Math 35300: Sections 161 and 162. Linear algebra II **Spring 2013**
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Homework 3

Definition 1. Let V be a vector space over a field $\mathbb{F}$ and suppose $S \subset V$ is a subset. A subspace U of V is called the *smallest subspace of V containing S* if (i) $U \supset S$ and (ii) if W is a subspace of V and $W \supset S$, then $W \supset U$.

Here, condition (i) is read as "U contains S" and condition (ii) is read as "if W is a subspace of V and W contains S, then W contains U."

Exercise 1. Let V be a vector space over a field $\mathbb{F}$. Prove the following.

(a) If $v_1, \ldots, v_r$ are elements of V, then $\mathrm{Span}(v_1, \ldots, v_r)$ is the smallest subspace of V containing $v_1, \ldots, v_r$. (See Definition 1.)
(b) The span of $v_1, \ldots, v_r$ is the same as the span of any reordering of $v_1, \ldots, v_r$.

Exercise 2. Let V be a vector space over a field $\mathbb{F}$. Prove the following.

(a) Any reordering of a linearly independent r-tuple of vectors $(v_1, \ldots, v_r)$ is linearly independent.
(b) An r-tuple of vectors $(v_1, \ldots, v_r)$ is linearly independent if and only if none of these vectors is a linear combination of the others.

Exercise 3. Let V be a finite dimensional vector space over a field $\mathbb{F}$.

(a) Show that any subset of a linearly independent set is linearly independent.
(b) Show that any reordering of a basis is also a basis.

Exercise 4. Prove Proposition 2 below.

Proposition 2. *Let V be a vector space over a field $\mathbb{F}$ and let $v_1, \ldots, v_r$, $w_1, \ldots, w_s$ be vectors of V. If $(v_1, \ldots, v_r)$ is linearly independent and $\mathrm{Span}(v_1, \ldots, v_r, w_1, \ldots, w_s) = V$, then by suitably chosen vectors from $(w_1, \ldots, w_s)$ one can extend $(v_1, \ldots, v_r)$ to a basis of V.*

Exercise 5. Let V be a vector space of dimension n over a field $\mathbb{F}$, and let $0 \leq r \leq n$. Prove that V contains a subspace of dimension r.

Exercise 6. Prove Proposition 3 below.

Proposition 3. *Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Let S, L be finite subsets of V.*

(a) *If $\mathrm{Span}(S) = V$, then $|S| \geq \dim V$ and equality holds only if S is a basis.*
(b) *If L is linearly independent, then $|L| \leq \dim V$ and equality holds only if L is a basis.*

Exercise 7. Prove Proposition 4 below.

Proposition 4. *Let V be a vector space over a field $\mathbb{F}$. If $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ are bases of V, then for each v_i there exists some w_j, so that on replacing v_i by w_j in $(v_1, \ldots, v_n)$ we still have a basis.*

Exercise 8. Let V be a real vector space and $a, b, c, d \in V$. Suppose that

$$\begin{aligned} v_1 &= a + b + c + d \\ v_2 &= 2a + 2b + c - d \\ v_3 &= a + b + 3c - d \\ v_4 &= a - c + d \\ v_5 &= -b + c - d \end{aligned}$$

Show that $(v_1, \ldots, v_5)$ is linearly dependent. (Remark: One can solve this exercise by expressing one of the v_i as a linear combination of the other four. But there is a proof in which one does not need to do any calculations.)

Definition 5. Let V be a vector space over a field $\mathbb{F}$ and let U_1, U_2 be subspaces of V. We say that U_1 and U_2 are *complementary subspaces* if $U_1 + U_2 = V$ and $U_1 \cap U_2 = \{0\}$.

For instance, consider the real vector space $V = \mathbb{R}^3$. It is easy to check that (i) $U_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$ and $U_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, x_2 = 0\}$ are complementary to each other and (ii) $U_1 = V$ and $U_2 = \{0\}$ are complementary to each other. The following exercise shows that there are many other examples.

Exercise 9. Let V be a vector space of dimension n over a field $\mathbb{F}$. Show that if U_1 is a subspace of dimension p, then there exists a subspace U_2 complementary to U_1, and each such subspace U_2 has dimension $n - p$.

Given a complex vector space V one can make a real vector space from it by simply restricting the scalar multiplication $\mathbb{C} \times V \longrightarrow V$ to $\mathbb{R} \times V \longrightarrow V$. Since on restriction the concepts "span" and "dimension" take on a new meaning, we sometimes write $\mathrm{Span}_{\mathbb{C}}$ and $\dim_{\mathbb{C}}$ (resp. $\mathrm{Span}_{\mathbb{R}}$ and $\dim_{\mathbb{R}}$), when regarding V as a complex (resp. real) vector space.

Exercise 10. For each $n \geq 0$ determine for which pairs (r, s) of numbers there exists a complex vector space and vectors $(v_1, \ldots, v_n)$ in it, such that

$$\begin{aligned} r &= \dim_{\mathbb{R}} \mathrm{Span}_{\mathbb{C}}(v_1, \ldots, v_n), \\ s &= \dim_{\mathbb{R}} \mathrm{Span}_{\mathbb{R}}(v_1, \ldots, v_n). \end{aligned}$$

Exercise 11. Let V be a finite dimensional vector space over a field $\mathbb{F}$, and let U_1, U_2 be subspaces of V. The formula $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$ is analogous to the formula $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$, which holds for sets. If three sets are given, then

$$\begin{aligned} |S_1 \cup S_2 \cup S_3| = {} & |S_1| + |S_2| + |S_3| \\ & - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|. \end{aligned}$$

Does the corresponding formula for dimensions of subspaces hold? Prove or find a counter-example.