Math 35300: Sections 161 and 162. Linear algebra II **Spring 2013**
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Handout 3

Subfields of the complex numbers. Instead of calculating with only $\mathbb{R}$ or $\mathbb{C}$, one can work with any subfield of the complex numbers. A *subfield* of $\mathbb{C}$ is any subset which is closed under the four operations addition, subtraction, multiplication, and division, and which contains 1. In other words, a subset $\mathbb{F} \subset \mathbb{C}$ is a subfield of $\mathbb{C}$ if the following properties hold:

(a) $a + b \in \mathbb{F}$ for all $a, b \in \mathbb{F}$.
(b) If $a \in \mathbb{F}$, then $-a \in \mathbb{F}$.
(c) $ab \in \mathbb{F}$ for all $a, b \in \mathbb{F}$.
(d) If $a \in \mathbb{F}$ and $a \neq 0$, then $a^{-1} \in \mathbb{F}$.
(e) $1 \in \mathbb{F}$.

The following are examples of subfields of $\mathbb{C}$.

(i) $\mathbb{F} = \mathbb{R}$ the field of real numbers.
(ii) $\mathbb{F} = \mathbb{Q}$ the field of rational numbers.
(iii) $\mathbb{F} = \mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

The abstract notion of a field. Instead of working only with subfields of $\mathbb{C}$, one can list the properties of the "scalars" that are needed axiomatically. This leads to the abstract notion of a "field", which contains important new classes of fields including the finite fields. Axioms (1)–(9) below are modeled on calculation with real or complex numbers. In other words, they are essentially designed so that one can calculate in a field "exactly as" one would calculate in $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. A *field* is a triple $(\mathbb{F}, +, \cdot)$ consisting of a set $\mathbb{F}$ and two operations

$$\mathbb{F} \times \mathbb{F} \xrightarrow{+} \mathbb{F}, \qquad (x, y) \longmapsto x + y \qquad \text{"addition"}$$

$$\mathbb{F} \times \mathbb{F} \xrightarrow{\cdot} \mathbb{F}, \qquad (x, y) \longmapsto xy \qquad \text{"multiplication"}$$

such that the following nine axioms hold:

(1) $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{F}$.
(2) $a + b = b + a$ for all $a, b \in \mathbb{F}$.
(3) There exists an element $0 \in \mathbb{F}$ with $a + 0 = a$ for all $a \in \mathbb{F}$.
(4) For each $a \in \mathbb{F}$ there exists an element $-a \in \mathbb{F}$ with $a + (-a) = 0$.
(5) $a(bc) = (ab)c$ for all $a, b, c \in \mathbb{F}$.
(6) $ab = ba$ for all $a, b \in \mathbb{F}$.
(7) There exists an element $1 \in \mathbb{F}$, $1 \neq 0$, such that $1a = a$ for all $a \in \mathbb{F}$.
(8) For all $a \in \mathbb{F}$ with $a \neq 0$ there exists an element $a^{-1} \in \mathbb{F}$ with $a^{-1}a = 1$.
(9) $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{F}$.