigenvalues" of a linear map $f\colon V \longrightarrow W$ at all, f must be

(a) epimorphic (surjective)
(b) isomorphic (bijective)
(c) endomorphic ($V = W$)

Exercise 2. The vector $v \neq 0$ is called an eigenvector for the eigenvalue c if $f(v) = cv$. If instead $f(-v) = cv$, then

(a) $-v$ is an eigenvector for the eigenvalue c.
(b) v is an eigenvector for the eigenvalue $-c$.
(c) $-v$ is an eigenvector for the eigenvalue $-c$.

Exercise 3. If $f\colon V \longrightarrow V$ is an endomorphism and c is an eigenvalue of f, then by the eigenspace E_c of f corresponding to the eigenvalue c, one understands

(a) the set of all eigenvectors for the eigenvalue c
(b) the set consisting of all eigenvectors for the eigenvalue c, together with the zero vector
(c) $\mathrm{Ker}(c\,\mathrm{Id})$

Exercise 4. Which of the following three vectors is an eigenvector of

$$f = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2?$$

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

Exercise 5. Let $f\colon V \longrightarrow V$ be an endomorphism of an n-dimensional vector space, and let $c_1, \dots, c_r$ be the distinct eigenvalues of f. Then

(a) $\dim E_{c_1} + \cdots + \dim E_{c_r} = c_1 + \cdots + c_r$.
(b) $\dim E_{c_1} + \cdots + \dim E_{c_r} \leq n$.
(c) $\dim E_{c_1} + \cdots + \dim E_{c_r} > n$.

Exercise 6. Let $f\colon V \xrightarrow{\cong} V$ be an automorphism of V and c an eigenvalue of f. Then

(a) c is also an eigenvalue of f^{-1}.
(b) $-c$ is an eigenvalue of f^{-1}.
(c) $\frac{1}{c}$ is an eigenvalue of f^{-1}.

Exercise 7. An endomorphism f of an n-dimensional vector space is diagonalizable if and only if

(a) f has n distinct eigenvalues.
(b) f has only one eigenvalue whose geometric multiplicity equals n.
(c) n equals the sum of the geometric multiplicities of the eigenvalues.

Exercise 8. The concepts of eigenvalue, eigenvector, eigenspace, geometric multiplicity, and diagonalizability have been defined for endomorphisms of (sometimes finite-dimensional) vector spaces V. Which further "general assumption" on V have we implicitly made here?

(a) V is always a real vector space.
(b) V is always a Euclidean vector space.
(c) no extra assumption; V is just a vector space over $\mathbb{F}$.

Exercise 9. The characteristic polynomial of $f = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is given by

(a) $P_f(c) = c^2 + c + 6$.
(b) $P_f(c) = c^2 - c + 6$.
(c) $P_f(c) = -c + 7$.

Exercise 10. If $f, g \colon V \longrightarrow V$ are endomorphisms and there exists some $\varphi \in GL(V)$ with $f = \varphi g \varphi^{-1}$, then f and g have

(a) the same eigenvalues
(b) the same eigenvectors
(c) the same eigenspaces

Exercise 11. Determine the eigenvalues and associated eigenspaces for the following 2×2 matrices over both the fields $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$:

$$\text{(a)} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{(d)} \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix} \qquad \text{(e)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{(f)} \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$$

Exercise 12. Prove the following proposition.

Proposition 1. *The following conditions on an endomorphism $f \colon V \longrightarrow V$ of a finite-dimensional vector space are equivalent:*

(a) $\operatorname{Ker} f > 0$.
(b) $\operatorname{Im} f < V$.
(c) *If A is the matrix of the endomorphism with respect to an arbitrary basis, then* $\det A = 0$.
(d) 0 *is an eigenvalue of f.*

Exercise 13. Prove the following: The eigenvalues of an upper or lower triangular matrix are its diagonal entries.

