Math 35300: Sections 161 and 162. Linear algebra II **Spring 2013**
John E. Harper
Purdue University

Homework 7

We would like to define the determinant not only for an $n \times n$ matrix, but also for an endomorphism $f\colon V \longrightarrow V$ of an n-dimensional vector space over $\mathbb{F}$. One possibility is to choose some basis $(v_1, \ldots, v_n)$ of V, consider the commutative diagram

$$\begin{tikzcd} V \arrow[r, "f"] & V \\ \mathbb{F}^n \arrow[u, "\Phi"', "\cong"] \arrow[r, "A"] & \mathbb{F}^n \arrow[u, "\cong"', "\Phi"] \end{tikzcd}$$

(i.e., A is the matrix associated to f relative to the basis $(v_1, \ldots, v_n)$ and Φ is the canonical basis isomorphism), and declare $\det f := \det A$. The only potential problem with this definition is that it appears to depend upon the choice of basis. In other words, could another choice of basis, and hence another matrix, mean another determinant for f? The following exercise settles this question.

Exercise 1. Prove the following: If $f\colon V \longrightarrow V$ is an endomorphism of an n-dimensional vector space, and if f is represented by the matrix A relative to a basis $(v_1, \ldots, v_n)$ and by the matrix A' relative to a basis $(v'_1, \ldots, v'_n)$ then $\det A = \det A'$.

Exercise 2. Compute $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}^n$

Exercise 3. Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

Exercise 4. Let A, B be square matrices.
(a) When is $(A+B)(A-B) = A^2 - B^2$?
(b) Expand $(A+B)^3$.

Exercise 5. Let D be the diagonal matrix

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

and let $A = (a_{ij})$ be any $n \times n$ matrix.
(a) Compute the products DA and AD.
(b) Compute the product of two diagonal matrices.
(c) When is a diagonal matrix invertible?

Exercise 6. A square matrix A is called *nilpotent* if $A^k = 0$ for some $k > 0$.
(a) Prove that if A is nilpotent then $\det A = 0$.
(b) Prove that if A is nilpotent then $I + A$ is invertible.

Exercise 7. A matrix A is called *symmetric* if $A = A^t$. Prove that for any matrix A, the matrix AA^t is symmetric and that if A is a square matrix then $A + A^t$ is symmetric.

Exercise 8.

(a) Prove that $(AB)^t = B^t A^t$ and that $A^{tt} = A$.
(b) Prove that if A is invertible then $(A^{-1})^t = (A^t)^{-1}$.

Exercise 9. Prove that the inverse of an invertible symmetric matrix is also symmetric.

Exercise 10. Let A and B be symmetric $n \times n$ matrices. Prove that the product AB is symmetric if and only if $AB = BA$.

Exercise 11. Let A be an $n \times n$ matrix. What is $\det(-A)$?

Exercise 12. Prove that $\det A^t = \det A$.

Exercise 13. Derive the formula $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ from the special properties (i)–(iii),(v) (proved in lecture) of the determinant.

Exercise 14. Let A and B be $n \times n$ matrices. Prove that $\det(AB) = \det(BA)$.

Exercise 15. Prove that $\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (\det A)(\det D)$, if A and D are square blocks.

Exercise 16. Let a $2n \times 2n$ matrix be given in the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where each block is an $n \times n$ matrix. Suppose that A is invertible and that $AC = CA$. Prove that $\det M = \det(AD - CB)$. Give an example to show that this formula need not hold when $AC \neq CA$.