Math 35300: Sections 161 and 162. Linear algebra II **Spring 2013**
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Homework 9

Exercise 1. Determine the eigenvalues and associated eigenspaces for the following 2×2 matrices over both the fields $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$:

$$(a) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (b) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (c) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(d) \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix} \qquad (e) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad (f) \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$$

Exercise 2. Prove the following proposition.

Proposition 1. *The following conditions on an endomorphism $f\colon V \longrightarrow V$ of a finite-dimensional vector space are equivalent:*

(a) $\operatorname{Ker} f > 0$.
(b) $\operatorname{Im} f < V$.
(c) *If A is the matrix of the endomorphism with respect to an arbitrary basis, then* $\det A = 0$.
(d) 0 *is an eigenvalue of f.*

Exercise 3. Prove the following: The eigenvalues of an upper or lower triangular matrix are its diagonal entries.

Exercise 4. Let $f\colon V \longrightarrow V$ be an endomorphism on a vector space of dimension 2. Assume that f is not multiplication by a scalar. Prove that there is a vector $v \in V$ such that $(v, f(v))$ is a basis of V, and describe the matrix of f with respect to that basis.

Exercise 5. Find all invariant subspaces of the real endomorphism whose matrix is as follows.

$$(a) \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \qquad (b) \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$$

Exercise 6. Let $f\colon V \longrightarrow V$ be an endomorphism of a vector space V. Recall that a subspace $U \subset V$ is *invariant* under f if $f(U) \subset U$. Show that the eigenspaces of $f^n := f \circ \cdots \circ f$ are invariant under f.

Exercise 7. An endomorphism $f\colon V \longrightarrow V$ on a vector space is called *nilpotent* if $f^k = 0$ for some k. Let f be a nilpotent endomorphism on a vector space V, and let $W^i := \operatorname{Im} f^i$.

(a) Prove that if $W^i \neq 0$, then $\dim W^{i+1} < \dim W^i$.
(b) Prove that if V has dimension n and if f is nilpotent, then $f^n = 0$.

Exercise 8. Prove that the matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ $(b \neq 0)$ are similar if and only if $a \neq d$.

Exercise 9.

(a) Use the characteristic polynomial to prove that a 2×2 real matrix A all of whose entries are positive has two distinct real eigenvalues.
(b) Prove that the larger eigenvalue has an eigenvector in the first quadrant, and the smaller eigenvalue has an eigenvector in the second quadrant.

Exercise 10.

(a) Find the eigenvectors and eigenvalues of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.
(b) Find a matrix P such that PAP^{-1} is diagonal.
(c) Compute $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{30}$.

Exercise 11. Prove that if A, B are $n \times n$ matrices and if A is invertible, then AB is similar to BA.

Exercise 12. Prove that an endomorphism $f\colon V \longrightarrow V$ on a finite-dimensional vector space is nilpotent if and only if there is a basis of V such that the matrix of f is upper triangular, with diagonal entries zero.

Exercise 13. Let $\mathbb{R}^{\mathbb{N}}$ denote the vector space of real sequences $(a_n)_{n\geq 1}$. Determine the eigenvalues and eigenspaces of the endomorphism $f\colon \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}^{\mathbb{N}}$ given by

$$(a_n)_{n\geq 1} \longmapsto (a_{n+1})_{n\geq 1}.$$

Exercise 14. Since we can both add and compose endomorphisms of V it makes sense to use the polynomial $P(t) = a_0 + a_1 t + \cdots + a_n t^n$, $a_i \in \mathbb{F}$ to define an endomorphism $P(f) = a_0 + a_1 f + \cdots + a_n f^n\colon V \longrightarrow V$. Show that if c is an eigenvalue of f, then $P(c)$ is an eigenvalue of $P(f)$.

Exercise 15. Let $f\colon V \longrightarrow V$ be an endomorphism on a real vector space V such that $f^2 = \mathrm{Id}$. Define subspaces as follows:

$$W^+ := \{v \in V \mid f(v) = v\}, \qquad W^- := \{v \in V \mid f(v) = -v\}.$$

Prove that V is isomorphic to the direct sum $W^+ \oplus W^-$.

