Math 35300: Sections 161 and 162. Linear algebra II **Spring 2013**
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Warm-up Questions 9

The following warm-up questions are intended to test your understanding of some of the basic definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction. They will not be collected or graded.

Question 1. In order to be able to discuss the "eigenvalues" of a linear map $f\colon V \longrightarrow W$ at all, f must be

(a) epimorphic (surjective)
(b) isomorphic (bijective)
(c) endomorphic ($V = W$)

Question 2. The vector $v \neq 0$ is called an eigenvector for the eigenvalue c if $f(v) = cv$. If instead $f(-v) = cv$, then

(a) $-v$ is an eigenvector for the eigenvalue c.
(b) v is an eigenvector for the eigenvalue $-c$.
(c) $-v$ is an eigenvector for the eigenvalue $-c$.

Question 3. If $f\colon V \longrightarrow V$ is an endomorphism and c is an eigenvalue of f, then by the eigenspace E_c of f corresponding to the eigenvalue c, one understands

(a) the set of all eigenvectors for the eigenvalue c
(b) the set consisting of all eigenvectors for the eigenvalue c, together with the zero vector
(c) $\mathrm{Ker}(c\,\mathrm{Id})$

Question 4. Which of the following three vectors is an eigenvector of

$$f = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2?$$

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

Question 5. Let $f\colon V \longrightarrow V$ be an endomorphism of an n-dimensional vector space, and let $c_1, \ldots, c_r$ be the distinct eigenvalues of f. Then

(a) $\dim E_{c_1} + \cdots + \dim E_{c_r} = c_1 + \cdots + c_r$.
(b) $\dim E_{c_1} + \cdots + \dim E_{c_r} \leq n$.
(c) $\dim E_{c_1} + \cdots + \dim E_{c_r} > n$.

Question 6. Let $f\colon V \xrightarrow{\cong} V$ be an automorphism of V and c an eigenvalue of f. Then

(a) c is also an eigenvalue of f^{-1}.
(b) $-c$ is an eigenvalue of f^{-1}.
(c) $\frac{1}{c}$ is an eigenvalue of f^{-1}.

Question 7. An endomorphism f of an n-dimensional vector space is diagonalizable if and only if

(a) f has n distinct eigenvalues.
(b) f has only one eigenvalue whose geometric multiplicity equals n.
(c) n equals the sum of the geometric multiplicities of the eigenvalues.

Question 8. The concepts of eigenvalue, eigenvector, eigenspace, geometric multiplicity, and diagonalizability have been defined for endomorphisms of (sometimes finite-dimensional) vector spaces V. Which further "general assumption" on V have we implicitly made here?
(a) V is always a real vector space.
(b) V is always a Euclidean vector space.
(c) no extra assumption; V is just a vector space over $\mathbb{F}$.

Question 9. The characteristic polynomial of $f = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is given by
(a) $P_f(c) = c^2 + c + 6$.
(b) $P_f(c) = c^2 - c + 6$.
(c) $P_f(c) = -c + 7$.

Question 10. If $f, g \colon V \longrightarrow V$ are endomorphisms and there exists some $\varphi \in GL(V)$ with $f = \varphi g \varphi^{-1}$, then f and g have
(a) the same eigenvalues
(b) the same eigenvectors
(c) the same eigenspaces

Question 11. Determine the eigenvalues and associated eigenspaces for the following 2×2 matrices over both the fields $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$:

$$\text{(a)} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{(d)} \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix} \qquad \text{(e)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{(f)} \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$$

Question 12. An endomorphism $f \colon V \longrightarrow V$ of a Euclidean vector space V is said to be *self-adjoint* (or *symmetric*) if, for all $v, w \in V$, we have
(a) $\langle f(v), f(w) \rangle = \langle v, w \rangle$.
(b) $\langle v, f(w) \rangle = \langle f(v), w \rangle$
(c) $\langle f(v), w \rangle = \langle w, f(v) \rangle$

Question 13. If $c_1, \dots, c_r$ are eigenvalues of a self-adjoint endomorphism, $c_i \neq c_j$ for $i \neq j$, and v_i is an eigenvector for c_i, $i = 1, \dots r$, then for $i \neq j$
(a) $c_i \perp c_j$
(b) $v_i \perp v_j$
(c) $E_{c_i} \perp E_{c_j}$

Question 14. Let V be a finite-dimensional Euclidean vector space. The assertion that for each f-invariant subspace $U \subset V$ also $U^\perp \subset V$ is invariant under f holds
(a) for each self-adjoint endomorphism $f \colon V \longrightarrow V$
(b) for each orthogonal endomorphism $f \colon V \longrightarrow V$
(c) for each endomorphism $f \colon V \longrightarrow V$

Question 15. Let A be a real $n \times n$ matrix and $z \in \mathbb{C}^n$ a complex eigenvector, $z = x + iy$ with $x, y \in \mathbb{R}^n$, for the real eigenvalue c. Suppose that $y \neq 0$. Then

(a) $y \in \mathbb{R}^n$ is an eigenvector of A for the eigenvalue c.
(b) $y \in \mathbb{R}^n$ is an eigenvector of A for the eigenvalue ic.
(c) if $x \neq 0$, then $y \in \mathbb{R}^n$ *cannot* be an eigenvector of A.

Question 16. Let V be an n-dimensional Euclidean vector space and $U \subset V$ be a k-dimensional subspace. When is the orthogonal projection $P\colon V \longrightarrow U \subset V$ self-adjoint?

(a) always
(b) only for $0 < k \leq n$
(c) only for $0 \leq k < n$

Question 17. Does there exist an inner product on $\mathbb{R}^2$ for which the shear is self-adjoint?

(a) No, because $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.
(b) Yes, let $\langle x, y \rangle := x_1 y_1 + x_1 y_2 + x_2 y_2$.
(c) Yes, because the standard inner product already has this property.

Question 18. Let $f\colon V \longrightarrow V$ be a self-adjoint endomorphism and let $(v_1, \ldots, v_n)$ be a basis of eigenvectors with $\|v_i\| = 1$ for $i = 1, \ldots, n$. Is $(v_1, \ldots, v_n)$ then already an orthonormal basis?

(a) Yes, by definition of an orthonormal basis.
(b) Yes, because the eigenvectors of a self-adjoint endomorphism are orthogonal to each other.
(c) No, because the eigenspaces do not need to be one-dimensional.

Question 19. If a symmetric real $n \times n$ matrix A has only one eigenvalue c, then

(a) A is already diagonal.
(b) $a_{ij} = c$ for all $i, j = 1, \ldots, n$.
(c) $n = 1$.

Question 20. How does the Jordan normal form of $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ look?

(a) $\begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & & \\ & 2 & 1 \\ & & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}$