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Homework 4

Exercise 1. Prove Proposition 1 below.

Proposition 1. Let V be a vector space over a field \mathbb{F} . If V is finite-dimensional and $U \subset V$ is a subspace, then U is also finite-dimensional and $\dim U \leq \dim V$, with equality if and only if U = V.

Exercise 2. Prove Proposition 2 below.

Proposition 2. Let V be a vector space over a field \mathbb{F} . If W_1, \ldots, W_n are subspaces of V, then the sum $W_1 + \cdots + W_n$ is the smallest subspace of V containing W_1, \ldots, W_n .

Exercise 3. Prove Proposition 3 below.

Proposition 3. Let V be a vector space over a field \mathbb{F} .

- (a) A single subspace W_1 is independent.
- (b) Two subspaces W_1, W_2 are independent if and only if $W_1 \cap W_2 = \{0\}$.

Exercise 4. Prove Proposition 4 below.

Proposition 4. Let V be a vector space over a field \mathbb{F} . If W_1, \ldots, W_n are subspaces of V, then $V = W_1 \oplus \cdots \oplus W_n$ if and only if every vector $v \in V$ can be written in the form

$$v = w_1 + \cdots + w_n$$
, (where w_i is a vector in W_i)

in exactly one way.

Exercise 5. Prove Proposition 5 below.

Proposition 5. Let W_1, \ldots, W_n be subspaces of a finite-dimensional vector space V, and let \mathbf{B}_i be a basis for W_i .

- (a) The ordered set **B** obtained by listing the bases $\mathbf{B}_1, \ldots, \mathbf{B}_n$ in order is a basis of V if and only if $V = W_1 \oplus \cdots \oplus W_n$.
- (b) $\dim(W_1 + \cdots + W_n) \leq \dim(W_1) + \cdots + \dim(W_n)$, with equality if and only if the subspaces W_1, \ldots, W_n are independent.

Exercise 6. Let V be a vector space over a field \mathbb{F} . Show that $V = V \oplus \{0\}$.

Exercise 7. Prove Proposition 6 below.

Proposition 6. Let V, W be vector spaces over a field \mathbb{F} . Let $f: V \longrightarrow W$ be a linear map. Then f is injective if and only if $\operatorname{Ker} f = 0$.

Exercise 8. Prove Proposition 7 below.

Proposition 7. Let V, W be vector spaces over a field \mathbb{F} . Let $f: V \longrightarrow W$ be an isomorphism. If (v_1, \ldots, v_r) is a linearly independent r-tuple of vectors in V, then the r-tuple of vectors $(f(v_1), \ldots, f(v_r))$ in W is also linearly independent.

Exercise 9. Prove Proposition 8 below.

Proposition 8. Let V, W be vector spaces over a field \mathbb{F} . If (v_1, \ldots, v_n) is a basis of V, then a linear map $f \colon V \longrightarrow W$ is an isomorphism if and only if $(f(v_1), \ldots, f(v_n))$ is a basis of W.

Exercise 10. Prove Proposition 9 below.

Proposition 9. Let V, W be finite-dimensional vector spaces over a field \mathbb{F} . If $\dim(V) = \dim(W)$, then a linear map $f \colon V \longrightarrow W$ is surjective if and only if it is injective.