Math 35300: Sections 161 and 162. Linear algebra II **Spring 2013**
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Homework 4

Exercise 1. Prove Proposition 1 below.

Proposition 1. *Let V be a vector space over a field $\mathbb{F}$. If V is finite-dimensional and $U \subset V$ is a subspace, then U is also finite-dimensional and* $\dim U \leq \dim V$*, with equality if and only if $U = V$.*

Exercise 2. Prove Proposition 2 below.

Proposition 2. *Let V be a vector space over a field $\mathbb{F}$. If $W_1, \ldots, W_n$ are subspaces of V, then the sum $W_1 + \cdots + W_n$ is the smallest subspace of V containing $W_1, \ldots, W_n$.*

Exercise 3. Prove Proposition 3 below.

Proposition 3. *Let V be a vector space over a field $\mathbb{F}$.*
(a) *A single subspace W_1 is independent.*
(b) *Two subspaces W_1, W_2 are independent if and only if $W_1 \cap W_2 = \{0\}$.*

Exercise 4. Prove Proposition 4 below.

Proposition 4. *Let V be a vector space over a field $\mathbb{F}$. If $W_1, \ldots, W_n$ are subspaces of V, then $V = W_1 \oplus \cdots \oplus W_n$ if and only if every vector $v \in V$ can be written in the form*

$$v = w_1 + \cdots + w_n, \qquad (\textit{where } w_i \textit{ is a vector in } W_i)$$

in exactly one way.

Exercise 5. Prove Proposition 5 below.

Proposition 5. *Let $W_1, \ldots, W_n$ be subspaces of a finite-dimensional vector space V, and let $\mathbf{B}_i$ be a basis for W_i.*
(a) *The ordered set $\mathbf{B}$ obtained by listing the bases $\mathbf{B}_1, \ldots, \mathbf{B}_n$ in order is a basis of V if and only if $V = W_1 \oplus \cdots \oplus W_n$.*
(b) $\dim(W_1 + \cdots + W_n) \leq \dim(W_1) + \cdots + \dim(W_n)$*, with equality if and only if the subspaces $W_1, \ldots, W_n$ are independent.*

Exercise 6. Let V be a vector space over a field $\mathbb{F}$. Show that $V = V \oplus \{0\}$.

Exercise 7. Prove Proposition 6 below.

Proposition 6. *Let V, W be vector spaces over a field $\mathbb{F}$. Let $f\colon V \longrightarrow W$ be a linear map. Then f is injective if and only if* $\operatorname{Ker} f = 0$.

Exercise 8. Prove Proposition 7 below.

Proposition 7. *Let V, W be vector spaces over a field $\mathbb{F}$. Let $f\colon V \longrightarrow W$ be an isomorphism. If $(v_1, \ldots, v_r)$ is a linearly independent r-tuple of vectors in V, then the r-tuple of vectors $(f(v_1), \ldots, f(v_r))$ in W is also linearly independent.*

Exercise 9. Prove Proposition 8 below.

Proposition 8. *Let V, W be vector spaces over a field $\mathbb{F}$. If $(v_1, \ldots, v_n)$ is a basis of V, then a linear map $f\colon V \longrightarrow W$ is an isomorphism if and only if $(f(v_1), \ldots, f(v_n))$ is a basis of W.*

Exercise 10. Prove Proposition 9 below.

Proposition 9. *Let V, W be finite-dimensional vector spaces over a field $\mathbb{F}$. If $\dim(V) = \dim(W)$, then a linear map $f\colon V \longrightarrow W$ is surjective if and only if it is injective.*