Exercise 16. Let $f\colon V \longrightarrow V$ be an endomorphism on a finite-dimensional vector space V. Prove that there is an integer n so that $(\mathrm{Ker}\, f^n) \cap (\mathrm{Im}\, f^n) = 0$.

Exercise 17. Consider the symmetric matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Find an orthogonal matrix $P \in O(3)$ so that PAP^t is diagonal.

Exercise 18. Prove the following proposition from lecture.

Proposition 2. *Let $(V, \langle\ ,\ \rangle)$ be a Euclidean vector space.*

(a) *If $(v_1, \ldots, v_n)$ is an orthonormal basis of V, then the corresponding matrix A of an endomorphism $f\colon V \longrightarrow V$ is given by*

$$a_{ij} = \langle f(v_j), v_i \rangle.$$

(b) *If $(v_1, \ldots, v_n)$ is an orthonormal basis of V, then an endomorphism $f\colon V \longrightarrow V$ is self-adjoint if and only if the corresponding matrix A is symmetric.*

Exercise 19. Prove the following proposition. (Hint: this may be argued similar to the proof of the Spectral Theorem given in lecture).

Proposition 3.

(a) *(Vector space form): Let $f\colon V \longrightarrow V$ be an endomorphism of a finite-dimensional complex vector space V. Then there is a basis $\mathbf{B}$ of V such that the corresponding matrix A of f is upper triangular (i.e., all entries below the diagonal are zero).*

(b) *(Matrix form): Let A be a complex $n \times n$ matrix. Then there is an invertible matrix $P \in \mathsf{M}(n \times n, \mathbb{C})$ such that PAP^{-1} is upper triangular.*

Exercise 20. Use the Spectral Theorem proved in lecture to give a short proof of the following proposition. (Hint: Even a one line proof can be given).

Proposition 4. *Let $f\colon V \longrightarrow V$ be a self-adjoint endomorphism on an n-dimensional $(n \geq 1)$ Euclidean vector space. Then there exists an orthogonal map*

$$P\colon \mathbb{R}^n \xrightarrow{\cong} V$$

so that the matrix of f with respect to P has the form

$$\begin{bmatrix} c_1 & & & & & & \\ & \ddots & & & & & \\ & & c_1 & & & & \\ & & & \ddots & & & \\ & & & & c_r & & \\ & & & & & \ddots & \\ & & & & & & c_r \end{bmatrix}$$

of the indicated diagonal matrix. Here $c_1, \ldots, c_r$ are the distinct eigenvalues of f, the number of each appearing on the diagonal being equal to the geometric multiplicity.

Exercise 21. Let $f\colon V \longrightarrow V$ be a self-adjoint endomorphism on a Euclidean vector space V. Prove the following: If v, w are eigenvectors of f corresponding to distinct eigenvalues $c \neq d$, then $v \perp w$.

Exercise 22. Use the Spectral Theorem proved in lecture to give a proof of the following proposition. (Hint: Exercise 21 should also be helpful).

Proposition 5. *Let $f\colon V \longrightarrow V$ be a self-adjoint endomorphism of a finite-dimensional Euclidean vector space, $c_1, \ldots, c_r$ its distinct eigenvalues, and $P_k\colon V \longrightarrow E_{c_k} \subset V$ the orthogonal projection onto the eigenspace E_{c_k}. Then*

$$f = c_1 P_1 + \cdots + c_r P_r.$$

Exercise 23. Let V be a finite-dimensional real vector space. Show that an endomorphism $f\colon V \longrightarrow V$ is diagonalizable if and only if there exists an inner product $\langle\ ,\ \rangle$ on V for which f is self-adjoint.

Exercise 24. Let V be a finite-dimensional Euclidean vector space and $U \subset V$ a subspace. Show that the orthogonal projection $P\colon V \longrightarrow U \subset V$ is self-adjoint, and determine its eigenvalues and eigenspaces.

Exercise 25. Let V be a finite-dimensional Euclidean vector space. Show that two self-adjoint endomorphisms $f, g \colon V \longrightarrow V$ can be diagonalized by the same orthogonal map $P \colon \mathbb{R}^n \xrightarrow{\cong} V$ if and only if they *commute* (i.e., $f \circ g = g \circ f$).