Exercise 14. Prove the following: The characteristic polynomial of an endomorphism $f \colon V \longrightarrow V$ on a finite-dimensional vector space does not depend on the choice of a basis.

Exercise 15. Let $f \colon V \longrightarrow V$ be an endomorphism on a vector space of dimension 2. Assume that f is not multiplication by a scalar. Prove that there is a vector $v \in V$ such that $(v, f(v))$ is a basis of V, and describe the matrix of f with respect to that basis.

Exercise 16. Find all invariant subspaces of the real endomorphism whose matrix is as follows.

$$\text{(a)}\quad \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \qquad \text{(b)}\quad \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$$

Exercise 17. Let $f\colon V \longrightarrow V$ be an endomorphism of a vector space V. Recall that a subspace $U \subset V$ is *invariant* under f if $f(U) \subset U$. Show that the eigenspaces of $f^n := f \circ \cdots \circ f$ are invariant under f.

Exercise 18. An endomorphism $f\colon V \longrightarrow V$ on a vector space is called *nilpotent* if $f^k = 0$ for some k. Let f be a nilpotent endomorphism on a vector space V, and let $W^i := \operatorname{Im} f^i$.

(a) Prove that if $W^i \neq 0$, then $\dim W^{i+1} < \dim W^i$.
(b) Prove that if V has dimension n and if f is nilpotent, then $f^n = 0$.

Exercise 19. Prove that the matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ $(b \neq 0)$ are similar if and only if $a \neq d$.

Exercise 20.

(a) Use the characteristic polynomial to prove that a 2×2 real matrix A all of whose entries are positive has two distinct real eigenvalues.
(b) Prove that the larger eigenvalue has an eigenvector in the first quadrant, and the smaller eigenvalue has an eigenvector in the second quadrant.

Exercise 21.

(a) Find the eigenvectors and eigenvalues of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.
(b) Find a matrix P such that PAP^{-1} is diagonal.
(c) Compute $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{30}$.

Exercise 22. Prove that if A, B are $n \times n$ matrices and if A is invertible, then AB is similar to BA.

Exercise 23. Prove that an endomorphism $f\colon V \longrightarrow V$ on a finite-dimensional vector space is nilpotent if and only if there is a basis of V such that the matrix of f is upper triangular, with diagonal entries zero.

Exercise 24. Let $\mathbb{R}^{\mathbb{N}}$ denote the vector space of real sequences $(a_n)_{n \geq 1}$. Determine the eigenvalues and eigenspaces of the endomorphism $f\colon \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}^{\mathbb{N}}$ given by

$$(a_n)_{n\geq 1} \longmapsto (a_{n+1})_{n \geq 1}.$$

Exercise 25. Since we can both add and compose endomorphisms of V it makes sense to use the polynomial $P(t) = a_0 + a_1 t + \cdots + a_n t^n$, $a_i \in \mathbb{F}$ to define an endomorphism $P(f) = a_0 + a_1 f + \cdots + a_n f^n \colon V \longrightarrow V$. Show that if c is an eigenvalue of f, then $P(c)$ is an eigenvalue of $P(f)$.

Exercise 26. Let $\pi\colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ be a bijective map (permutation). Let $f_\pi \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be defined by $f_\pi(x_1, \ldots, x_n) := (x_{\pi(1)}, \ldots, x_{\pi(n)})$. Determine the set of eigenvalues of f_π.

Exercise 27. Let $f\colon V \longrightarrow V$ be an endomorphism on a real vector space V such that $f^2 = \mathrm{Id}$. Define subspaces as follows:

$$W^+ := \{v \in V \mid f(v) = v\}, \qquad W^- := \{v \in V \mid f(v) = -v\}.$$

Prove that V is isomorphic to the direct sum $W^+ \oplus W^-$.

Exercise 28. Let $f\colon V \longrightarrow V$ be an endomorphism on a finite-dimensional vector space V. Prove that there is an integer n so that $(\mathrm{Ker}\, f^n) \cap (\mathrm{Im}\, f^n) = 0$.